International conference
Recent Advances in Theoretical Physics of Fundamental Interactions

Renormalizations, RG equations and high energy behaviour in non-renormalizable theories

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Motivation:

- The Standard Model is renormalizable
- Gravity is not renormalizable

Non-renormalizable theories are not accepted due to:

- UV divergences are not under control - infinite number of new types of divergences
- The amplitudes increase with energy (in PT) and violate unitarity

Suggestion (novel approach to NR interactions):

- To replace the multiplicative renormalization procedure by a new one, where the renormalization constant Z is replaced by an operator $\hat{Z}$, which depends on kinematics
- To sum up the leading asymptotics in all orders of PT (generalized RG) and to study the high-energy behavior

Renormalization

Consider 2->2 scattering amplitude on shell

$$\Gamma_4(s,t,u) = \Gamma_4^{tree}\bar{\Gamma}_4(s,t,u) \qquad \bar{\Gamma}_4 = 1 + \lambda ... + \lambda^2 ...$$

Renormalization (dimensional regularization)

$$\bar{\Gamma}_4 = Z_4(\lambda)\bar{\Gamma}_4^{bare}|_{\lambda_{bare}->\lambda Z_4},$$
$$\lambda_{bare} = \mu^\epsilon Z_4(\lambda)\lambda$$

BPHZ R-operation

$$RG = (1-K)R'G$$

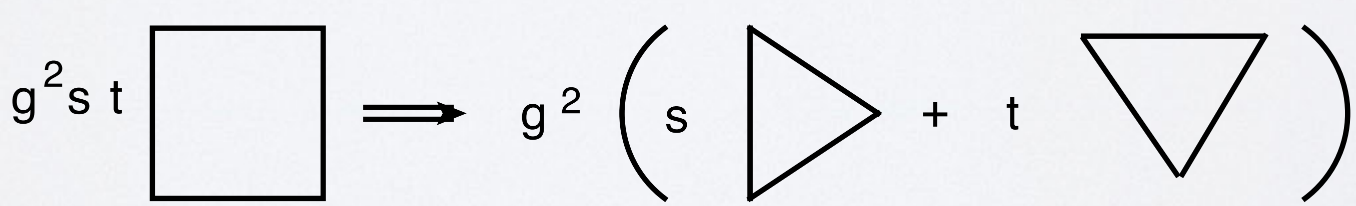
$$Z = 1 - \sum_i KR'G_i$$

In NR theories $Z \to \hat{Z}$ $\hat{Z}$ is a function (polynomial) of s,t,u acting as an operator

Example (taken from D=8 YM theory)

Exactly follows the BPHZ R-operation

$$\hat{Z} = 1 + g^2 \frac{st}{\epsilon}$$



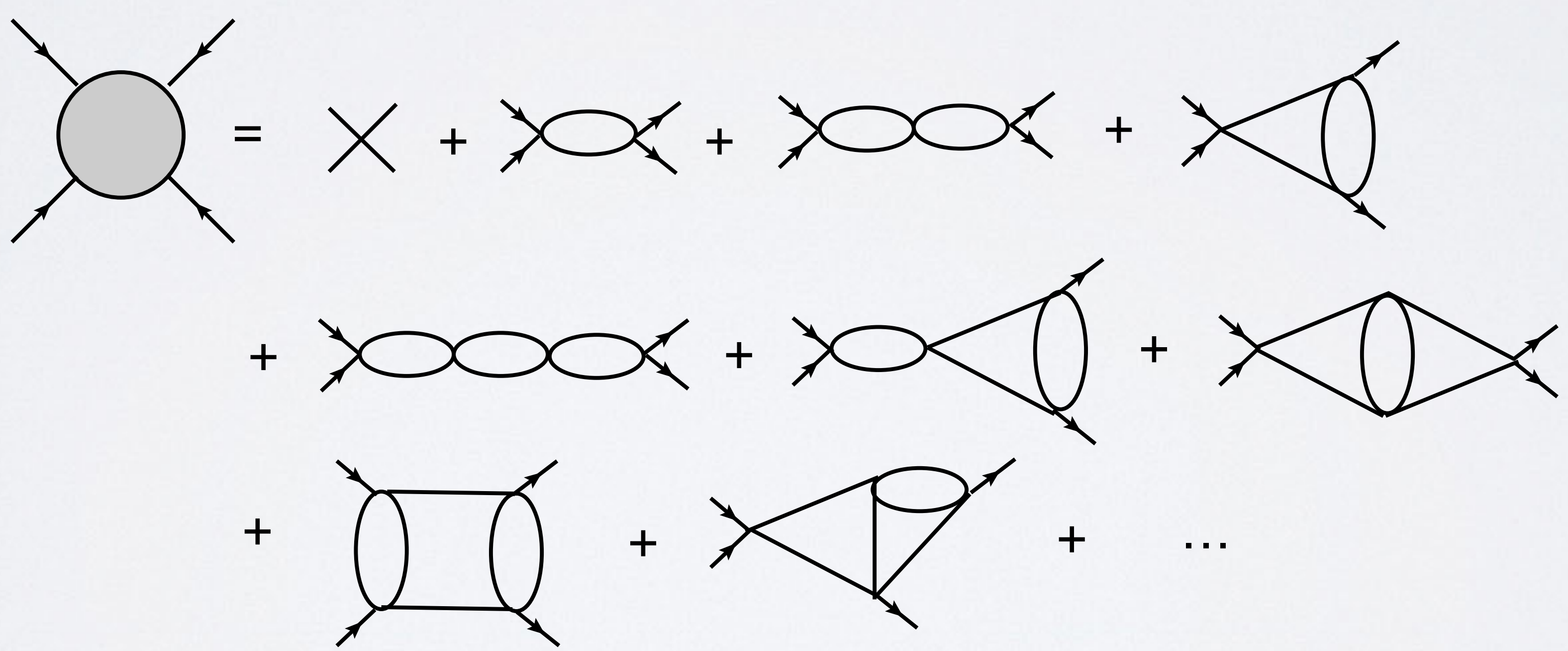
Either s or t are to be inserted into the loop and integrated

ϕ^4_D $\quad D = 4, 6, 8, 10 \quad [\lambda] = 2 - D/2$

2->2 scattering amplitude on shell $\quad m = 0 \quad s + t + u = 0$

$$\Gamma_4(s,t,u) = \lambda(1 + \Gamma_s(s,t,u) + \Gamma_t(s,t,u) + \Gamma_u(s,t,u))$$

PT: $$\Gamma_s = \sum_{n=1}^{\infty}(-z)^n S_n, \quad \Gamma_t = \sum_{n=1}^{\infty}(-z)^n T_n, \quad \Gamma_u = \sum_{n=1}^{\infty}(-z)^n U_n, \quad z \equiv \frac{\lambda}{\epsilon}$$



PT expansion (only s-channel is shown)

BPHZ R-operation

$$\mathcal{R}'G_n = \frac{A_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + ... + \frac{A_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^n}$$

$$+ \qquad \text{lower pole terms}$$

$A_k^{(n)}(\mu^2)^{k\epsilon}$ terms appear after subtraction of (n-k) loop counter terms

<u>Statement:</u> $R'G_n$ is local, i.e. terms like $\log^k \mu^2/\epsilon^m$ should cancel for any k and m

<u>Consequence:</u> $A_n^{(n)} = (-1)^{n+1}\dfrac{A_1^{(n)}}{n}$

$$KR'G_n = \sum_{k=1}^{n}\left(\frac{A_k^{(n)}}{\epsilon^n}\right) \equiv \frac{A_n^{(n)'}}{\epsilon^n} \qquad A_n^{(n)'} = (-1)^{n+1}A_n^{(n)} = \frac{A_1^{(n)}}{n}.$$

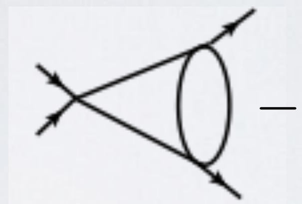
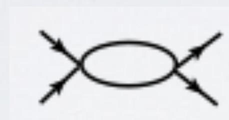
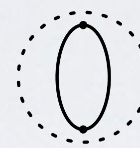
$A_1^{(n)}$ is the contribution to the leading pole in n-loops from the diagrams appearing in due corse of R-operation after subtraction of (n-1) loop counter terms

The leading divergences are governed by 1 loop diagrams!

Two loop example

ϕ^4_4

 $= \left(\frac{A_2^{(2)}}{\epsilon^2} + \frac{A_1^{(2)}}{\epsilon}\right) (\frac{\mu^2}{s})^{2\epsilon}$

$\mathcal{R}'$  $=$  $-$   $= \left(\frac{A_2^{(2)}}{\epsilon^2} + \frac{A_1^{(2)}}{\epsilon}\right) (\frac{\mu^2}{s})^{2\epsilon} - \frac{A_1^{(1)}}{\epsilon}(\frac{\mu^2}{s})^{\epsilon} \frac{A_1^{(1)}}{\epsilon}$

$$= \frac{A_2^{(2)}}{\epsilon^2} - \frac{(A_1^{(1)})^2}{\epsilon^2} + 2\frac{A_2^{(2)}}{\epsilon}\log(\mu^2/s) - \frac{(A_1^{(1)})^2}{\epsilon}\log(\mu^2/s) \quad = -\frac{1}{2}\frac{(A_1^{(1)})^2}{\epsilon^2} + \ldots$$

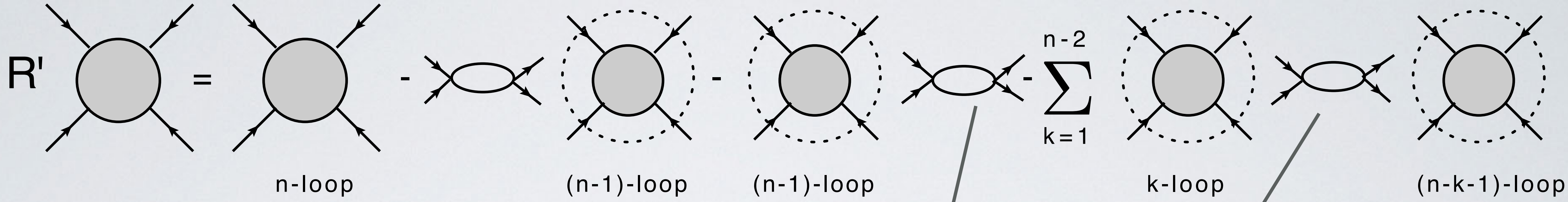
non-local terms to be cancelled

Leading divergence is given by the one-loop term $A_2^{(2)} = \frac{1}{2}(A_1^{(1)})^2$

ϕ^4_D

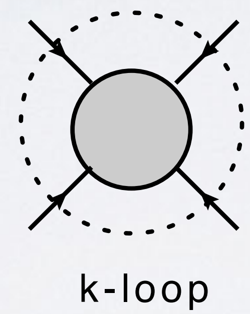
- These statements are universal and are valid in non-renormalizable theories as well.
- The only difference is that the counter term $A_1^{(1)}$ depends on kinematics and has to be integrated through the remaining one-loop graph.
- As a result $A_2^{(2)}$ is not the square of $A_1^{(1)}$ anymore but is the integrated square (see below).
- This last statement is the general feature of any QFT irrespective of renormalizability

Recurrence Relations for the Leading Poles



- ··· terms with higher loop remaining diagrams

KR'



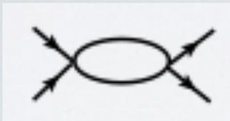
$$= \frac{A_k^{(k)'}}{\epsilon^k} = (-1)^{k+1}\frac{A_k^{(k)}}{\epsilon^k}$$

$$\begin{aligned}
nS_n(s,t,u) \quad &= \quad \frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\left(S_{n-1}(s,t',u')+T_{n-1}(s,t',u')+U_{n-1}(s,t',u')\right) \\
&+ \quad \frac{1}{2}\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\sum_{k=1}^{n-2}\sum_{p=0}^{(D/2-2)k}\sum_{l=0}^{p}\frac{1}{p!(p+D/2-2)!}\times \\
&\times \quad \frac{d^p}{dt'^l du'^{p-l}}(S_k+T_k+U_k)\frac{d^p}{dt'^l du'^{p-l}}(S_{n-k-1}+T_{n-k-1}+U_{n-k-1})s^p[x(1-x)]^p t^l u^{p-l}
\end{aligned}$$

$$t' = -xs, u' = -(1-x)s$$

Solution of Recurrence Relations

Starting point:



$$S_1 = \frac{1}{2}\frac{\Gamma(D/2-1)}{\Gamma(D-2)} s^{D/2-2}, \quad etc$$

Use the recurrence relation

$$S_2 = \frac{1}{4}\frac{\Gamma(D/2-1)}{\Gamma(D-2)}\left[\frac{\Gamma(D/2-1)}{\Gamma(D-2)} + 2(-)^{D/2}\frac{\Gamma(D-3)}{\Gamma(3D/2-4)}\right] s^{D-4}, \quad etc$$

Notice the difference with renormalizable theory: S_1 depends on kinematics!

To get S_2 one has to integrate T_1 and U_1 over the loop

This is exactly what we do in writing the recurrence relation

$$\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\left(S_{n-1}(s,t',u')+T_{n-1}(s,t',u')+U_{n-1}(s,t',u')\right)$$

Differential Equation

Summing up the recurrence relation $\sum_{n=2}^{\infty}(-z)^n$ one gets the diff equation

$$-\frac{d\Gamma_s(s,t,u)}{dz} = \frac{1}{2}\frac{\Gamma(D/2-1)}{\Gamma(D-2)}s^{D/2-2} \qquad \Gamma_s(z=0)=0$$

$$+\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\left[\Gamma_s(s,t',u')+\Gamma_t(s,t',u')+\Gamma_u(s,t',u')\right]\Big|_{\substack{t'=-xs,\\ u'=-(1-x)s}}$$

$$+\quad \frac{1}{2}\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\sum_{p=0}^{\infty}\sum_{l=0}^{p}\frac{1}{p!(p+D/2-2)!}\times$$

$$\times\quad \left(\frac{d^p}{dt'^l du'^{p-l}}(\Gamma_s+\Gamma_t+\Gamma_u)\Big|_{\substack{t'=-xs,\\ u'=-(1-x)s}}\right)^2 s^p[x(1-x)]^p t^l u^{p-l}$$

$$\frac{d\Gamma_s(s,t,u)}{d\log\mu^2} = -\frac{\lambda}{2}\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\sum_{p=0}^{\infty}\sum_{l=0}^{p}\frac{1}{p!(p+D/2-2)!}\times$$

$$\times\left(\frac{d^p\bar{\Gamma}_4(s,t',u')}{dt'^l du'^{p-l}}\Big|_{\substack{t'=-xs,\\ u'=-(1-x)s}}\right)^2 s^p[x(1-x)]^p t^l u^{p-l}$$

$$\Gamma_s(\log\mu^2=0)=0$$

Solution of RG Equation

$D = 4$

$s \sim t \sim u \sim E^2$

$$\frac{d\bar{\Gamma}_4}{d\log\mu^2} = -\lambda\frac{3}{2}\bar{\Gamma}_4^2, \quad \bar{\Gamma}_4(\log\mu^2 = 0) = 1 \quad \rightarrow \quad \bar{\Gamma}_4 = \frac{1}{1+\frac{3}{2}\lambda\log(\mu^2/E^2)}$$

General Solution for any D

$$\bar{\Gamma}_4(s,t,u) = \mathcal{P}\frac{1}{1+\frac{1}{2}\frac{\Gamma(D/2-1)}{\Gamma(D-2)}\lambda(s^{D/2-2}+t^{D/2-2}+u^{D/2-2})\log(\mu^2/E^2)}$$

$\mathcal{P}$ is the symbol of ordering in a sense of recurrence relation

$$\Gamma_4(s,t,u) = \mathcal{P}\frac{\lambda}{1+\lambda A_1^{(1)}\log(\mu^2/E^2)} = \mathcal{P}\sum_{n=0}^{\infty}(-\lambda)^n\log^n(\mu^2/E^2)(A_1^{(1)})^n$$

$$\mathcal{P}(A_1^{(1)})^n = \int_0^1 dx\sum_{k=0}^{n-1}\overrightarrow{\mathcal{P}(A_1^{(1)})^k}\;A_1^{(1)}\;\overleftarrow{\mathcal{P}(A_1^{(1)})^{n-1-k}},$$

Solution of RG Equation

To get S_2 for instance one has to take $s^{D/2-2}(s^{D/2-2}+t^{D/2-2}+u^{D/2-2})$ terms

and integrate s, t and u over the s-loop. This is exactly given by

$$S_2 = \frac{1}{2}\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\left(S_1(s,t',u')+T_1(s,t',u')+U_1(s,t',u')\right) \qquad (*)$$

$$t'=-xs, u'=-(1-x)s.$$

To get S_3 one has to take $s^{D/2-2}(s^{D/2-2}+t^{D/2-2}+u^{D/2-2})^2$ terms

and integrate s, t and u over the s-loop. This is given by expr $(*)$ and by the terms

emerging when t and u come from the diagrams standing to the left and right of the s-channel one

$$S_3 = \frac{1}{3}\Bigg[\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\left(S_2(s,t',u')+T_2(s,t',u')+U_2(s,t',u')\right)$$

$$+\frac{1}{2}\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\sum_{p=0}^{(D/2-2)}\sum_{l=0}^{p}\frac{1}{p!(p+D/2-2)!}$$

$$\frac{d^p}{dt'^l du'^{p-l}}(S_1+T_1+U_1)\frac{d^p}{dt'^l du'^{p-l}}(S_1+T_1+U_1)s^p[x(1-x)]^p t^l u^{p-l}\Bigg] \qquad (**)$$

High Energy Behaviour of the scattering amplitude in ϕ^4_D theory

$$\Gamma_4(s,t,u) = \mathcal{P}\frac{\lambda}{1+\frac{1}{2}\frac{\Gamma(D/2-1)}{\Gamma(D-2)}\lambda(s^{D/2-2}+t^{D/2-2}+u^{D/2-2})\log(\mu^2/E^2)}$$

$$s \sim t \sim u \sim E^2$$

$D=4$ $\quad 3/2>0$ $\quad$ As a result one has a Landau pole as $E \to \infty$

$D=6$ $\quad s+t+u=0$ $\quad$ All the leading divergences (logs) cancel in all loops

One can explicitly check that S_2 given above vanishes

$D=8$ $\quad s^2+t^2+u^2>0$ $\quad$ has a Landau pole as $E \to \infty$

$D=10$ $\quad s^3+t^3+u^3=3stu>0$ $\quad s>0, t,u<0$ $\quad$ has a Landau pole as $E \to \infty$

Conclusion: ϕ^4_D has a Landau pole as $E \to \infty$