Magnificent Four with Colors

NIKITA NEKRASOV

Moscow, June 10, 2019

♢♢♢

Magnificent Four with Colors, and Beyond Eleven Dimensions

Nikita Nekrasov

Simons Center for Geometry and Physics

ITMPh Moscow June 10, 2019

♢♢♢

♢♢♢♢

A popular approach to quantum gravity

is to approximate the space-time geometry by some discrete structure

♢♢♢♢

♢♢♢♢

A popular approach to quantum gravity

is to approximate the space-time geometry by some discrete structure

Then develop tools for summing over these discrete structures

♢♢♢♢

♢♢♢♢

A popular approach to quantum gravity

is to approximate the space-time geometry by some discrete structure

Then develop tools for summing over these discrete structures

Tuning the parameters so as to get, in some limit

Smooth geometries

♢♢♢♢

◇◇◇◇

To some extent

two dimensional quantum gravity

is successfully solved in this fashion

using matrix models

$$\log \int_{N\times N} dM\, e^{-N\mathrm{tr}\, V(M)} \sim \sum_{\text{fat graphs}\leftrightarrow\text{triangulated Riemann surfaces}}$$

◇◇◇◇

♢♢♢♢

To some extent

two dimensional quantum gravity

is successfully solved in this fashion

using matrix models

$$\log \int_{N\times N} dM\, e^{-N \mathrm{tr} V(M)} \sim \sum_{g=0}^{\infty} N^{2-2g} \sum_{\text{genus } g \ \mathit{Riemann\,surfaces}}$$

♢♢♢♢

♢♢♢♢

Going up in dimension

proves difficult

♢♢♢♢

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

Going up in dimension

proves difficult – the obvious generalization of a matrix model

is the so-called tensor theory

$$M_{ij} \longrightarrow \Phi_{ijk}$$

There is no analogue of genus expansion for general three-manifolds

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

◊◊◊◊

Going up in dimension

proves difficult – the obvious generalization of a matrix model

is the so-called tensor theory

$$M_{ij} \longrightarrow \Phi_{ijk}$$

There is no analogue of genus expansion for general three-manifolds

However an interesting large N scaling has been recently found

In the context of the SYK model, $g \mapsto$ Gurau index

◊◊◊◊

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

In this talk

I will discuss two models of random three dimensional geometries

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

♢♢♢♢

In this talk

I will discuss two models of random three dimensional geometries

I do not claim they quantize three dimensional Einstein gravity

♢♢♢♢

♢♢♢♢

In this talk

I will discuss two models of random three dimensional geometries

They may teach us about eleven dimensional super-gravity

♢♢♢♢

◊◊◊◊

In this talk

I will discuss two models of random three dimensional geometries

They may teach us about eleven dimensional super-gravity

M-theory, and beyond

◊◊◊◊

♢♢♢♢

Geometries from partitions

One way to generate a d-dimensional random geometry

Is from some local growth model in $d+1$-dimensions

♢♢♢♢

♢♢♢♢

Geometries from partitions

For example, start with the simplest "mathematical" problem

♢♢♢♢

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

Geometries from partitions

For example, start with the simplest problem: counting natural numbers

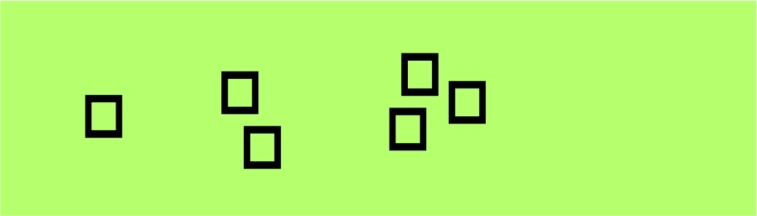
$$1,\ 2,\ 3, \ldots$$

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

Geometries from partitions

For example, start with the simplest problem: accounting

$$1, \ 2, \ 3, \ldots$$

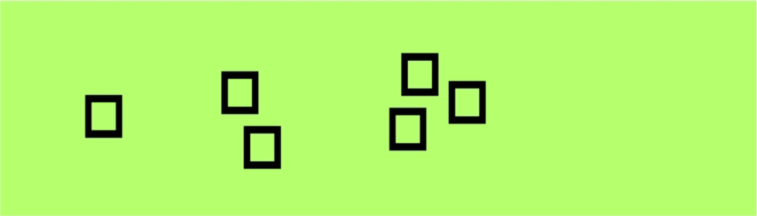


◊◊◊◊

Geometries from partitions

Accounting for objects without structure

$$1, \; 2, \; 3, \ldots$$



◊◊◊◊

♢♢♢♢

Geometries from partitions

Now add the simplest structure: partitions of integers

$$(1)\,;\ (2)\,,\ (1,1)\,;\ (3)\,,\ (2,1)\,,\ (1,1,1)\,;\ \ldots$$

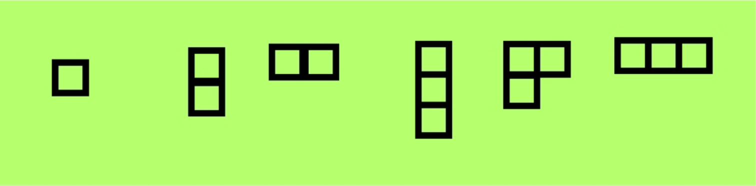
♢♢♢♢

♢♢♢♢

Geometries from partitions

Now add the simplest structure: partitions of integers

$$(1)\,; \qquad (2)\,,\ (1,1)\,; \qquad (3)\,,\ (2,1)\,,\ (1,1,1)\,;\ \ldots$$

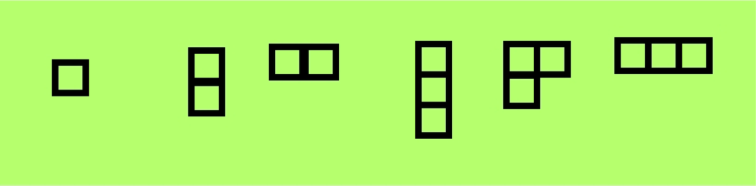


♢♢♢♢

Geometries from partitions

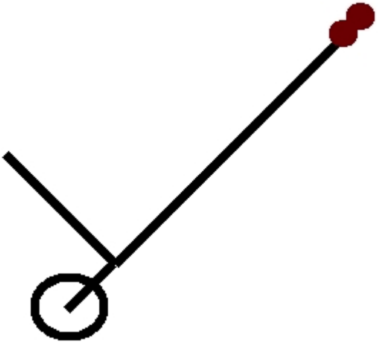
The structure: partitions of integers as bound states

$$(1)\,; \qquad (2)\,,\ (1,1)\,; \qquad (3)\,,\ (2,1)\,,\ (1,1,1)\,;\ \ldots$$



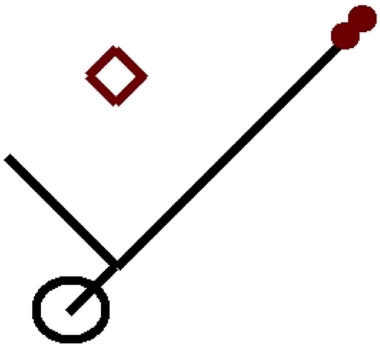
Partitions as growth model

Gardening and bricks



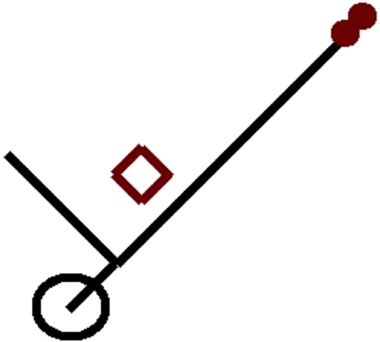
Partitions as growth model

Gardening and bricks



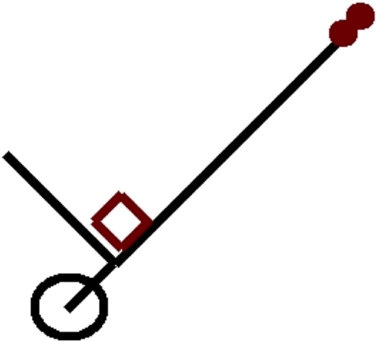
Partitions as growth model

Gardening and bricks



Partitions as growth model

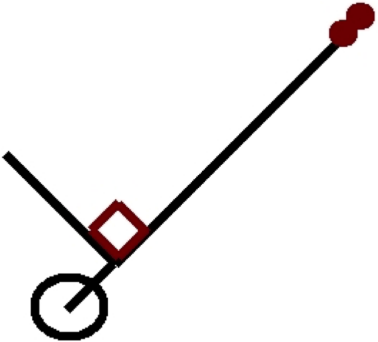
Gardening and bricks



♢♢♢♢

Partitions as growth model

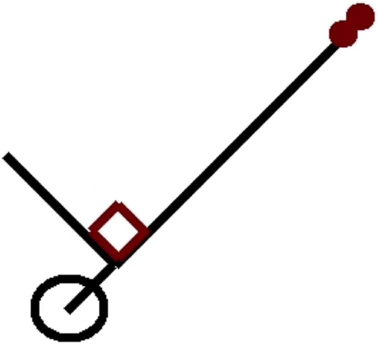
Gardening and bricks



♢♢♢♢

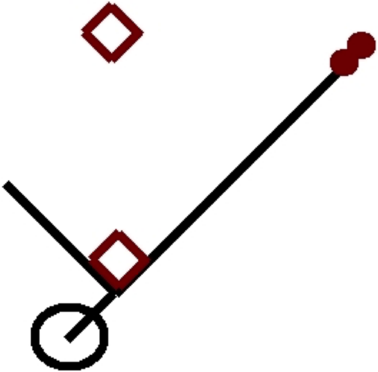
Partitions as growth model

Partition (1) made of brick



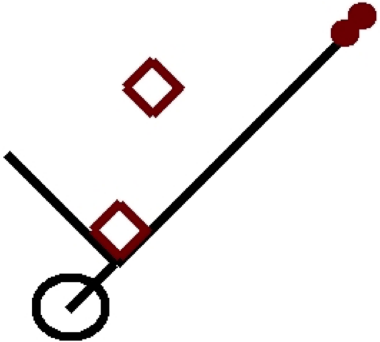
Partitions as growth model

Gardening and bricks



Partitions as growth model

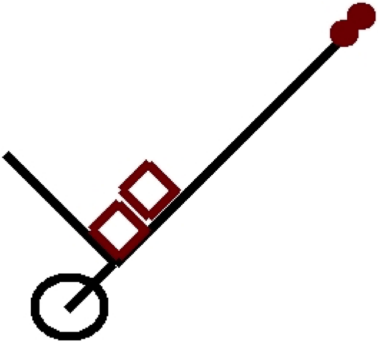
Gardening and bricks



◊◊◊◊

Partitions as growth model

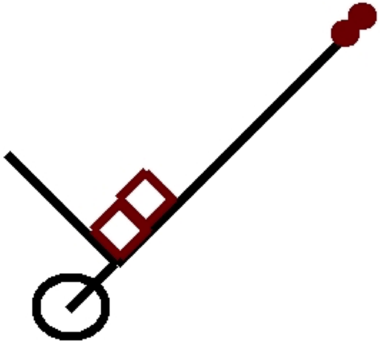
Gardening and bricks



◊◊◊◊

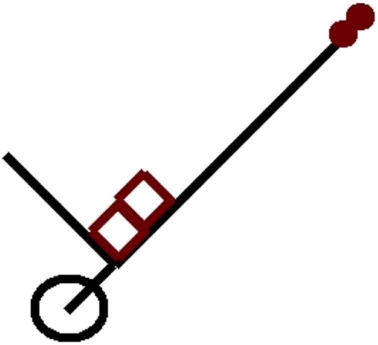
Partitions as a growth model

Gardening and bricks



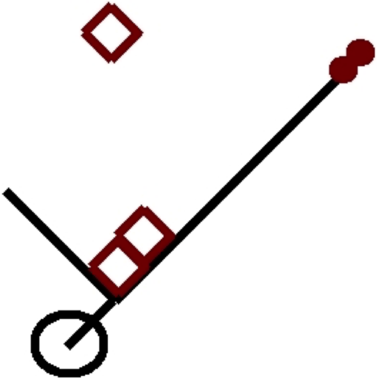
Partitions as a growth model

Partition (2) made of bricks



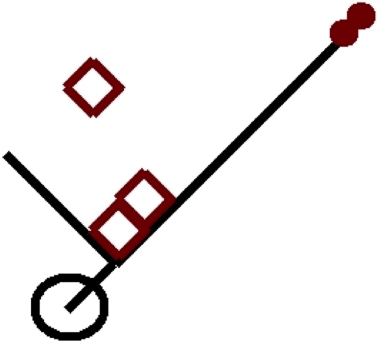
Partitions as a growth model

Gardening and bricks



Partitions as a growth model

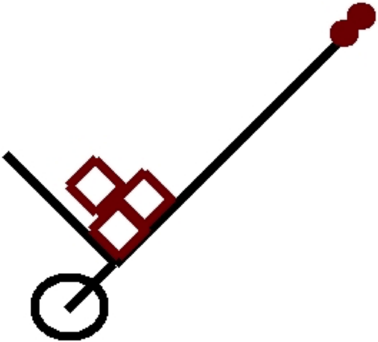
Gardening and bricks



◊◊◊◊

Partitions as a growth model

Gardening and bricks

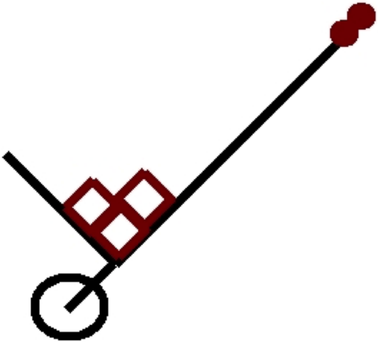


◊◊◊◊

◊◊◊◊

Partitions as a growth model

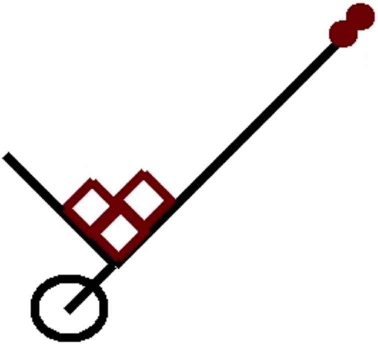
Gardening and bricks



◊◊◊◊

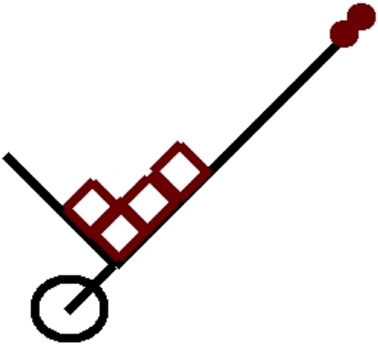
Partitions as a growth model

Partition $(2, 1)$ made of bricks



Partitions as a growth model

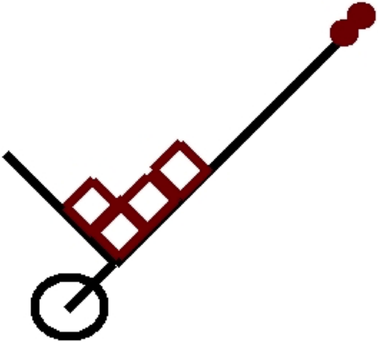
Partition $(3, 1)$ made of bricks



◊◊◊◊

Partitions as a growth model

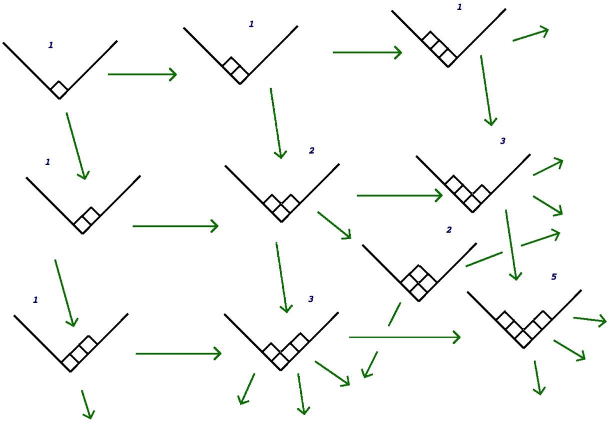
The probability of a given partition, e.g. $(3, 1)$, is determined by the equality



of the chances of jumps from one partition
e.g. from $(3, 1)$ to another, e.g. $(3, 2)$ or $(4, 1)$, or $(3, 1, 1)$

◊◊◊◊

Possibilities of growth: Young graph



♢♢♢♢

Partitions as a growth model

Thus the probability p_λ of a given partition λ

is proportional to the $\#$ of ways it can be built out of the nothing

times the $\#$ of ways it can be reduced to nothing

♢♢♢♢

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

Partitions as a growth model

Thus the probability p_λ of a given partition λ

is proportional to the # of ways it can be built out of the nothing

times the # of ways it can be reduced to nothing: quantum bricks

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

Plancherel measure: symmetry factors

One can calculate this to be equal to

$$p_\lambda = \left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2 \Lambda^{2|\lambda|} e^{-\Lambda^2}$$

$$= \ e^{-\Lambda^2} \ \left(\prod_{\square\in\lambda} \frac{\Lambda}{\mathrm{hook-length\ of\ }\square}\right)^2$$

For example, $p_{3,1} = \frac{1}{1^2 2^2 4^2 1^2} = \frac{1}{64}$

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

Supersymmetric gauge theory

Remarkably, p_λ is the simplest example

of an instanton measure

$$p_\lambda = (\mathrm{sDet}\Delta_{A_\lambda})^{-\frac{1}{2}}$$

i.e. the one-loop (exact) contribution of an instanton $A = A_\lambda$

in $\mathcal{N} = 2$ supersymmetric gauge theory

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

◊◊◊◊

Supersymmetric gauge theory and random partitions

Consider $\mathcal{N} = 2$ supersymmetric gauge theory in four dimensions

The fields of a vector multiplet are
$A_m,\ m = 1, 2, 3, 4;\ \lambda_{\alpha i},\ \alpha = 1, 2$ and $i = 1, 2;\ \phi, \bar{\phi}$

with the supersymmetry transformations, schematically

$$\delta A \sim \lambda + \bar{\lambda}\,, \qquad \delta\phi \sim \lambda\,, \qquad \delta\bar{\phi} \sim \bar{\lambda}$$

$$\delta(\lambda, \bar{\lambda}) \sim (F^{+} + D_A\phi, F^{-} + D_A\bar{\phi}) + [\phi, \bar{\phi}]$$

◊◊◊◊

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

Supersymmetric gauge theory and random partitions

Supersymmetric partition function of the theory can be computed exactly

by localizing on the δ-invariant field configurations, i.e. $F_A^+ = 0$

$$Z = \sum_k \Lambda^{2Nk} \int_{\mathcal{M}_k^+} \text{instanton measure}$$

of some effective measure, including the regularization factors

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

♢♢♢♢

Supersymmetric gauge theory and random partitions

The integral over the moduli space can be further simplified by

by deforming the supersymmetry using the rotational symmetry of $\mathbb{R}^4$

$$Z = \sum_k \Lambda^{2Nk} \sum_{\lambda, |\lambda|=k} p_\lambda$$

The deformed path integral is computed by exact saddle point analysis

with λ enumerating the saddle points

♢♢♢♢

♢♢♢♢

Supersymmetric gauge theory and random partitions

Generic rotation of $\mathbb{R}^4$: $g_{\mathrm{rot}} = \exp \begin{pmatrix} 0 & \varepsilon_1 & 0 & 0 \\ -\varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \\ 0 & 0 & -\varepsilon_2 & 0 \end{pmatrix}$

$$Z = \sum_k \Lambda^{2Nk} \sum_{\lambda,\, |\lambda|=k} p_\lambda(\varepsilon_1, \varepsilon_2)$$

The deformed path integral is computed by exact saddle point

Exact saddle point approximation

for $U(N)$ gauge theory: $\lambda =$ an N-tuple of partitions $\lambda^{(1)}, \ldots, \lambda^{(N)}$

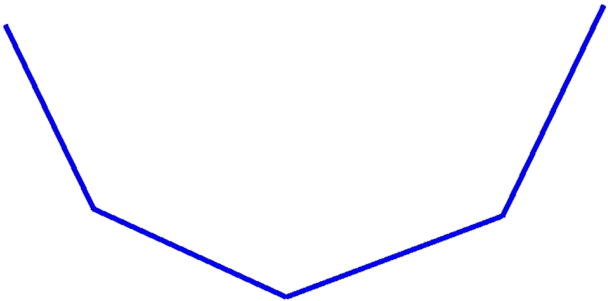
♢♢♢♢

♢♢♢♢

Supersymmetric gauge theory and random partitions

In this way supersymmetric gauge theory becomes a model of

random partitions = random piecewise linear geometries



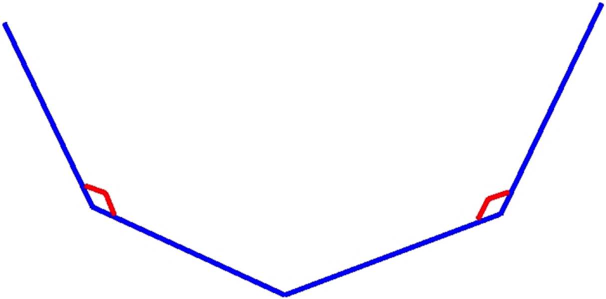
♢♢♢♢

◊◊◊◊

Supersymmetric gauge theory and random partitions

In this way supersymmetric gauge theory becomes a model of

random partitions = random piecewise linear geometries



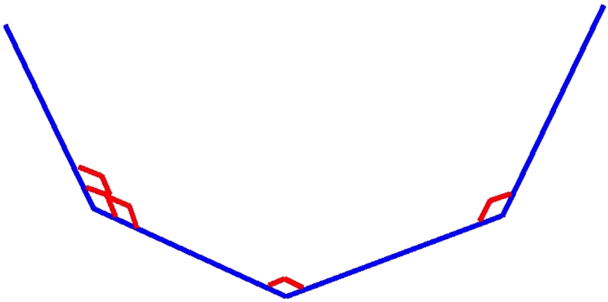
◊◊◊◊

◊◊◊◊

Supersymmetric gauge theory and random partitions

In this way supersymmetric gauge theory becomes a model of

random partitions = random piecewise linear geometries

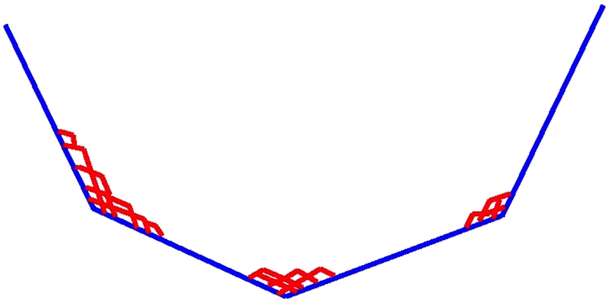


◊◊◊◊

Supersymmetric gauge theory and random partitions

In this way supersymmetric gauge theory becomes a model of

random partitions = random piecewise linear geometries

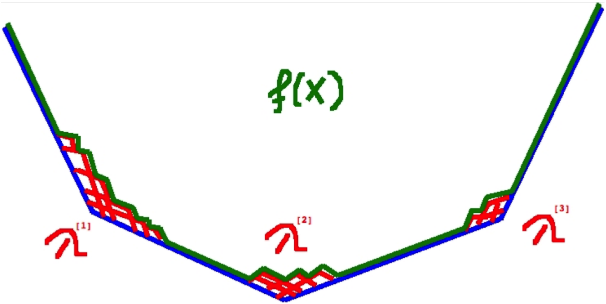


◊◊◊◊

Supersymmetric gauge theory and random partitions

In this way supersymmetric gauge theory becomes a model of

random partitions = random piecewise linear geometries

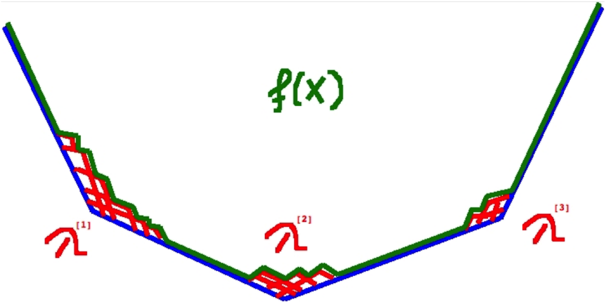


◊◊◊◊

◊◊◊◊

Supersymmetric gauge theory and random partitions

In this way supersymmetric gauge theory becomes a model of random partitions = random piecewise linear geometries



$$p_\lambda(\varepsilon_1, \varepsilon_2) = \exp \int\int dx_1 dx_2 \; f''(x_1) f''(x_2) K(x_1 - x_2; \varepsilon_1, \varepsilon_2)$$

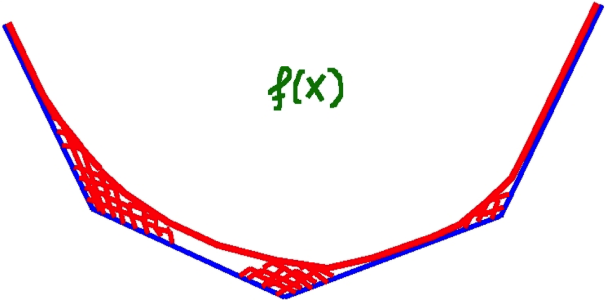
◊◊◊◊

◊◊◊◊

Emergent spacetime geometry

In the limit $\varepsilon_1, \varepsilon_2 \to 0$ (back to flat space supersymmetry)

The sum over random partitions is dominated by the so-called limit shape



$$p_\lambda(\varepsilon_1, \varepsilon_2) \sim \exp \frac{1}{\varepsilon_1 \varepsilon_2} F_\lambda$$

◊◊◊◊

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

Higher dimensional gauge theories

The analogous supersymmetric partition functions

can be defined for $d = 4, 5, 6, 7, 8, 9$ dimensional gauge theories

using embedding in string theory for $d > 4$

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

Extra dimension

These computations can be used to test some

of the most outstanding predictions of mid-90s, e.g. that

sum over the $D0$-branes = lift to one higher dimension

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

◊◊◊◊

Extra dimension

E.g. the max susy gauge theory in $4+1$ dim's

$$Z_{4+1}^{N=1} = \mathrm{Tr}_{\mathcal{H}_{\mathbb{R}^4}}\, g_{\mathrm{rot}}\ g_{\mathrm{R-sym}}\ g_{\mathrm{flavor}}\, (-1)^F$$

$$= \exp \sum_{k=1}^{\infty} \frac{1}{k} F_5(q_1^k, q_2^k, \mu^k, p^k)$$

Free energy $F_5(q_1, q_2, \mu, p) = \frac{p}{1-p}\frac{(1-\mu q_1)(1-\mu q_2)}{\mu(1-q_1)(1-q_2)}$

$q_1 = e^{\mathrm{i}\beta\varepsilon_1}$, $q_2 = e^{\mathrm{i}\beta\varepsilon_2}$, $\beta\varepsilon_1$, $\beta\varepsilon_2$ are the angles of the spatial $\mathbb{R}^4$ rotation

$\mu = e^{\mathrm{i}\beta m}$, m is the mass of the adjoint hypermultiplet,

β is the circumference of the temporal circle

p is the fugacity for the # of instantons =

of $D0$ branes bound to a $D4$ brane in the IIA string picture

◊◊◊◊

♢♢♢♢

Extra dimension

Remarkably,

$$Z_{4+1}^{N=1}(q_1, q_2, \mu, p) \; = \; \exp \sum_{k=1}^{\infty} \frac{1}{k} F_5 \left((\cdot)^k \right) =$$

$= $ Partition function of a minimal $d = 6$, $\mathcal{N} = (0, 2)$ multiplet

On space-time $\mathbb{R}^4 \tilde{\times} \mathbb{T}^2$

$$p = e^{2\pi \mathrm{i} \tau} \, , \qquad \tau = \text{complex modulus of the} \quad \mathbb{T}^2$$

♢♢♢♢

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

Extra dimension

Remarkably,

$$Z^{N=1}_{4+1}(q_1, q_2, \mu, p) = \exp \sum_{k=1}^{\infty} \frac{1}{k} F_5 \left((\cdot)^k \right) =$$

$=$ Partition function of a minimal $d = 6$, $\mathcal{N} = (0, 2)$ multiplet

On space-time $\mathbb{R}^4 \widetilde{\times} \mathbb{T}^2$

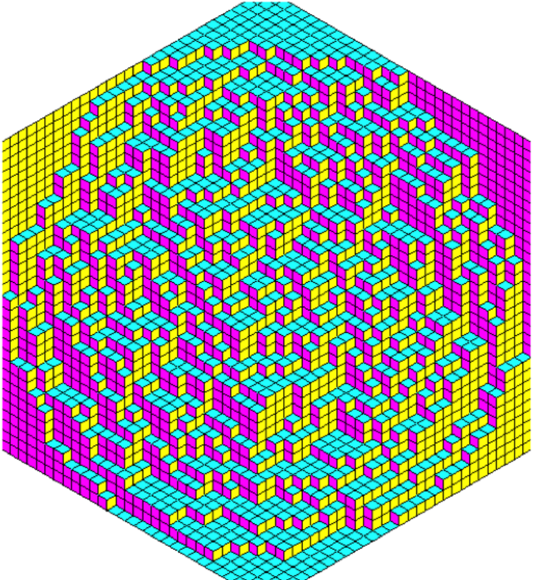
$$p = e^{2\pi \mathrm{i} \tau}\,, \qquad \tau = \text{complex modulus of the} \quad \mathbb{T}^2$$

In agreement with $D4$ brane $=$ $M5$ brane on S^1

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

Even higher dimensions: $6+1$

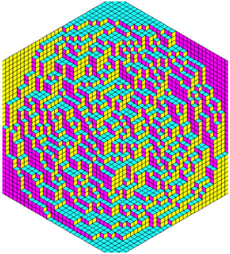
SYM in $6+1$ dim's – $\mathrm{Tr}(-1)^F g$ is expressed as a sum over plane partitions



Even higher dimensions: $6+1$

SYM in $6+1$ dim's – $\mathrm{Tr}(-1)^F g$ is expressed as a sum over plane partitions

$$g_{\mathrm{rot}} = \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{pmatrix}, \quad R_i = \exp \mathrm{i}\beta \begin{pmatrix} 0 & \varepsilon_i \\ -\varepsilon_i & 0 \end{pmatrix}$$



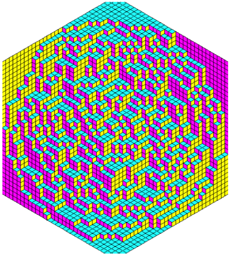
◊◊◊◊

Even higher dimensions: $6+1$

SYM in $6+1$ dim's – $\mathrm{Tr}(-1)^F g$ is expressed as a sum over plane partitions

$$Z_{6+1}^{N=1} = \exp \sum_{k=1}^{\infty} \frac{1}{k} F_7(q_1^k, q_2^k, q_3^k, p^k)$$

Again, p counts instantons $=$ $D0$ branes bound to a $D6$



◊◊◊◊

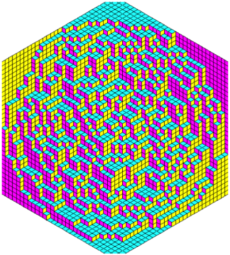
♢♢♢♢

Even more higher dimensions: $6+1 \to 10+1$

It turns out, that the supersymmetric free energy of plane partitions

$$F_7(q_1,q_2,q_3,p) = \frac{\sum_{a=1}^{5}\left(Q_a - Q_a^{-1}\right)}{\prod_{a=1}^{5}\left(Q_a^{\frac{1}{2}} - Q_a^{-\frac{1}{2}}\right)}$$

$$Q_1 = q_1,\, Q_2 = q_2,\, Q_3 = q_3,\, Q_4 = p(q_1q_2q_3)^{-\frac{1}{2}},\, Q_5 = p^{-1}(q_1q_2q_3)^{-\frac{1}{2}}$$



$S(3)$-symmetry enhanced to $S(5)$ symmetry
Twisted Witten index of 11d supergravity!
Plane partitions = 3d Young diagrams
know about (super)gravity in $10+1$ dimensions!
In agreement with: $D6 \to Taub - Nut \approx \mathbb{R}^4$,
$IIA \to$ M-theory

♢♢♢♢

◊◊◊◊

From 2d and 3d to 4d Young diagrams

It turns out, one can set up a count of solid partitions

$=$ 4d Young diagrams

How to visualize them?

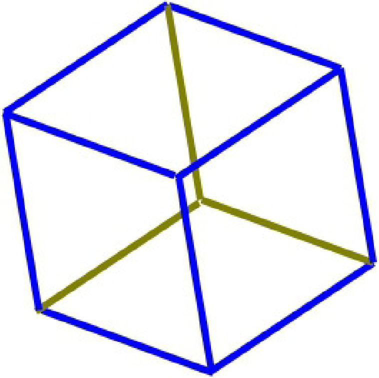
◊◊◊◊

◇◇◇◇

From 2d and 3d to 4d Young diagrams

How to visualize 4d Young diagrams?

Use the projection from $\mathbb{R}^4 \to \mathbb{R}^3$ along the $(1,1,1,1)$ axis

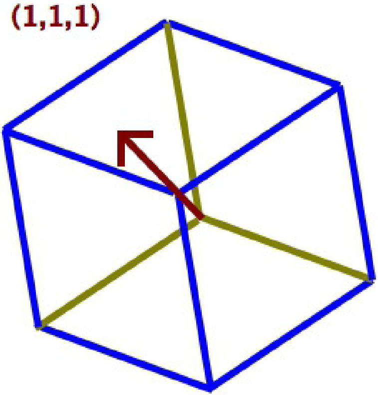


◇◇◇◇

From 2d and 3d to 4d Young diagrams

How to visualize 4d Young diagrams?

Just like the projection from $\mathbb{R}^3 \to \mathbb{R}^2$ along the $(1,1,1)$ axis

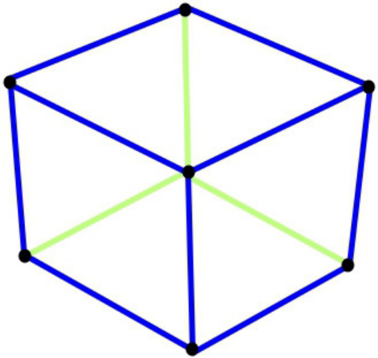


◊◊◊◊

From 2d and 3d to 4d Young diagrams

How to visualize 4d Young diagrams?

The projection from $\mathbb{R}^3 \to \mathbb{R}^2$ gives the tesselation of $\mathbb{R}^2$



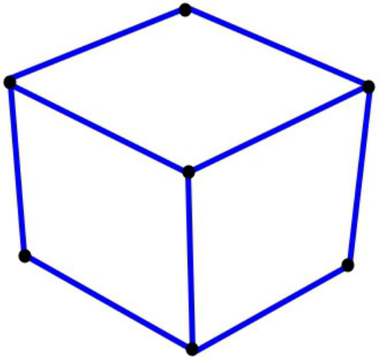
By rombi of three orientations

◊◊◊◊

From 2d and 3d to 4d Young diagrams

How to visualize 4d Young diagrams?

The projection from $\mathbb{R}^3 \to \mathbb{R}^2$ gives the tesselation of $\mathbb{R}^2$

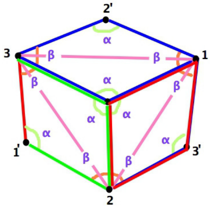


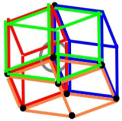
By rombi of three orientations

From 2d and 3d to 4d Young diagrams

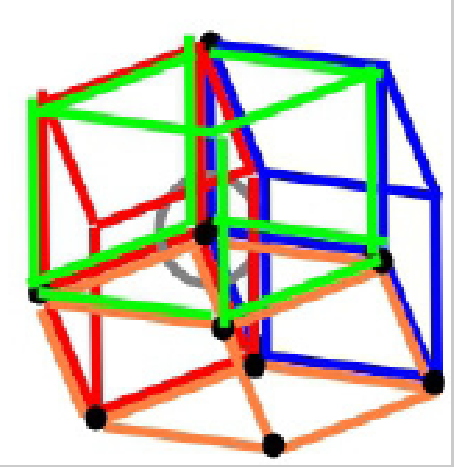
Projection from $\mathbb{R}^4 \to \mathbb{R}^3$ along $(1,1,1,1)$

Get the tesselation of $\mathbb{R}^3$ by squashed cubes





Random 3d geometries!



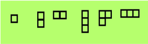
From 2d and 3d to 4d Young diagrams

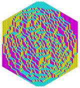
It turns out, one can set up a count of solid partitions

= 4d Young diagrams

Previous famous attempts due to P. MacMahon, 1916

$Z_2(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$, L. Euler





$Z_3(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$, MacMahon

$Z_4(q) =^? \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{n(n+1)/2}}$

Gives $1, 4, 10, 26, 59, 141, \dots, 217554, 424148 \dots$

Instead of $\mathbf{1}, \mathbf{4}, \mathbf{10}, \mathbf{26}, \mathbf{59}, \mathbf{140}, \dots, \mathbf{214071}, \mathbf{416849} \dots$

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

Beyond eleven dimensions ?!?!

Supersymmetric count of solid partitions

Four dimensional Young diagrams

as instanton configurations in

super-Yang-Mills theory on $\mathbb{R}^8 \tilde{\times} S^1$

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

◊◊◊◊

Magnificent Four Partition Function

$$Z^{U(n|n)}_{8+1}(q_1, q_2, q_3, q_4; \vec{a}|\vec{b}; p) \;=\; \exp \sum_{k=1}^{\infty} \frac{1}{k} F_9(q_1^k, q_2^k, q_3^k, q_4^k, \mu^k, p^k)$$

$$q_1, q_2, q_3, q_4, \qquad \prod_{a=1}^{4} q_a = 1\,,$$

parameters of an $SU(4) \subset Spin(8)$ rotation of $\mathbb{R}^8$

◊◊◊◊

◊◊◊◊

Magnificent Four Partition Function

$$Z_{8+1}^{U(n|n)}(q_1, q_2, q_3, q_4; \vec{a}|\vec{b}; p) \;=\; \exp \sum_{k=1}^{\infty} \frac{1}{k} F_9(q_1^k, q_2^k, q_3^k, q_4^k, \mu^k, p^k)$$

$$\vec{a}|\vec{b} = (a_1, \ldots, a_n | b_1, \ldots, b_n), \qquad \prod_{a=1}^{4} q_a = 1\,, \qquad \mu = \prod_{i=1}^{n} b_i / a_i$$

eigenvalues of the complexified $U(n|n)$ holonomy on S^1

◊◊◊◊

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

Magnificent Four Partition Function

$$Z_{8+1}^{U(n|n)}(q_1, q_2, q_3, q_4; \vec{a}|\vec{b}; p) = \exp \sum_{k=1}^{\infty} \frac{1}{k} F_9(q_1^k, q_2^k, q_3^k, q_4^k, \mu^k, p^k)$$

$$\text{Free energy } F_9 = \frac{[q_{12}][q_{13}][q_{23}][\mu]}{[q_1][q_2][q_3][q_4][\sqrt{\mu p}][\sqrt{\mu/p}]}$$

$$[x] := x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$

Our formula has been checked for up to $n = 16$ instantons

with N. Piazzalunga

R. Poghossian −−up to n=17

Works in all **1, 4, 10, 26, 59, 140, . . . , 214071, . . .** cases!

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

◊◊◊◊

Magnificent Four Partition Function

For special values of μ our partition function

Reduces to the previously known lower dimensional ones

In particular, for if all $b_i = q_4$

we get the partition function of $U(n)$ theory in $6+1$ dimensions

which matches sugra on $\mathbb{R}^4/\mathbb{Z}_n \times \mathbb{R}^6 \times S^1$

◊◊◊◊

♢♢♢♢

Magnificent Four Partition Function

For general values of μ our partition function

Coincides with that of some system of free bosons and fermions

Which contains (cohomologically) 11d linearized supegravity

What is its minimal number of spacetime dimensions?

♢♢♢♢

Beyond eleven dimensions ?!?!

Non-Poincare supersymmetry?

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$

THANK YOU

$\diamondsuit\diamondsuit\diamondsuit\diamondsuit$