Walks in the Hilbert space

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Outline of the talk

- «Path integrals in the Hilbert space». Duality and explicit examples
- Exponential random graphs as model for the Hilbert spaces. Phase transitions,eigenvalue instantons, spectral statistics and finite-size effects
- Dirac operator spectrum in QCD and holography

Spectral duality

- There is the duality in the integrable many-body systems which relates two systems with the coordinates and spectral variables get interchanged classically
- The quantum wave function Q(X,E) serves for two many-body systems simultaneously
-

$$Z(T) = \int dx(t) e^{-S(x(t))/\hbar} = Tr e^{-H/T}$$

Closed paths in X and E spaces. What about open paths in X and E ?

Examples of classical duality

2-body problem

$$H(p,q) = \frac{1}{2}p^2 + \frac{\nu^2}{2\sin^2(q)}.$$

$$H_D(I,\varphi) = \cos(q) = \cos\varphi\sqrt{1 - \frac{2\nu^2}{I^2}}$$

Many-body problem

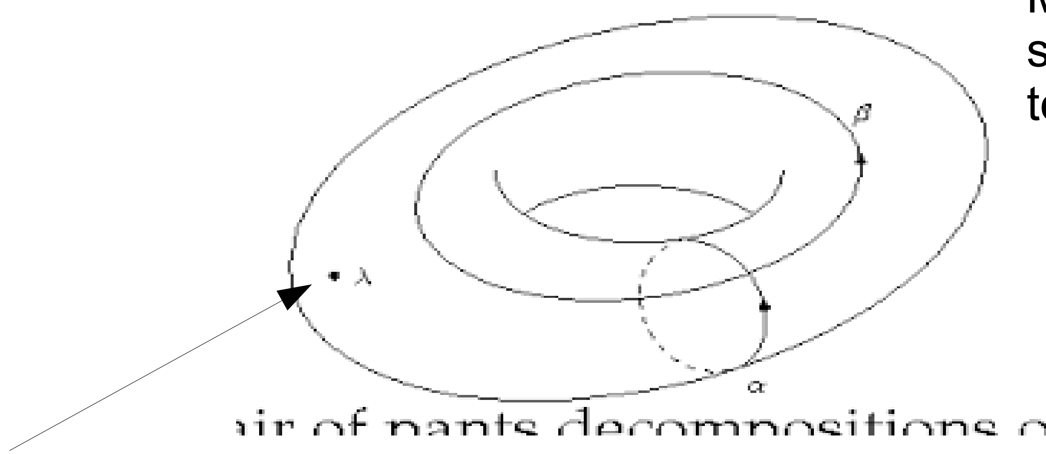
$$H_2 = \sum_i p_i^2 + \sum_{i<j} \frac{R^2\nu^2}{4\sin^2\left(\frac{R(q_i-q_j)}{2}\right)}.$$

Perturbed 2d Yang-Mills
Trigonometric Calogero

$$H = \frac{1}{2}\mathrm{Tr}(L + L^{-1}) = \sum_i \cos(\beta p_i) \prod_{j\neq i} \sqrt{1 - \frac{(\beta\nu)^2}{q_{ij}^2}}.$$

Perturbed 3d Chern-Simons
Rational Ruijsenaars-Schneider

Geometrical interpretation of duality



Ruijsenaars system- intergrable Many-body system on the moduli space of flat connections on the torus with the marked point

Fock,Nekrasov,Roubtsov, A.G 99

Marked point with Wilson Line insertion

$$ABA^{-1}B^{-1} = \exp(R\beta J)$$

Different names

Bispectrality=
Automorphism of the Hamiltonian reduction

$$J_{ij} = i\nu(\delta_{ij} - e_i e_j^*)$$

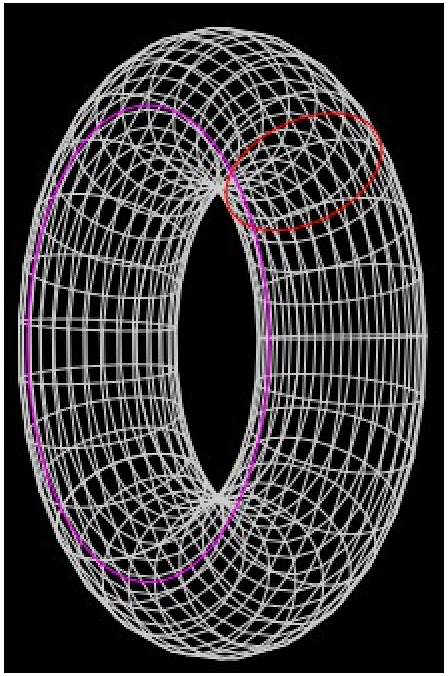
$$\sum_i |e_i|^2 = N$$

How the classical dynamics in the dual system(non-integrable) goes?

- Degrees of freedom? Motion across the energy levels(or all sets of integrals of motion for many-body system)
- Symplectic structure?
- What are times? Parameters of potential
- General theory of Whitham dynamics. Works both for holomorphic and real dynamical systems

Classical dynamics in the Hilbert space. Holomorphic system. Whitham. Prepotential

$$dI^i \wedge dI_i^D$$

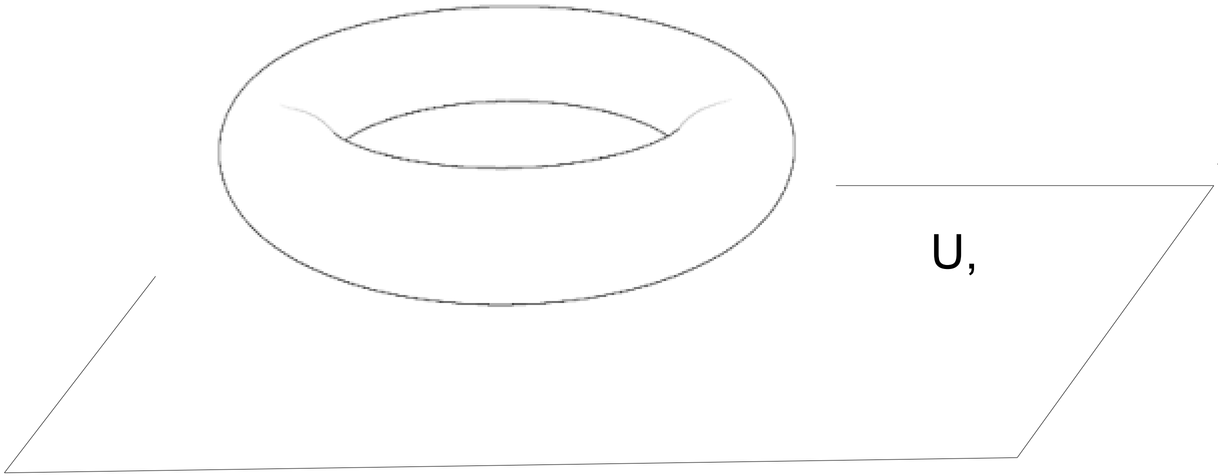


$$I_i^D = \frac{\partial \mathcal{F}}{\partial I^i}$$

Prepotential = action S in the Whitham Hamiltonian system

Whitham hierarchy describes the dynamics on the moduli space of solutions to the classical equations of motion. For holomorphic systems- dynamics on the moduli space of complex structures of the Riamann surfaces Krichever- 80-th

Whitham in SU(2) SYM.Example



Seiberg-Witten ,94
Krichever,Marshkov,Mironov,Morozov
A.G/ 95

$$H = u = p^2 + \Lambda^2 cos x, (x,p) \in C$$

Action

$$a_D = \frac{\partial \mathcal{F}}{\partial a} \qquad \frac{\partial \mathcal{F}}{log \Lambda} = const H$$

momentum — coordinate — Time variable

$$a_D = \oint_{\gamma_1} \lambda$$

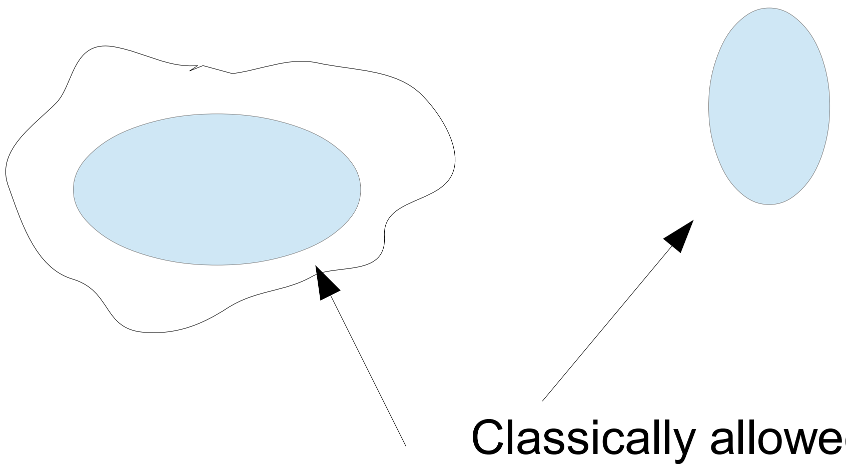
$$a = \oint_{\gamma_2} \lambda.$$

$$\lambda = \frac{\sqrt{2}}{2\pi} \frac{dx\,\sqrt{x-u}}{\sqrt{x^2-1}} :$$

General remarks on Whitham dynamics

- Is there any meaning of the classical Whitham equations at the quantum level? Yes - it is P/NP relation (Alvares et al , Dunne-Unsal, Milekhin-A.G, Marino et al)
- Whitham equation from the field theory viewpoint — anomalous Ward identity+ RG
- For the holomorpic system relation of Whitham with the knot invariants near the degeneration points Det(M-t)=A(t) — Alexander polynomial, M- monodromy matrix near Argres-Douglas point

Dynamics in the Hilbert space. Real phase space. Whitham. Prepotential



(x, p)

Classically allowed regions in the phase (x,p) space

Whitham dynamics described the deformation of the classically allowed regions as function of integrals of motion and parameters of the potential.

Prepotential F = Action in the Whitham dynamics

Mineev,Zabrodin,Wiegman - 2000

$$\bar{z} = S(z)$$

Feynman path integral in a nutshell

REVIEWS OF

MODERN PHYSICS

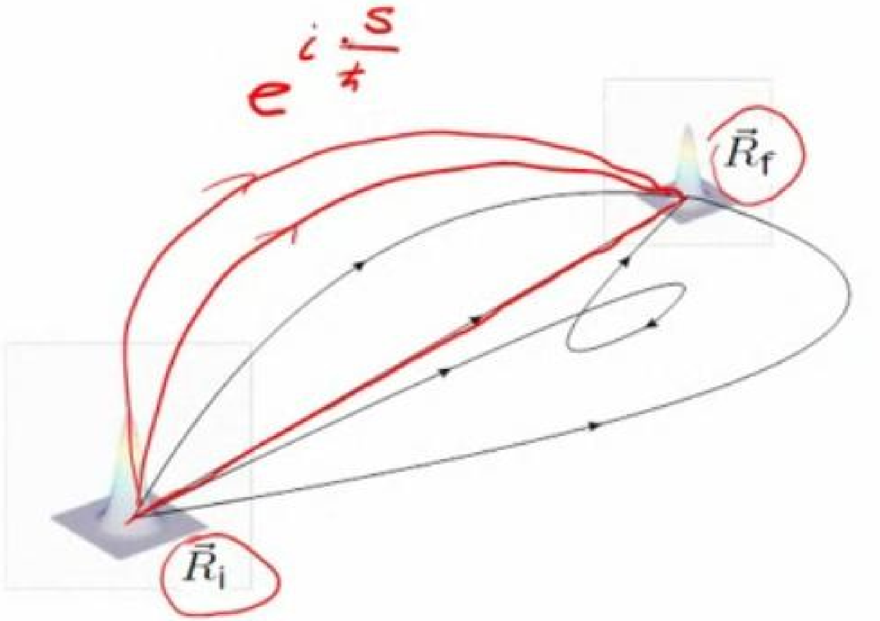
VOLUME 20, NUMBER 2 APRIL, 1948

Space-Time Approach to Non-Relativistic Quantum Mechanics

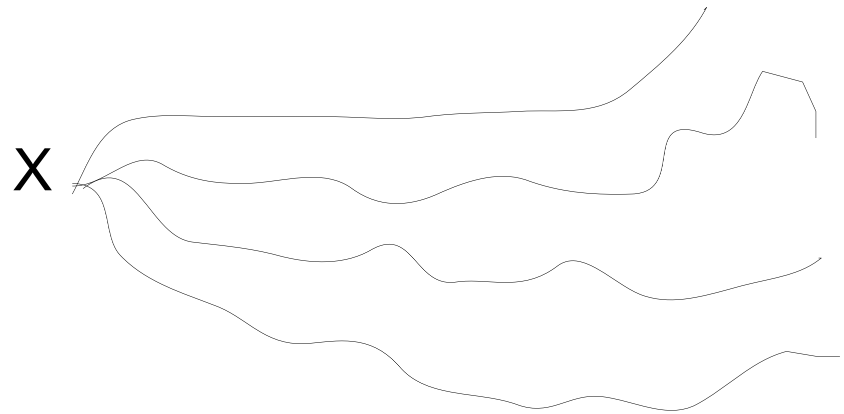
R. P. FEYNMAN

Cornell University, Ithaca, New York

Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be

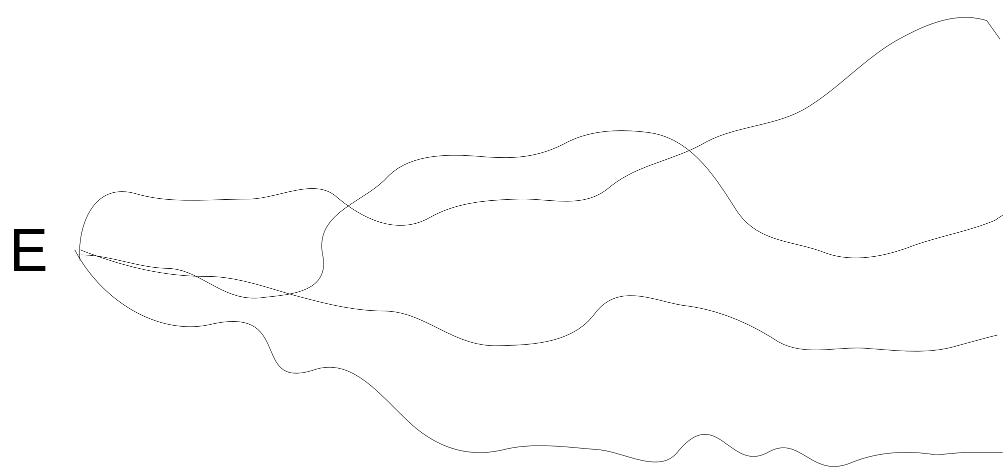


Q(X,E) wave function



Sum over open paths in X space

Feynman path integral



Sum over open paths in E space

How parameterize?

What is the measure?

Does it have any meaning at all?

Quantum Oscillator. Example

$$\hat{H}\psi_n = \omega(n+\frac{1}{2})\psi_n$$

$$\psi_n(q) = (\frac{\omega}{\pi})^{1/4}\frac{e^{-\frac{\omega q^2}{2}}}{2^{n/2}\sqrt{n!}}H_n(q\sqrt{\omega})$$

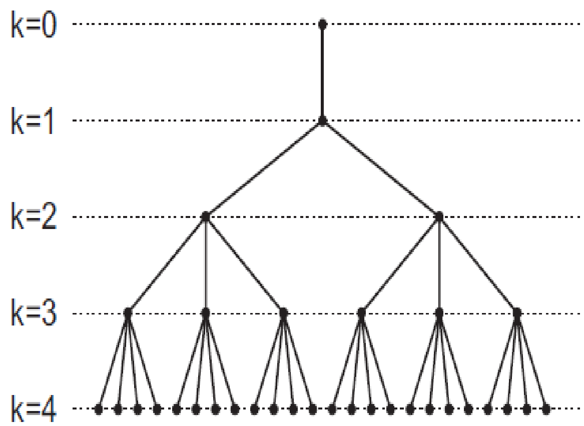
Differential equation in coordinate q

Dual Hamiltonian yields the recurrence for Hermite's. Difference equation

$$\hat{X}\Psi_x(n) = x\Psi_x(n)$$

$$\hat{H}_D = T_+\sqrt{n} + \sqrt{n}T_-, \quad T_\pm = e^{\pm\frac{\partial}{\partial n}}$$

Degree variance. Oscillator



$$p_k = \begin{cases} p_0, & \text{for } k = 0 \\ p_0 + ak, & \text{for } k \geq 1,\ a \leq 0 \end{cases}$$

Degree of the tree for oscillator Hilbert space grows linearly with the level

Sum over all paths on the tree of length N up to k level

$$\begin{cases} Z_{N+1}(k) = (p_{k-1} - 1) Z_N(k-1) + Z_N(k+1) & \text{for } 2 \leq k \leq K-1 \\ Z_{N+1}(k) = Z_N(k+1), & \text{for } k = 0 \\ Z_{N+1}(k) = p_{k-1} Z_N(k-1) + Z_N(k+1) & \text{for } k = 1 \\ Z_{N+1}(k) = (p_{k-1} - 1) Z_N(k-1), & \text{for } k = K-1 \\ Z_{N=0} = \delta_{k,0} \end{cases}$$

$$\mathbf{Z}_{N+1} = \hat{T}\mathbf{Z}_N; \qquad \hat{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ p_0 & 0 & 1 & 0 & & \\ 0 & p_1 - 1 & 0 & 1 & & \vdots \\ 0 & 0 & p_2 - 1 & 0 & & \\ \vdots & & & & \ddots & \\ 0 & & \dots & & p_{K-2} - 1 & 0 \end{pmatrix}; \qquad \mathbf{Z}_{N=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$P_k(\lambda) = \det(\hat{T} - \lambda \hat{I}),$$

Play the role of wave function!

Argument of the wave function weights the length

Comparison with ODE/IS duality

$$V(x) = x^{2m} + \frac{l(l+1)}{x^2}$$

Dorey,Tateo 98-99
Bazhanov,Lukyanov,Zamolodchikov
2001

Q(E) <-----------------> det (H - E)

Solution to the Baxter
equation in the statistical model

Spectral determinant

In our case

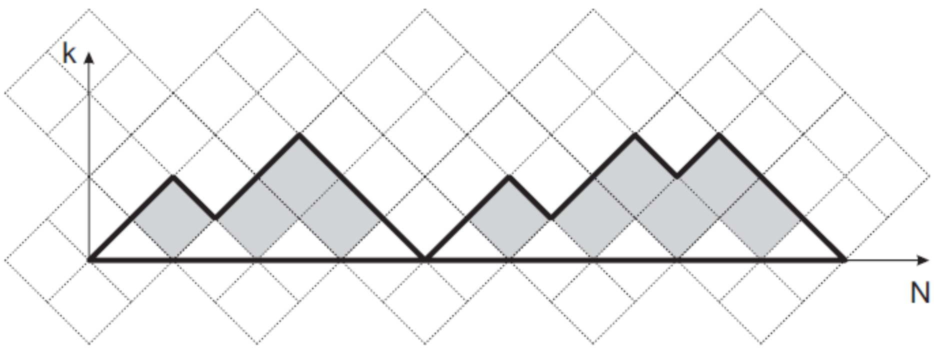
Det (T-x) <-------------------> Q(x)

Determinant of transfer matrix of
some statistical model

Wave function

Varying degree. Fredkin model.

$$\begin{cases} W_{N+1}(k,q) = q^{k-1} W_N(k-1,q) + W_N(k+1,q) & \text{for } 1 \le k \le K-1 \\ W_{N=0}(k,q) = \delta_{k,0} \end{cases}$$



Ground state of the generalized Fredkin model = sum over the magnetic Dyck paths. Dyck path corresponds to the path on the tree with the varing non-integer degree like for oscillator.

Magnetic field= velocity of the degree change (a version of T-duality)

Korepin et al (16)

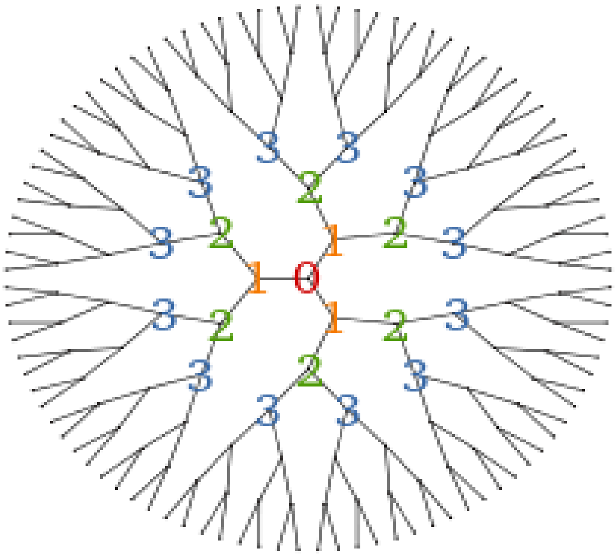
How parameterize the Hilbert space of non-integrable system?

- If the system is chaotic the matrix model approach works (Wigner,Dyson...)
- More general situation — not chaotic necessarily. Parametrization by some weighted

  random graphs(Altshuler,Gefen, Kamenev, Levitov 97).
- Instead of integrating over the random matrix ensemble with some weight summation over some graph ensemble with some weight. So-called exponential random graphs

Parameterizing the quantum Hilbert space of interacting many-body system.

Bethe tree - discretization of the hyperbolic plane



Vertex= state

Link = two states related
By resonant condition

No cycles ?

Constant vertex degree ?

Altshuler,Gefen,Kamenev,Levitov (97)

Bethe tree and many-body localization(non-thermalization)

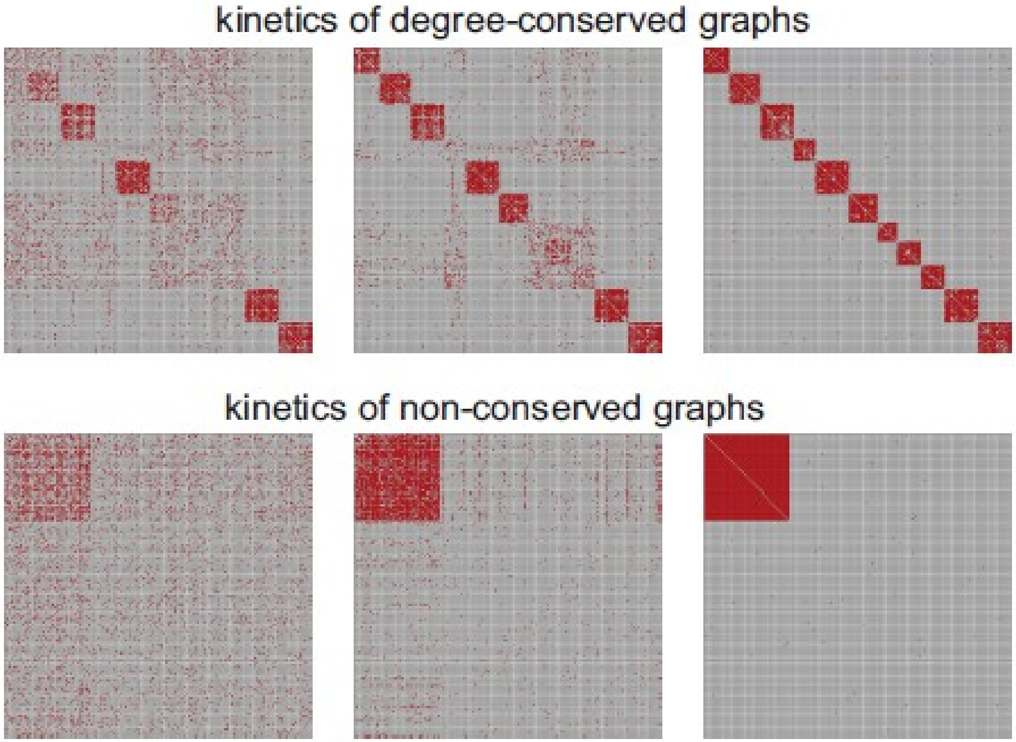
- The model was very rough but provides starting playground for many-body localization.
- Strong disorder on the cites — one- body localization in the Hilbert space(=tree)
- One-body localization in the Hilbert space =many-body localization in the physical space
- Localized states in the Hilbert space corresponds to the states in the initial system

Some general aspects of graph representation of Hilbert spaces

- Role of cycles(singlet sector). Critical behavior when cycles start to dominate at some chemical potential(resonant triples...)
- Open motifs(open strings) also induces the phase transitions
- Clustering and localization on the graph
- Strong finite-size 1/N effects can be analyzed

Phase transitions in exponential random graphs

Behavior above the critical value of the chemical potential



RRG

ER

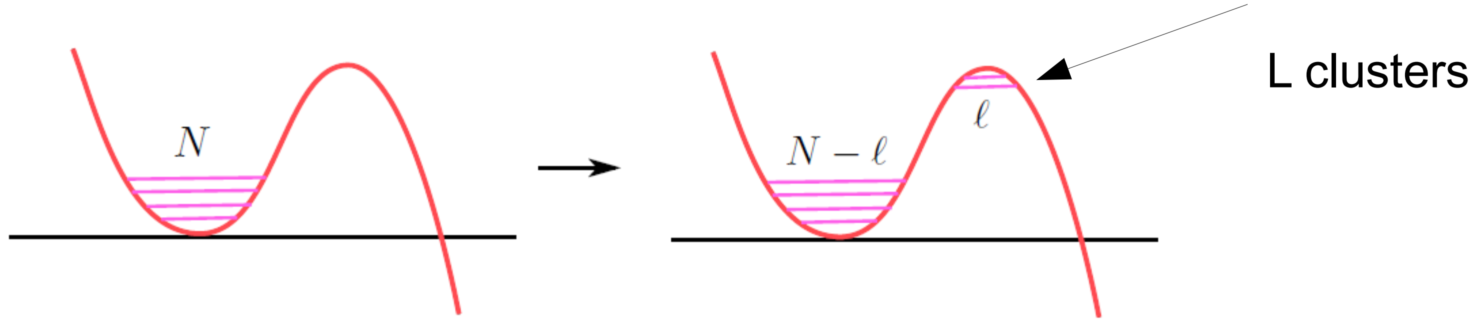
$$Z(\mu) = \sum_{\{\text{states}\}}{}' e^{-\mu n_{\triangle}}$$

Avetisov, Nechaev, Hovhannisyan, Valba, A.G.(16)

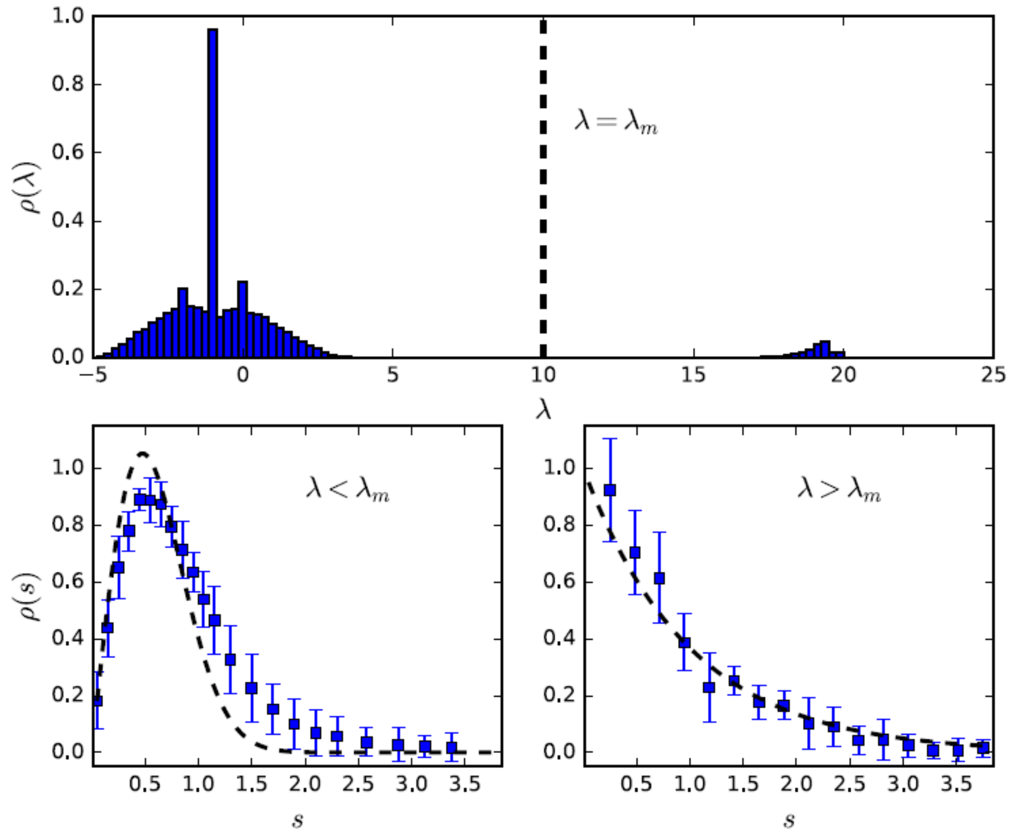
Can be chemical potentials for open triples, for 4-cycles etc. In all cases- phase transitions

Cluster formation=eigenvalue instanton

The matrix model counterpart of the cluster formation. Eigenvalue tunneling — instanton, nonperturbative phenomena in many physical situations. Formation of stable D-branes in the string theory from unstable D0,s. Baby-universes in 2d gravity, domain walls in SUSY YM



Clusters and localization.



Clusters due to the 3-cycles strongly influence the localization properties in the network describing the Hilbert space. Modes corresponding to the Clsuters in the Hilbert space are localized.

Avetisov, Nechaev, Valba ,A.G (18)

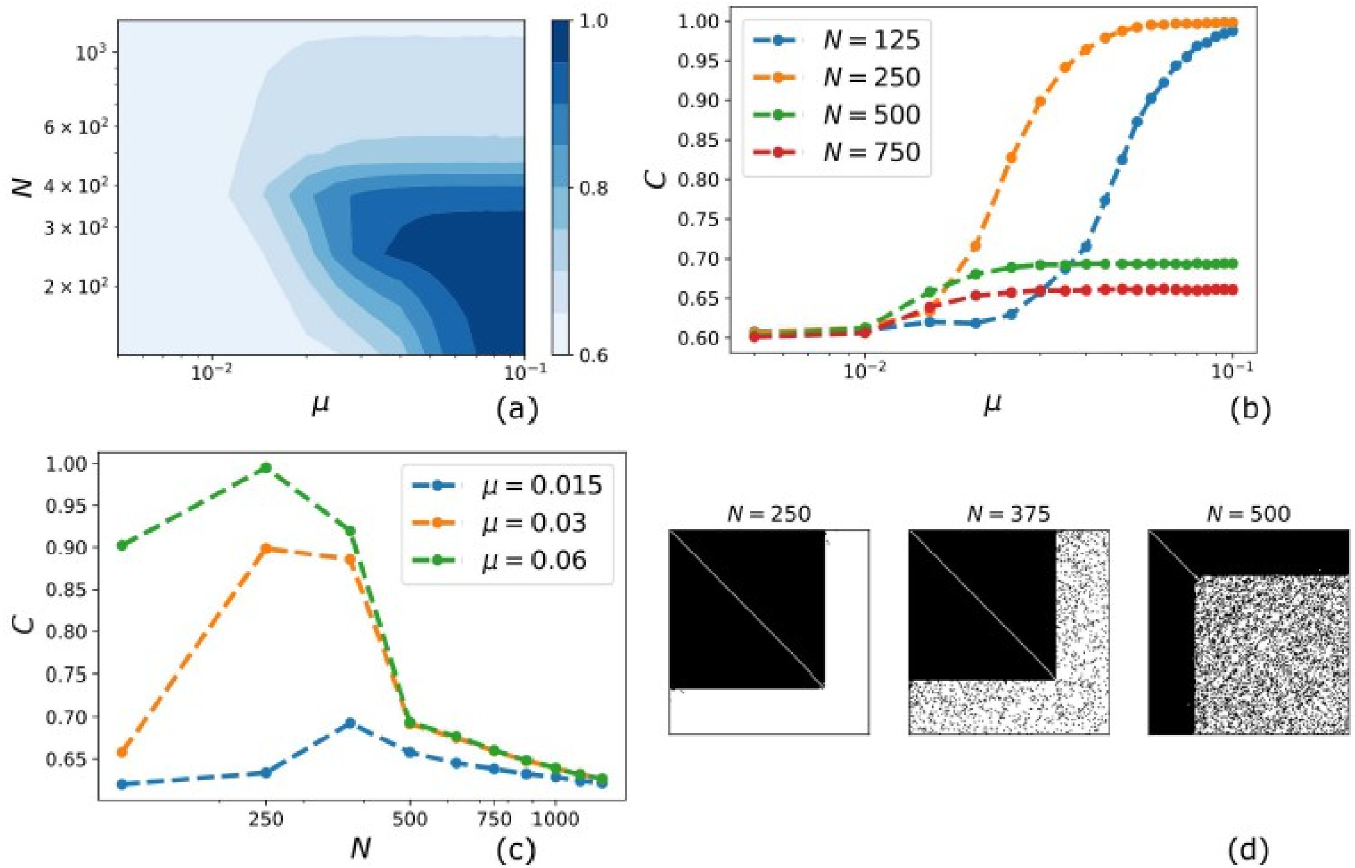
Finite-size effects and phase transitions

$$Z = \sum_G e^{-H(G)}$$

$$H = \theta L(G) - \mu s(G),$$

Number of triples

The model has been solved in mean-field approximation (Newman, Park 2005)
At large N. It turns our that there are non-trivial finete-size effects (Valba,A.G 19)

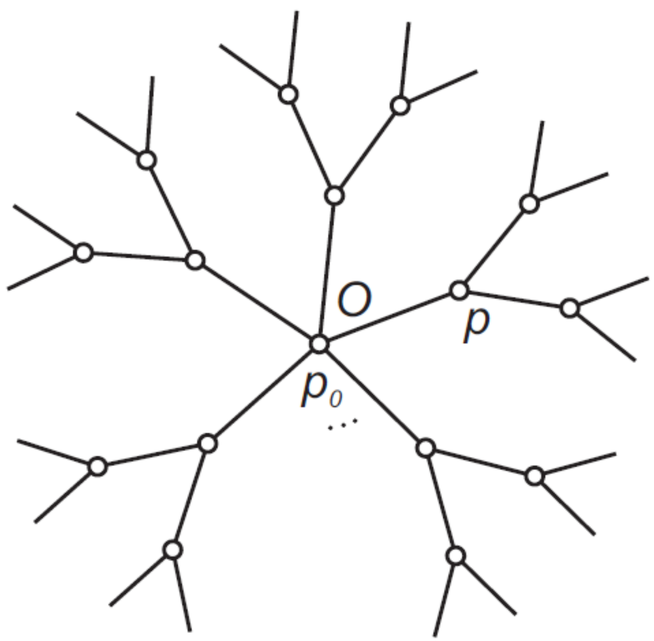


Crossover at some finite N!

Full block is the model for the black hole (Susskind, Hanada …..) buit from DO branes.
With this interpretation the BH starts to evaporate then «hair» gets emerged
And final stage of the evolution is the random network «vapor» and the single
Star. Recall that g=1/N , the gravity becomes weaker

Entropic trap phase transition

Nechaev,Tamm,Valba (16)

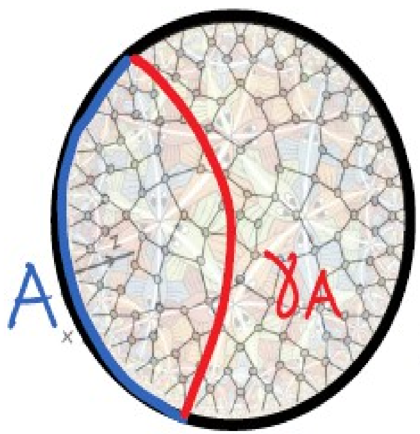


$$\bar{p}_0 = p^2 - p.$$

Critical degree of the heavy root degree when the typical walkers will be localized around a root. A kind of horizon gets created. This fits parametrically with the Crossover found (Valba, A.G)

Radial coordinate in the bulk — RG energy scale

AdS/CFT duality: conjectural realization

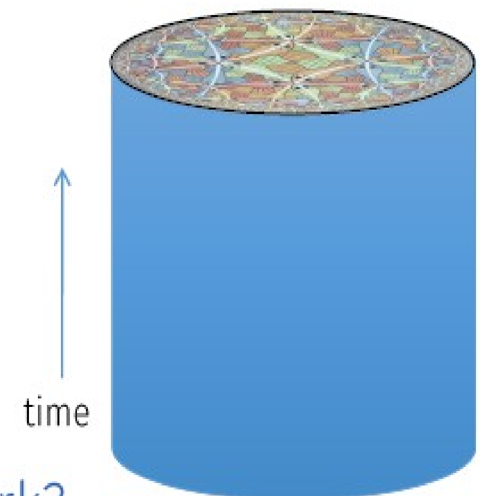


$$S(A) = \frac{1}{4G_N} \min |\gamma_A|$$

[Ryu-Takayanagi]

Space-time as a tensor network?

[Swingle]



boundary:
d-dim CFT

bulk: (d+1)-dim string
(gravity) theory

Operator-state correspondence. Operator graph counting in SYK

Stanford et al (18)

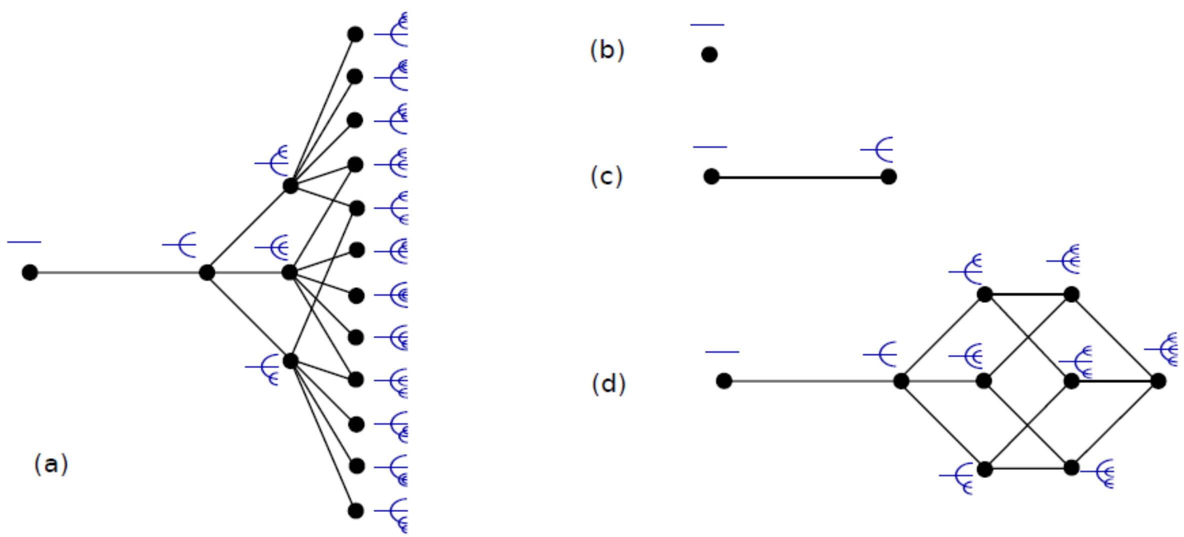


Figure 2: The graph of operators. In (a) we show the first four layers. Vertices correspond to basis operators, whose associated fan diagrams are indicated in blue. The problem of the time evolution of $\psi_1(t)$ in the large N theory is equivalent to the motion of a quantum particle on this graph (extended to further layers). In (b), (c), and (d), we show versions of the graph where we limit the recursive depth of the fan diagrams. The return amplitude on these graphs gives the zeroth, first and second iterations of the real-time Schwinger-Dyson equations. For any finite cutoff these amplitudes oscillate in time, but for the infinite graph the return amplitude decays exponentially.

Operator counting = particle falling in the radial coordinate in holographic picture

Holography and mobility edge

- In many disordered systems there can be mobility edge which separates the localized and delocalized modes at some energy scale
- What is the holographic conterpart of the mobility edge? There shoud be some specific scale at the radial coordinate
- A bit surprizingly the Dirac operator in QCD provides the clear-cut example

Spectral properties of Dirac operator in low-energy QCD

– Casher-Banks relation(82)

$$D\psi_n = i\lambda_n\psi_n$$

$$\rho(\lambda) =< \sum_n \delta(\lambda - \lambda_n) >_{QCD}$$

$$< \bar{\Psi}\Psi >= \Sigma = \frac{\pi\rho(0)}{V}$$

– Smilga-Stern linear correction(93)

$$\frac{\rho'(0)}{\rho(0)} = \frac{(N_f - 2)(N_f + 1)\Sigma}{16\pi N_f F_\pi^2}$$

--Microscopic and macroscopic spectral densities and correlators from two different matrix models (Verbaarschot -Shuryak 94 and zero modes in ChL).

QCD as chiral disordered system

- Consider the QCD Dirac operator in the Euclidean 4d space-time as Hamiltonian of disordered system in 4+1 space
- Disorder — vacuum fluctuations of gluon field
- Hamiltonian in Euclidean 4d is conjugated to the Schwinger proper time variable(Janik, Nowak,Zahed — 98, Osborn -Verbaarchoot 98)

- All modes are delocalized in the confinement phase!(Garsia-Garsia,Osborn 2006)
- --There is mobility edge in the deconfinement phase ! (2006)

Tools for diagnostics of localization transition

– Two-point spectral correlator and spectral formfactor

$$\rho(\lambda) = Im < Tr\frac{1}{H-\lambda} >$$

$$R(\lambda_1, \lambda_2) = < \rho(\lambda_1)\rho(\lambda_2) > - < \rho(\lambda_1 >< \rho(\lambda_2) >$$

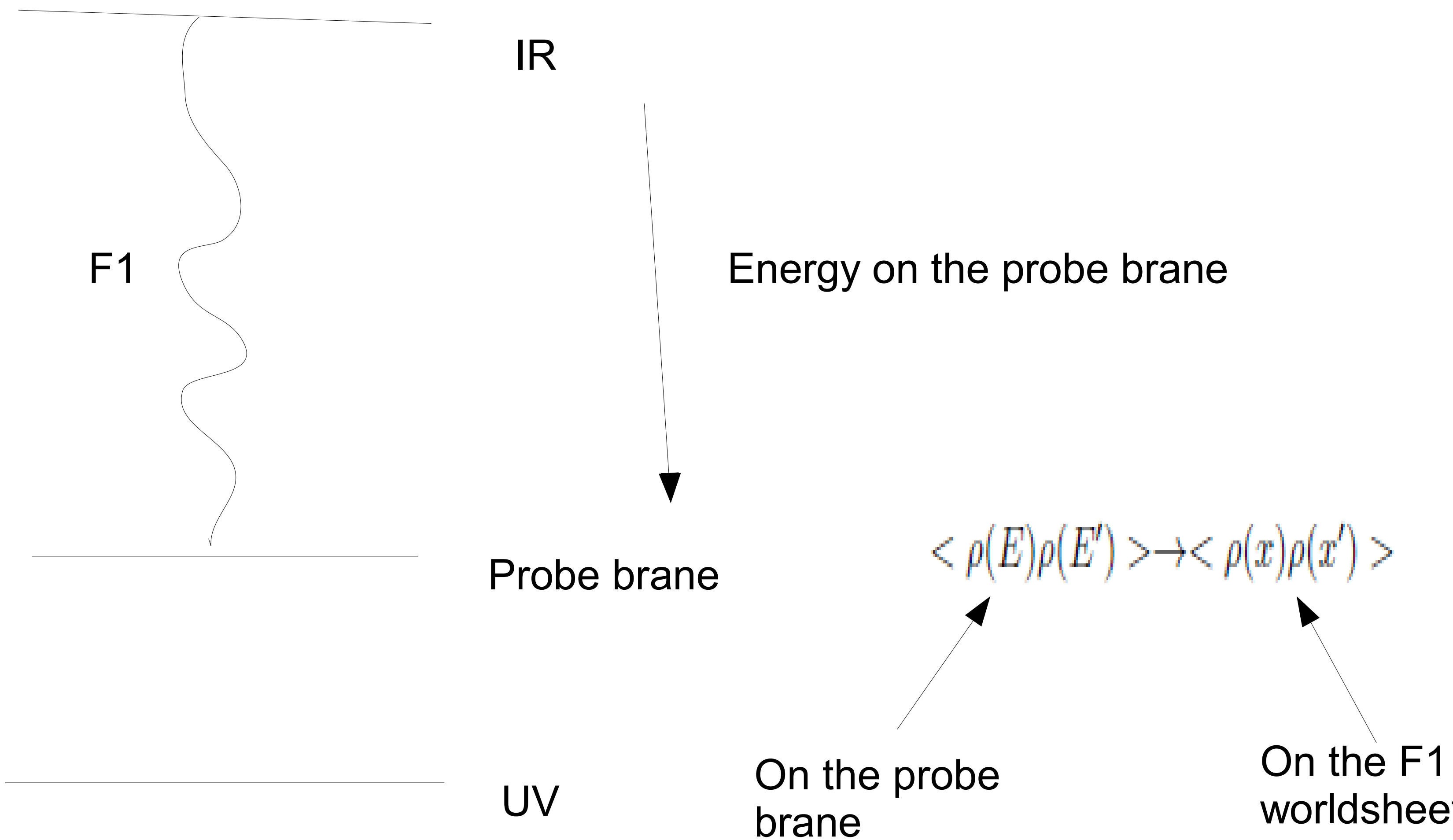
In our case t — RG scale

$$K(E,t) = \frac{1}{2\pi\hbar}\int d\lambda R(E+\lambda, E-\lambda)e^{\frac{i\lambda t}{\hbar}}$$

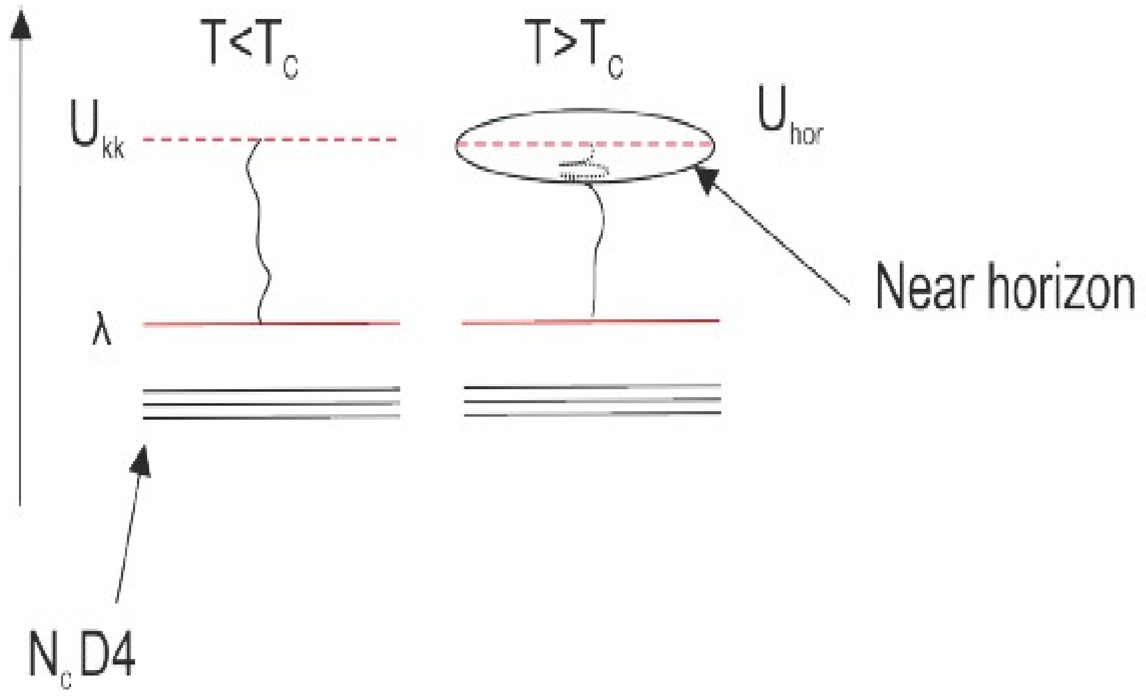
– Level spacing distribution

$$\begin{cases} P_{deloc}(s) = A\, s\, e^{-Bs^2} & \text{below mobility edge, } \lambda_m \\ P_{loc}(s) = e^{-s} & \text{above mobility edge, } \lambda_m \end{cases}$$

Mobility edge and black hole horizon

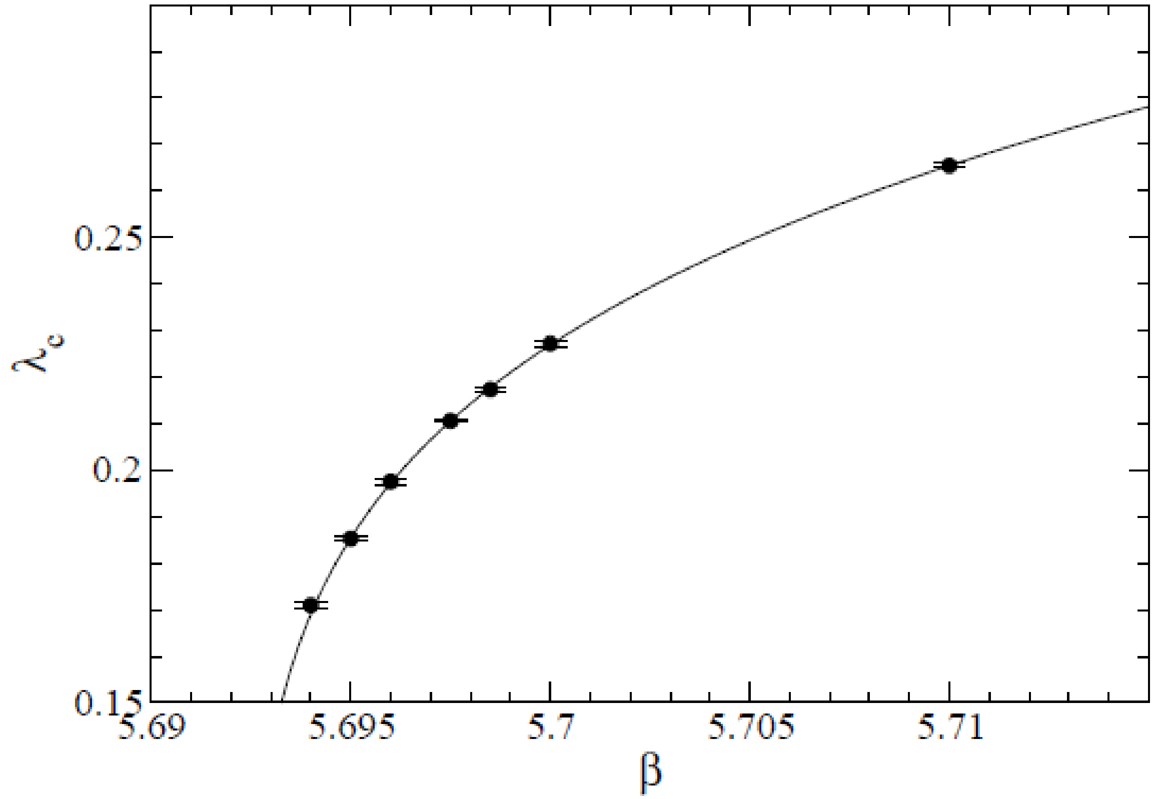


Spectral correlator = space-time correlator



At the deconfinement phase transition the black hole emerges .
Remind that the confinement phenomena=horizon decay.

The idenfification of the mobility edge in the Dirac operator spectrum and the near-horizon region passes the simplest checks(Litvinov- AG, 19).



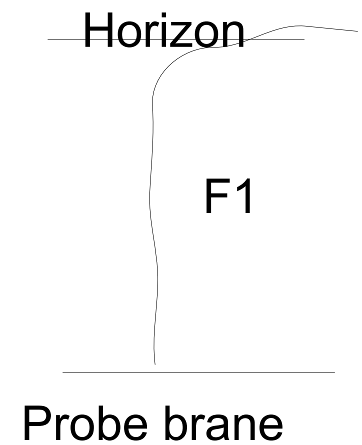


FIG. 3. The mobility edge λ_c as a function of the inverse gauge coupling for temporal lattice size $N_t = 4$. The solid

N_t	β_c^{deconf}	β_c^{loc}
4	5.69254(24)	5.69245(17)
6	5.8941(5)	5.8935(16)

TABLE II. The critical gauge couplings where the localization and the deconfining transition happen for $N_t = 4$ and $N_t = 6$ temporal lattice sizes.

The temperature of deconfinement phase transition equals with high accuracy to the temperature when mobility edge appears(Kovacs et al,'17)

Mobility edge and Black hole horizon

- Localization= nearby energy levels do not know about each other

- Energy = coordinate on the string worldsheet.

– Hence non-interacting nearby energies correspond to the non-interaction nearby string bits. How it could be?

Conjecture ,String tension vanishes at the BH horizon.
Hence the string bits at the near horizon region do not interact.
Mobility edge= near horizon region

- Very recently similar emergence of the mobility edge in SYK perturbed by massive term (Garsia-Garsia et al, 18)

Conclusion

- In some cases the very explicit representation of the «path integral» over the Hilbert space can be formulated. More work is needed
- The graph representation of the Hilbert space of system provides some new generic features
- Holographic approach provides the bulk gravity interpretation of some spectral effects like the emergence of mobility edge.

The collaborators with whom I shared a pleasure to work on these issues

Nikita Nekrasov, Vladimir Fock, Vladimir Roubtsov, Sergei Nechaev, Kseniya Bulycheva, Alexei Milekhin, Nikita Sopenko, Olga Valba, Vladik Avetisov, Minas Hovhanessyan, Mikhail Tamm, Mikhail Litvinov,