Thermalization: from SYK to QGP (via Random Matrices)

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Thermalization

- One of great problems in T & MP is the problem of thermalization.
- The problem is how to explain the appearance of the Gibbs distribution for many body system if we start from the microscopic Newton's or Schrodinger equations.
- This problem was studied by Boltzmann, Maxwell, Poincare, von Neumann, Pauli, Bogolyubov and many others and we are still far from its solution.

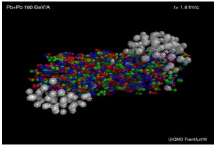
Thermalization

Thermalization in a quantum system is a major theoretical challenge.

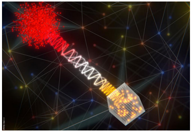
It is involved in many problems of physics which involve initial states out of equilibrium:

- Early Universe

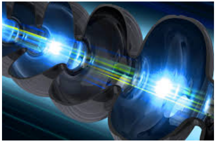
- Heavy ion collisions



- Dynamics of cold atomic gas



- XRay Laser, etc.



In the holographic approach

Thermalization $\leftrightarrow$ Black hole (BH) creation in AdS

- Via holographic duality BHs are described by thermal states of a dual QFT.
- The process of BH formation corresponds to the process of thermalization of certain (unitary) QFTs, evolving from nonequilibrium initial states towards thermal equilibrium.
  I'll discuss these two problems.
- As usual, very instructive starting points are the simplest (TOY) models, which contain enough structure to take into account the subtleties that are present in more complex realistic theories.
- By this reason low-dimensional holographic models, such as AdS_3/CFT_2, and $NAdS_2/NCFT_1$ came to the stage again.
- I'll speak on a recent progress in this direction related with random matrices and SYK model and I'll make few comment concerning applications to quark-gluon plasma (QGP).

Outlook

- Introduction
  - Thermalization
  - Quantum quench
- Thermalization in holographical approach
  - Thermalization after bilocal quench
  - Colliding particle and BH creation in AdS_3
  - Colliding shocks in deformed AdS_5 and QGP formation
- Sachdev-Ye-Kitaev (SYK) model
  - Randomness and replica in SYK model
  - Physics of SYK model
  - Non-diagonal saddles and phase structure
  - Spontaneous symmetry breaking vs quenches
- Matrix ensembles and gravity
  - GUE
  - Sinh-UE and JT
  - Gas of baby Universes in JT
- Conclusion

Thermalization

Let me remind the basic fact about thermalization.
The problem of quantum thermalization can be stated as:

- Given microscopic unitarity, how do Gibbs ensembles emerge?
  - If a many body quantum system is set in an initial pure state $|\psi(0)\rangle$, the evolved state $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ is pure and time dependent, so it can never become a time independent mixed density matrix, such as Gibbs ensembles:

$$|\psi(t)\rangle\langle\psi(t)| \neq \rho_{Gibbs}$$

- The dynamical emergence of Gibbs ensembles from unitary dynamics is the problem of quantum thermalization.
- This problem is almost as old as quantum mechanics itself.
- There is a similar problem in BH physics (information loss paradox in gravity) due to Hawking radiation

Thermalization

Since exact thermality cannot be attained within unitary evolution, we can expect approximate thermality

$$\langle\psi(t)|\,\mathcal{O}\,|\psi(t)\rangle = \mathrm{Tr}(\rho_{Gibbs}\,\mathcal{O}) + error\ .$$

for large t
The error – negligible in the thermodynamic limit.

$$\langle E_a|\,\mathcal{O}\,|E_a\rangle = \mathrm{Tr}(\rho_{Gibbs}\,\mathcal{O}) + error$$

This almost equivalent to Eigenstate Thermalzation Hypothesis.

In classical mechanics the emergence of thermal behavior is related with dynamical chaos, or more formally of ergodicity:
the delocalization of a general initial state over phase space = dynamical chaos.

Thermalization from quantum quench

A natural setup to study thermalization in *closed* quantum systems is **quantum quench**:

- Prepare a pure state $|\psi\rangle$ - ground state of a Hamiltonian H_1;
- Evolve it with $H_2 \neq H_1$, and consider $|\psi(t)\rangle = \mathrm{e}^{-iH_2 t}|\psi\rangle$.

Let A be a subsystem, with density matrix $\rho_A(t) = \mathrm{Tr}_{\bar{A}}|\psi(t)\rangle\langle\psi(t)|$.
The system **thermalizes**, if for any subsystem A it is true that

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \rho_A(t)dt = \rho_\beta = \frac{1}{Z}\mathrm{e}^{-\beta H_2} \quad \text{for some } \beta\,;$$

How do we probe it?

- Entanglement entropy: $S(A) = -\mathrm{Tr}_A\, \rho_A \log \rho_A$ - our primary tool
- More "fine-grained" observables: Renyi entropies, correlation functions of specific operators, ...

Quenches in CFT

CFT is a convenient arena to study quench dynamics (Cardy, Calabrese '05).

- **Global quench** - excites every point of the circle uniformly in the initial state - **popular setup for studies thermalization**: Balasubramanian et. al., Lopez et. al.; Liu et. al.; Hartman et. al.; Mezei, Stanford; I.A., Ageev and many others

Quenches in CFT/gravity

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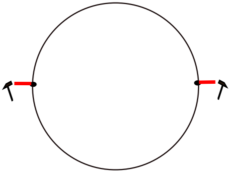
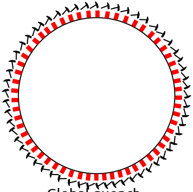
- **Local quench**: excites a point of the circle in the initial state
- **bilocal quench**: excites two antipodal points of the circle in the initial state - I.A., Khramtsov,Tikhanovskaya, 1706.07390(from Gr.side), and recently Caputa, Numasawa, Shimaji, Takayanagi, Wei, 1905.08265(from CFT)
  multiple by Guo, He, Luo,1802.08815,



Bilocal quench



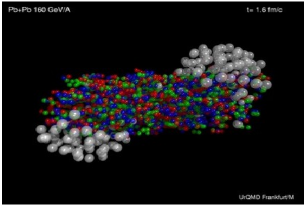
Global quench

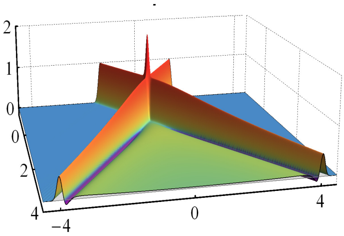
A local quench by a hammer

QGP creation in HIC

Within AdS/CFT correspondence:

Thermalization $\leftrightarrow$ Black hole creation in AdS



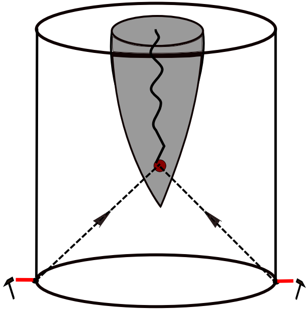


- Heavy Ions collisions $\Longleftrightarrow$ Shock wave collisions (IA, UFN'14)
- There is a possibility to check in d=2 case

This is a main motivation to study $d = 2$.

The bulk dual of the bilocal quench

- We interpret an injection of massless particles from the boundary as *local* excitations in the boundary CFT_2
- We are interested in thermalization $\Rightarrow$ we study the case when the colliding particles **produce a black hole**.



Holographic entanglement entropy: results

- Static thermal equilibrium regime:

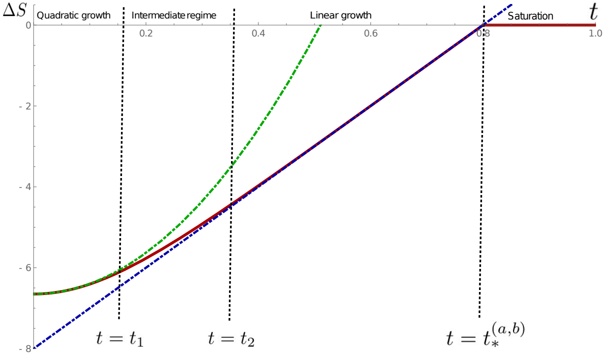
$$S_{\text{eq}}(a,\ b) = \frac{c}{3}\log\left(\frac{2}{\epsilon}\sinh\left(R\frac{\Delta\varphi}{2}\right)\right)\ ;$$

- Dynamic non-equilibrium regime: crossing geodesic dominates (only for segments containing one excitation).

$$S_{\text{non-eq}}(a,\ b|t) = \frac{c}{6}\log\left[\frac{2}{\epsilon}\Big\{ -1+\mathcal{E}^2+(1+\mathcal{E}^2)\cosh R\Delta\varphi + \right.$$
$$+\mathcal{E}^2\cosh 2R\varphi_0 + \mathcal{E}^2\cosh 2Rt - 2\mathcal{E}\sinh R\Delta\varphi +$$
$$\left. +4\mathcal{E}\cosh Rt\cosh R\varphi_0(\sinh R\frac{\Delta\varphi}{2} - \mathcal{E}\cosh R\frac{\Delta\varphi}{2})\Big\}\right]\ .$$

Notations: $\Delta\varphi = \varphi_b - \varphi_a$, $\varphi_0 = \frac{1}{2}(\varphi_a + \varphi_b)$; $\quad \mathcal{E} = \cosh\frac{\pi R}{2}$

Evolution of entanglement entropy



Red curve is the function $\Delta S(t) = S_{\text{non-eq}}(t) - S_{\text{eq}}$, green dashed curve is the quadratic approximation, and blue dashed line is the linear asymptotic. Main feature - sharp transition to saturation.

Conclusions & open questions for bilocal quench-part

- Non-trivial non-equilibrium dynamics are shown by susbsystems which contain one of the excitations.
- Significant difference from the global quench: sharp transition
- Linear growth, loss of memory about the initial state and black hole interior are intimately connected. Linear growth of entanglement is a diagnostic of the information loss in the bulk
- Corrections to the holographic limit
- Generalization to the *n*-local case (collision of *n* particles in the bulk)

Sachdev-Ye-Kitaev (SYK) model

Hamiltonian Sachdev, Ye, '93; Kitaev, '15

$$H = (i)^{\frac{q}{2}} \sum_{i_1<i_2<\dots<i_{q-1}<i_q} J_{i_1\dots i_q} \chi_{i_1} \dots \chi_{i_q}$$

- χ_j Majorana fermions: $\{\chi_j, \chi_k\} = \delta_{jk}, \quad D = 1+0$
- $J_{i_1\dots i_q}$ independent Gaussian random variables, $i_1, \dots i_q = 1, 2, \dots, N$.

$$\overline{J_{i_1 i_2 \dots i_q}} = 0\,, \qquad \overline{J^2_{i_1 i_2 \dots i_q}} = \frac{(q-1)!J^2}{N^{q-1}}$$

- The free energy of the model with quenched disorder is given by

$$F = -\frac{1}{\beta}\overline{\log Z}, \quad Z = \mathrm{Tr}\, \mathrm{e}^{-\beta H}$$

- Dimensionless coupling: βJ
  SYK possess conformal symmetry in the strong coupling limit

Physics of SYK model

Outstanding feature of SYK models – maximal chaos in

- early time (Ehrenfest time)

Polchinski,Rosenhaus'16
Maldacena, Stanford '16
Gross,Rosenhaus '17
Gu, Qi, Stanford,'17
Bagrets '16

- late time (Heisenberg time)

Fu,Sachdev'16
García-García, Verbaarschot'16
Cotler et al '16
You et al '17, ...

Physics of SYK model (early time)

- There is an exponential grow in OTOC (out of time correlation)
- *SYK shows the maximal quantum chaos with the largest possible Lyapunov exponent*

$$\lambda_L = 2\pi/\beta$$

  (characterizes the quantum information scrambling.)
- It saturates the quantum chaos bound

  Maldacena, Shenker, Stanford, '15?
- Since quantum black holes are the fastest quantum information scramblers in Nature $\Rightarrow$
  SYK may be a boundary theory of some sort of bulk dilaton gravity theory,
  2d JT (Jackiw-Teitelboim) gravity .

Physics of SYK model (late time)

- In the late time SYK is described by RM (Random Matrix)
  Mainly numerical results

  Fu.1 at al, 17, You et al '17,
  Cotler et al,'16,
  Garcia-Garcia, Verbaarschot '16,'18,...
  Kanazawa et al 17,

- The RMT has also been employed to study the quantum chaotic behaviours of event horizon fluctuations of black holes

Disorder averaging and replica trick

The free energy of the model with quenched disorder is given by

$$F = -\frac{1}{\beta}\overline{\log Z}$$

Replica trick:

$$\overline{\ln Z} = \lim_{M\to 0} \frac{\ln \overline{Z^M}}{M}$$

χ_i^α for each replica $\alpha = 1, ..., M$, then analytic continuation to $M \to 0$.

Can we trust to replica trick?

Verbaarschot, Zirnbauer, Critique of the replica trick, J. Phys. A 18 (1985)1093

Integer M corresponds to annealed quantities

Example $|Z(\beta)|^2$ - spectral form factor $(\beta \in \mathbb{C})$ Cotler at al.'16

We study $\overline{Z^M}$ at arbitrary M.

Replica partition function

Integrate out the fermions — Kitaev; Maldacena, Stanford

$$\overline{Z(\beta)^M} = \int DGD\Sigma\, \mathrm{Pf}[\delta_{\alpha\beta}\partial_\tau - \hat{\Sigma}_{\alpha\beta}]^N \times$$

$$\exp\left[-\frac{N}{2}\int_0^\beta\int_0^\beta d\tau_1 d\tau_2 \left(\Sigma_{\alpha\beta}(\tau_1,\tau_2)G_{\alpha\beta}(\tau_1,\tau_2) - \frac{J^2}{q}G_{\alpha\beta}(\tau_1,\tau_2)^q\right)\right]$$

Saddle points equation

Saddle point equations (at large N):

$$\partial_\tau G_{\alpha\gamma}(\tau,\tau'') - \int_0^\beta d\tau' G_{\alpha\beta}(\tau,\tau')\Sigma_{\beta\gamma}(\tau',\tau'') = \delta_{\alpha\gamma}\delta(\tau-\tau'')$$

$$\Sigma_{\alpha\beta}(\tau,\tau') = J^2 G_{\alpha\beta}(\tau,\tau')^{q-1}$$

Diagonal ansatz: $G_{\alpha\beta}(\tau,\tau') = G(\tau,\tau')\delta_{\alpha\beta}$

Non-diagonal solutions: Conformal limit: $G_{\alpha\beta}(\tau,\tau') = G(\tau,\tau')P_{\alpha\beta}$

- **Strong coupling (IR, conformal) limit**: $\partial_\tau \to 0 \;\Leftrightarrow\; \beta J \gg 1$
- $P_{\alpha\beta}$ Parisi matrix

- Replica Non-diagonal Solutions (RNS)

I.A., Khramtsov, Tikhanovskaya, I.Volovich, 1811.0483

Separation of variables in the conformal limit

SYK for general q in strong coupling limit $\partial_\tau \to 0$.

- We separate the variables

$$\begin{aligned} G_{\alpha\beta}(\tau,\tau') &= g(\tau,\tau')P_{\alpha\beta} \quad (*) \\ \sum_\beta P^q_{\alpha\beta} &= 1 \quad \forall\alpha \quad (**) \end{aligned}$$

- $P_{\alpha\beta} = P_{\beta\alpha} \quad (***)$.

Solution:

$$g(\tau,\tau') = b\left(\frac{\pi}{\beta J}\right)^{2\Delta} \frac{\mathrm{sgn}(\tau-\tau')}{\left|\sin\frac{\pi}{\beta}(\tau-\tau')\right|^{2\Delta}}, \quad \Delta = 1/q$$

Equation for P:

$$\sum_\beta P_{\alpha\beta}P^{q-1}_{\beta\gamma} = 0, \quad \alpha \neq \gamma$$

Non-diagonal replica-symmetric

$q = 4$.

- Non-diagonal replica-symmetric ansatz

$$\begin{pmatrix} A_0 & A_1 & ...A_1 \\ A_1 & A_0 & ...A_1 \\ ... & ... & ... \\ A_1 & A_1 & A_0 \end{pmatrix}$$

symmetric replica nondiagonal solution

$$A_0 = 0, \quad A_1^2 + 1 - 2A_1^3 = 0$$

3 roots: $A_1^{(1)} = 1$ (does not admit the normalization),
$A_1^{(2,3)} = \frac{1}{4}\left(-1 \pm i\sqrt{7}\right)$

I.A., Khramtsov, Tikhanovskaya, I.Volovich, 1811.0483;
Wang, Bagrets, Chudnovskiy, Kamenev, 1812.02666

Free energy for $A_0 = 0, \mu = 0$

$$A_1^{(2,3)} = \frac{1}{4}\left(-1 \pm i\sqrt{7}\right)$$

$$\begin{aligned} f &\equiv \lim_{M\to 0}\frac{1}{M}\frac{1}{NJ}\Delta F = \frac{1}{NJ}(F_{RND} - F_{RD}) \\ &= \left(\frac{3}{4}\log|1 - A_1^4| - \log|1 - A_1^3| - \Re\frac{A_1^3}{1 - A_1^3}\right)\Big|_{A_1 = A_1^{(2,3)}} \\ &= -\frac{3}{8} + \frac{\log 2}{2} = -0.028 < 0 \end{aligned}$$

Nondiagonal saddles in the $q = 4$ model

We use the same replica-symmetric ansatz. The saddle point equations read

$$-i\omega G_0(\omega) - G_0(\omega)\Sigma_0(\omega) - (M-1)G_1(\omega)\Sigma_1(\omega) = 1\,;$$
$$-i\omega G_1(\omega) - G_1(\omega)\Sigma_0(\omega) - G_0(\omega)\Sigma_1(\omega) - (M-2)G_1(\omega)\Sigma_1(\omega) = 0\,;$$
$$\Sigma_{0,1}(\tau,\tau') = J^2 G_{0,1}(\tau,\tau')^{q-1}\,.$$

a system of two coupled integral equations.
We solve it numerically using the method of iterations at finite temperature, with $\beta = 2\pi$.
The initial condition is a $q = 2$ solution, and we increase q during iterations.

Nondiagonal saddles in the $q = 4$ model

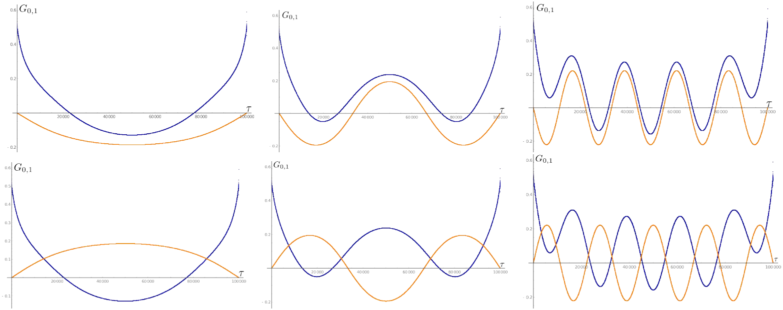


Figure: G_0 (blue curve) and G_1 (orange curve) as a function of Euclidean time on replica-nondiagonal solutions for $q = 4$. $\beta = 2\pi$, $M = 2$ and $J = 10$.

An infinite family of nondiagonal replica-symmetric solutions for $M > 0$
At $M = 0$, no solution found other than the standard diagonal saddle.

Conclusions for RND saddles

- RND saddles for $q = 2$(exact) and $q = 4$(numerically) at finite replica number are found.
- RND in strong coupling regime at $M \to 0$ are found.
  The regularized free energy of RND solutions is lower than that of the standard conformal solution.
- We have not found a plausible UV completion for the $M = 0$ solutions at strong coupling. However, rich structure and symmetry hints at possible holographic interpretation.

Random matrices etc.

- In the past years, there have been fascinating development in RMT, and its relationship with combinatorics, integrable systems etc. from math part),
  and condensed matter, low dimensional quantum gravity, string theories etc. from theor. - phys. part.

- Many of this developments can be described by using the topological recursion introduced by Eynard and Orantin, 07,....
  Loop eqs. Double scaling limit (Brézin-Kazakov, Douglas-Shenker, Gross-Migdal,1990)

JT gravity and matrix models

- It has been shown by Saad, Shenker and Stanford (SSS, '19) that the genus expansion of a certain matrix integral generates the partition functions of JT, Jackiw, 1984,Teitelboim,1983, quantum gravity on Riemann surfaces of arbitrary genus with an arbitrary fixed number of boundaries.

- It is shown that an important part of JT quantum gravity is reduced to computation of the Weil-Petersson volumes of the moduli space of hyperbolic Riemann surfaces with various genus and number of boundaries, for which Mirzakhani'13 established recursion relations.

JT gravity and matrix models

- The results of Saad, Shenker and Stanford provide a nonperturbative approach to JT quantum gravity on Riemann surfaces of various genus and perturbative description of boundaries.
- We use an extension of this result for nonperturbative studying of gas of baby universes in JT gravity.
- To investigate the boundaries nonperturbatively we explore the generating functional of boundaries in the matrix model and in JT gravity.

JT quantum gravity/matrix models duality. I

- The following remarkable relation between correlation functions in MM and JT gravity holds: Saad, Shenker and Stanford, '19

$$\mathcal{Z}_n^{matrix,d.s.}(\beta_1,...,\beta_n;\gamma) \simeq Z_n^{grav}(\beta_1,...,\beta_n;\gamma)$$

LHS:

- $\mathcal{Z}_n^{matrix,d.s.}(\beta_1,...,\beta_n;\gamma)$ is the double scaling (d.s.) limit of the correlation function in a matrix model with a spectral curve.

- The form of the curve is obtained by computing the JT path integral for the disc.

JT quantum gravity/matrix models duality. II

RHS

- $\mathcal{Z}_n^{grav}(\beta_1,...,\beta_n;\gamma) \simeq \sum_{g=0}^{\infty} \gamma^{-\chi} \mathcal{Z}_{g,n}^{grav}(\beta_1,...,\beta_n)$

  χ is the Euler characteristic $\chi = 2 - 2g - n$, $\gamma = \mathrm{e}^{-S_0}$

- $Z_{g,n}^{grav}(\beta_1,...,\beta_n,;\gamma)$ is the JT gravity path integral for Riemann surface $M_{g,n}$ of genus g with n boundaries with lengths $\beta_1,...,\beta_n$

$$Z_{g,n}^{grav}(\beta_1,...,\beta_n) = e^{-S_0\chi} \int \frac{\mathcal{D}g_{\mu\nu}\mathcal{D}\phi}{\mathrm{Vol(diff)}} \, e^{-I_{JT}[g_{\mu\nu},\phi]}$$

- I_{JT} is the JT action.

$$I_{JT} = -\left[\frac{1}{2}\int_M \sqrt{g}\phi(R+2) + \int_{\partial M} \sqrt{h}\phi(K-1)\right]$$

$g_{\mu\nu}$ a metric on a two dimensional manifold M, ϕ a dilaton

JT quantum gravity/matrix models duality. III

LHS

We consider ensemble of $N \times N$ Hermitian matrices with potential $V(M)$.

$$Z_n^{matrix}(\beta_1, ..., \beta_n) = \frac{1}{Z_0} \int Z(\beta_1)...Z(\beta_n) \exp(-N \mathrm{Tr} V(M)) dM$$

$$Z(\beta) = \mathrm{Tr} e^{-\beta M} = \sum_{i=1}^{N} \exp(-\beta \lambda_i), \quad \beta > 0.$$

where λ_i are eigenvalues of the matrix M.

Generating functional

I.A., I.Volovich, 1905.08207

- Generating functional for the gravitational correlation functions $Z_n^{grav}(\beta_1,...,\beta_n;\gamma)$

$$\mathcal{Z}^{grav}(J;\gamma)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\infty}d\beta_1...\int_0^{\infty}d\beta_n Z_n^{grav}(\beta_1,...,\beta_n;\gamma)J(\beta_1)...J(\beta_n)$$

where $J(\beta)$ is a source function.

- An appropriate generating functional in matrix theory

$$\begin{aligned}\mathfrak{Z}^{matrix}(J) &= <\mathrm{e}^{N\int Z(\beta)J(\beta)d\beta}>=<\mathrm{e}^{N\int \mathrm{Tr}\,\mathrm{e}^{-\beta M}J(\beta)d\beta}> \\ &= \sum_{n=0}^{\infty}\frac{N^n}{n!}\int d\beta_1...\int d\beta_n \mathcal{Z}_n^{matrix}(\beta_1,...\beta_n)J(\beta_1)...J(\beta_n)\end{aligned}$$

Generating functional in matrix models. Details

$$\mathfrak{Z}^{matrix}(J) = \int \exp\{N\sum_{j=1}^{N}\tilde{J}(\lambda_j)\}\, d\mu_N(\lambda_1, ...\lambda_N)$$

where

$$\begin{aligned} d\mu_N(\lambda_1, ...\lambda_N) &= \frac{1}{Z_N}\prod_{j>k}(\lambda_j - \lambda_k)^2 \prod_{j=1}^{N} e^{-NV(\lambda_j)}, \\ \tilde{J}(\lambda) &= \int d\beta J(\beta)\mathrm{e}^{-\beta\lambda} \end{aligned}$$

This amounts to shift the potential $V(x) \to V(x) - \tilde{J}(x)$.

One expands $\mathfrak{G}^{matrix}(J) = \log \mathfrak{Z}^{matrix}(J)$ to get the connected correlation functions

$$\mathfrak{G}^{matrix}(J) = \sum_{n=0}^{\infty}\frac{1}{n!}\int Z^{matrix}_{n,conn}(\beta_1, ...\beta_n)\prod_i J(\beta_1)d\beta_i$$

Generating functional in JT gravity. Details

$$\mathcal{Z}_n^{grav}(\beta_1, ..., \beta_n) \simeq \sum_{g=0}^{\infty} (e^{-S_0})^{2g+n-2} \mathcal{Z}_{g,n}^{grav}(\beta_1, ..., \beta_n)$$

According to SSS

$$Z_{g,n}^{grav}(\beta_1, ..., \beta_n) = \int_{b_i > 0} V_{g,n}(b_1, ..., b_n) \prod_i Z_{\text{Sch}}^{\text{trumpet}}(\beta_i, b_i) b_i db_i$$

where $(g \geq 2)$, $V_{g,n}(b_1, ..., b_n)$ is the Weil-Petersson (WP) volume of the moduli space of a genus g Riemann surface with n geodesic boundaries of lengths $b_1, ..., b_n$ and

$$Z_{\text{Sch}}^{\text{trumpet}}(\beta, b) = \frac{1}{2\pi^{1/2}\beta^{1/2}} \mathrm{e}^{-\frac{b^2}{4\beta}}.$$

The IR dynamics of SYK model is described by the Schwarzian theory: Maldacena,Stanford '16; Bagrets,Atland,Kamenev'16; Belokurov,Shavgulidze, '17, '18
Schwarzian describes the boundary behaviour of 2D JT gravity Jensen'16; Maldacena, Stanford, Yang 16; Engelsoy, Mertens, H.Verlinde '16

Partition functions in JT and the WP volume of the moduli space

$$\mathcal{Z}_g^{grav}(J) = \sum_{n=0}^{\infty} \frac{1}{n!}(e^{-S_0})^{2g+n-2} \int_{\beta_i>0} Z_{g,n}^{grav}(\beta_1, ..., \beta_n) \prod J(\beta_i) d\beta_i$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!}(e^{-S_0})^{2g+n-2} \int V_{g,n}(b_1, ..., b_n) \prod_i \hat{J}(b_i) b_i db_i,$$

where

$$\hat{J}(b) = \int_0^{\infty} d\beta \frac{1}{2\pi^{1/2}\beta^{1/2}} e^{-\frac{b^2}{4\beta}} J(\beta)$$

and the generating functional

$$\mathfrak{Z}^{grav}(J) \simeq \sum_{g=0}^{\infty} \mathcal{Z}_g^{grav}(J)$$

Finally,

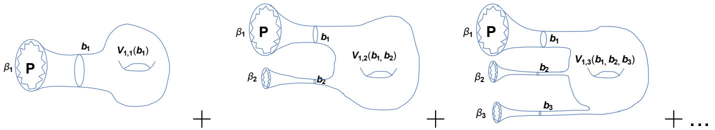
$$d.s. \lim \log \mathfrak{Z}^{matrix}(J) \simeq \mathfrak{G}^{grav}(J)$$

Baby universes and wormholes

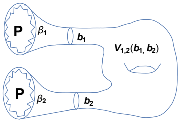
- In cosmology, one usually deals with baby universes that branch off from, or join onto, the parent(s) Universe(s).
- In matrix theories one parent is a connected Riemann surface with arbitrary number of handles and at least one boundary.
- Hawking, Rubakov-Lavrelashvili-Tenyakov, Giddings-Strominger, Coleman,...
- We take $J(\beta) = -J\delta(\beta + \omega)$

Baby universes.

In cosmology, one usually deals with baby universes that branch off from, or join onto, the parent(s) Universe(s). In matrix theories one parent is a connected Riemann surface with arbitrary number of handles and at least one boundary. Hawking, 1987



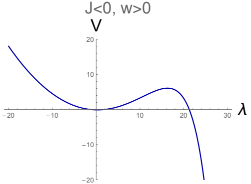
Gas of baby universes. $|\beta_1| >> |\beta_i|$ for $i \geq 2$ and $b_i \leq b_c$ for $i \geq 2$



Two parents connected by the **wormhole**

Deformation of MM by an exponential potential

$$U(x) = V_0(x) + V_1(x) = \frac{m^2 x^2}{2} + Je^{\omega x}$$

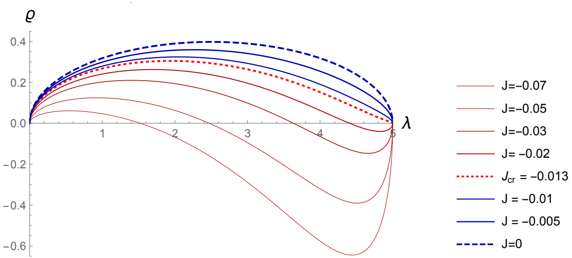


Perturbations of the gaussian ensemble by the exp. potential $V_1 = Je^{\omega x}$

To shift $\lambda \to \lambda + \delta\lambda$ add linear term $V(x) = m^2\left(\frac{1}{2}x^2 + Cx + J\,\mathrm{e}^{\omega x}\right)$

Deformation of Wigner density by an exponential potential

$$\rho_{nn}(\lambda) = \rho_W(\lambda, \Lambda) + J\rho_{\exp}(\lambda, \Lambda, \omega)$$



The plot of non–normalized density for the quadratic potential deformed by the exp. potential for different values of the regularization parameter J, $\Lambda = 5$, $\omega = 1$

Double scaling limit. Universality (Backup)

The correlation functions in RMT can be written as a determinant

$$\varrho_{kN}(x_1, ..., x_k) = \det(K_N(x_i, x_j))_{i,j=1}^{k} \quad \text{where} \quad K_N(x,y) = \sum_{n=0}^{N-1} \psi_n(x)\psi_n(y),$$

$$\psi_n(x) = \frac{1}{\sqrt{h_n}} P_n(x) e^{-NV(x)/2}, \quad P_n(x) = x^n + ... - \text{orthog.polyn.}$$

$\int_{\mathbb{R}} P_n(x)P_m(x)e^{-NV(x)}dx = h_n\delta_{nm},\ n, m = 0, 1, ...$
Suppose the parameter $g \to g_c$. How to scale the eigenvalues and $g - g_c$ to get nontrivial limit of corr.functions. Brezin, Hikami; Bleher, Its,...

$$\lim_{N\to\infty} \frac{1}{N^{(k-1)\eta}} \rho_{kN}\left(\frac{x_1}{N^\eta}, \dots, \frac{x_k}{N^\eta}; g_c + \frac{y}{N^\xi}\right) = \rho_k(x_1, \dots, x_k; y),$$

or

$$\lim_{n,N\to\infty:\ (n/N)=f(g_c)+y/N^\xi} \frac{1}{C_n} \psi_n\left(\frac{x}{N^\eta}\right) = \psi_\infty(x; y),$$

with some critical exponents ξ and η,

Matrix model for JT gravity

- Potentials for non-normalized density distribution

For fixed Λ we consider the eigenvalue distribution normalized to 1

$$\rho_{norm,1}(E) = n(\Lambda)\rho_{nn}(E), \quad \rho_{nn}(E) = \frac{1}{(2\pi)^2}\sinh(2\pi\sqrt{E}), \qquad E > 0.$$

This form of the eigenvalue distribution in the SYK model has been obtained in Bagrets et all, Stanford, Witten and is nothing but the Bethe formula for the nuclear level density Bethe.

For large Λ one has $n(\Lambda) \approx 8\pi^3 e^{-2\pi\sqrt{\Lambda}}/\sqrt{\Lambda}$.

The distribution normalized to N has the form

$$\rho^{\text{nnorm}}_{d.s.}(E) \equiv \rho_{norm,N}(E) = \frac{\mathrm{e}^{S_0}}{(2\pi)^2}\sinh(2\pi\sqrt{E}), \qquad E > 0.$$

It is evident that $\mathrm{e}^{S_0} = Nn(\Lambda)$

Matrix model for JT gravity

- To recover the potential that supports the distribution $\rho_{norm,1}(E)$, we write

$$V(\mu) = n(\Lambda) V_{nn}(\mu), \tag{1}$$

where $V'_{nn}(\mu)$ is defined by

$$V'_{nn}(\mu) = \frac{1}{(2\pi)^2} \mathcal{P}\int_0^\Lambda \frac{\sinh 2\pi\sqrt{\lambda}}{\mu-\lambda} d\lambda. \tag{2}$$

This gives

$$\begin{aligned} V'_{nn}(\mu) &= \frac{1}{(2\pi)^2}\Big[e^{-2\pi\sqrt{\mu}}\left(e^{4\pi\sqrt{\mu}}\text{Ei}\left(2\pi\left(\sqrt{\Lambda}-\sqrt{\mu}\right)\right)+\text{Ei}\left(2\pi\left(\sqrt{\Lambda}+\sqrt{\mu}\right)\right)\right) \\ &- e^{-2\pi\sqrt{\mu}}\left(\text{Ei}\left(2\pi\left(\sqrt{\mu}-\sqrt{\Lambda}\right)\right)+e^{4\pi\sqrt{\mu}}\text{Ei}\left(-2\pi\left(\sqrt{\Lambda}+\sqrt{\mu}\right)\right)\right)\Big] \end{aligned} \tag{3}$$

Here Ei is the exponential integral that for real non zero values of x is defined as

$$\text{Ei}(x) = -\mathcal{P}\int_{-x}^{\infty} \frac{e^{-t}}{t} dt. \tag{4}$$

Matrix model for JT gravity (Backup)

Ei has an expansion

$$\mathrm{Ei}(x) = \gamma + \ln x + \sum_{k=1}^{\infty} \frac{x^k}{k\ k!},$$

and faster converging Ramanujan's series has the form

$$\mathrm{Ei}(x) = \gamma + \ln x + \exp(x/2) \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n!\, 2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1} \qquad (5)$$

The potential up to a constant is

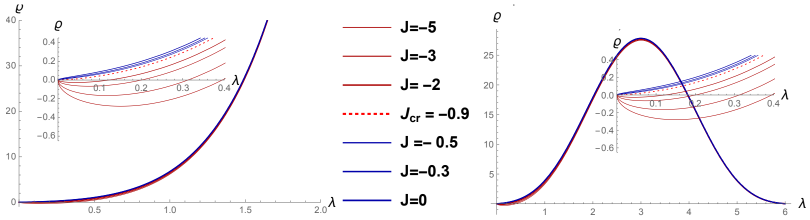
$$V(\mu) = \int^{\mu} V'(\lambda) d\lambda$$

Matrix model for JT gravity. Critical points

Deformation by ρ_{exp}

$$\rho_{nn}(\lambda) = \frac{n(\Lambda)}{(2\pi)^2} \sinh 2\pi\sqrt{\lambda} + J\rho_{\text{exp}}(\lambda, \Lambda, \omega)$$

$$\rho_{nn,s}(\lambda) = \frac{n_s(\Lambda)}{(2\pi)^2} \sinh 2\pi\sqrt{\lambda(1-\lambda/\Lambda)} + J\rho_{\text{exp}}(\lambda, \Lambda, \omega).$$



$$\Lambda = 6, \omega = -1$$

SYK and JT gravity

- We have found the critical point in the MM related with JT gravity.
- It is interesting to know what this critical point means for SYK?

Conclusion (for RM/JT part)

- Generating functionals for MM and the partition functions in JT quantum gravity are studied.
- It is shown numerically that there is the critical point $J_c < 0$ for the generating functional.
- One expects that RM/JT duality will help to construct a dual gravity for the SYK model.

Conclusion

- **The problems are still open**

- One can hope that holography and the RM/JT duality put a new light to the fundamental problem of thermalization and information loss in black holes.

Thank you for attention!

Join to solve the open problems!