Quantum KdV hierarchy in 2d CFTs

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What is Conformal Field Theory?

If the conformal blocks g are known, (4 18) yields a system of equations, deterrmn- σ the dimensions σ conformal blocks (4.16) for general values of \sim importance for the conformal quantum field theory. The first few terms of the power expansion for the case $f(x)$ \sim A is considered for the same of slmphcly. Although the conformal blocks are \sim not yet known for the general case, there are the special values of the dimensions A \mathcal{A} associated with the degenerate representation of the Virasoro algebra, see section algebra, see section \mathcal{A} such that the corresponding conformal blocks can be computed exactly, being the solutions of certain linear differential equations The s~mplest example is the hypergeometnc function. In these special cases the bootstrap eq (4 18) can be solved **5. Degenerate conformal families** $T_{\rm eff}$ and $T_{\rm eff}$ is interest the value of the dimension algebra is interest the dimension of the dimensional $T_{\rm eff}$ A takes some special values [6, 7]. For these values the vector space V a proves to contain a special vector $\mathcal{L}_{\mathcal{A}}$ is the equation of equations of equations of equations of equations of equations of equations of the equations of *L,,]X)* = 0, If n > 0, \mathbf{L} , and \mathbf{L} is \mathbf{L} , \mathbf{L} , \mathbf{L} characteristic of the particle of the particle of the post-

 $\mathbf{y} = \mathbf{y} - \mathbf{y} - \mathbf{y}$

on the venty that the vector

- relativistic theory of massless quantum fields
- algebra of operators

- continuous limit of lattice systems

- theory of quantum gravity

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Motivation to study thermalization in CFTs

- Thermalization in 1D systems after a quantum quench
- CFT thermal physics \Leftrightarrow black hole physics
- Eigenstate Thermalization Hypothesis in CFT
	- new property of CFT correlation functions
	- fundamental difference between small and large c theories?

Conformal symmetry & KdV hierarchy

- Conformal symmetry controls many aspects of 2d CFT dynamics
- Holographically, many aspects of classical gravity emerge from conformal symmetry
- Solving CFTs in 2d requires understanding KdV hierarchy
	- $-$ ETH $=$ property of "correct" gKdV basis of CFT descendants

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Modular invariance may mix symmetry protected and unprotected data

Origin of qKdV charges

• Verma module (free action of L_n)

$$
L_{-m_1} \dots L_{-m_k} |\Delta\rangle, \qquad m_1 \ge m_2 \ge \dots m_k \ge 0
$$

• quantum KdV charges Q_{2k-1}

$$
H \equiv Q_1 = \int_0^{\ell} du \, T(u), \quad Q_3 = \int_0^{\ell} du \, T(u)^2, \quad \dots
$$

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 \bullet Q_{2k-1} commute and act within the Verma module

Classical vs quantum

co-adjoint orbit of Virasoro algebra

$$
-\psi'' + u \psi = 0 \implies -\tilde{\psi}'' + \tilde{u} \tilde{\psi} = 0
$$

$$
\tilde{u}(\theta) = \left(\frac{d\theta}{d\varphi}\right)^{-2} \left(u + \frac{1}{2}\{\theta, \varphi\}\right)
$$

$$
\frac{c}{6\pi}\left\{u(\varphi),u(\varphi')\right\} = 4u(\phi)\delta'(\varphi-\varphi') + 2u'(\varphi)\delta(\varphi-\varphi') - \delta'''(\varphi-\varphi')
$$

stress-tensor in 2d conformal field theory

$$
\tilde{T}(w) = \left(\frac{dw}{dz}\right)^{-2} \left(T + \frac{c}{12}\{w, z\}\right)
$$

$$
[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m}
$$

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What are KdV charges?

• classical KdV hierarchy

$$
Q_1 = \int d\varphi \, u(\varphi), \quad Q_3 = \int d\varphi \, u(\varphi)^2, \quad Q_5 = \int d\varphi \, u(\varphi)^3 + 2u'(\varphi)^2,
$$

charges generate Hamiltonian dynamics

$$
\dot{u} = \{Q_1, u\} = \partial u, \quad \dot{u} = \{Q_3, u\} = 6u \partial u - \partial^3 u
$$

quantum KdV hierarchy

$$
Q_1 = \int d\varphi T
$$
, $Q_3 = \int d\varphi T^2$, $Q_5 = \int d\varphi T^3 + \frac{c+2}{12}T'^2$,

$$
\dot{T} = [Q_1, T] = \partial_{\varphi} T, \quad \dot{T} = [Q_3, T] = -3\partial(TT) - \frac{c-1}{6}\partial^3 T
$$

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Quantum KdV charges of 2d CFTs

• infinite tower of charges in involution

$$
H \equiv Q_1 = \int_0^{\ell} d\varphi \, T(\varphi), \quad Q_3 = \int_0^{\ell} d\varphi \, T(\varphi)^2, \quad \dots
$$

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Quantum KdV charges in 2d CFTs

- infinite tower of charges Q_{2k-1} in involution
- spectrum of $Q_1 = L_0 c/24$ is highly degenerate, spectrum of all other charges is not
- **quantum KdV hierarchy defines preferred eigenbasis on** the Verma module

$$
|\Psi\rangle = L_{-m_1} \dots L_{-m_k} |\Delta\rangle + \dots, \qquad Q_{2k-1} |\Psi\rangle = \lambda_{2k-1} |\Psi\rangle
$$

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and hence on the space of descendants of 2d CFT

• a priory quantum numbers to index $|\Psi\rangle$ are not known

Spectrum of Q_3

- spectrum of Q_3 can be parametrized by Young diagrams
	- $\ell^3 Q_3 | m_i, \Delta \rangle = \lambda_3 | m_i, \Delta \rangle, \qquad | m_i, \Delta \rangle = L_{-m_1} \dots L_{-m_r} | \Delta \rangle + \dots$

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$$
\lambda_3 = P_2(\Delta + n) + 4\Delta \sum_i m_i + \frac{c}{6} \sum_i (m_i^3 - m_i) + \dots
$$

• free boson representation,

$$
\sum_{i} (m_i)^p = \sum_{k} n_k k^p
$$

• free shift of $\Delta' = \Delta - c/24$ (inspired by holography)

Spectrum of Q_3

 \bullet spectrum of Q_3 in terms of "boundary gravitons"

$$
\ell^3 Q_3 | n_k, \Delta \rangle = \lambda | n_k, \Delta \rangle
$$

$$
\lambda_3 = \Delta'^2 + \Delta' \left(\sum_k n_k 6k - \frac{1}{6} \right) + c \left(\sum_k n_k \frac{k^3}{6} + \frac{1}{1440} \right) +
$$

$$
\sum_k n_k \left(\frac{(5k^3 - 9k^2)}{6} + 36(2k - 1) \frac{\Delta'}{c} \right) -
$$

$$
\sum_k n_k^2 \left(3\frac{k^2}{2} + 36\frac{\Delta'}{c} \right) + 2 \left(\sum_k n_k k \right)^2 + O(1/c)
$$

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qKdV hierarchy at large c

- Spectrum of Q_{2k-1} at first two orders in $1/c$ was calculated in terms of "boundary gravitons" guesswork supported by quasi-classical quantization
- Spectrum of Q_3, Q_5 is known up to $1/c^2$

Generalized partition function in cylinder limit is known including leading $1/c$ corrections

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Generalized Eigenstate Thermalization in 2d CFTs

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Eigenstate Thermalization Hypothesis

Eigenstate Thermalization Hypothesis - ansatz for the matrix elements in the energy eigenbasis

$$
\langle E_i|\mathcal{O}|E_j\rangle = \delta_{ij}f_{\mathcal{O}}(E_i) + e^{-S/2}g_{\mathcal{O}}(E_i, E_j)R_{ij}
$$

ETH predicts emergence of thermal equilibrium under the assumption that the latter is equal to the diagonal ensemble

$$
\langle \psi | \mathcal{O}(t) | \psi \rangle = \sum_{i} |C_i|^2 \langle E_i | \mathcal{O} | E_i \rangle + \sum_{i \neq j} C_i^* C_j \langle E_i | \mathcal{O} | E_j \rangle e^{-i(E_i - E_j)t}
$$

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Generalized Eigenstate Thermalization

what is the minimal set of physical quantities characterizing an equilibrium state?

Eigenstate Thermalization - energy is the only thermodynamically-relevant quantity

$$
\langle E_i|\mathcal{O}|E_i\rangle = f_{\mathcal{O}}(E_i)
$$

Generalized Eigenstate Thermalization - infinite tower of thermodynamically-relevant quantities

$$
\langle E_i|\mathcal{O}|E_i\rangle = f_{\mathcal{O}}(E_i,\mathcal{I}_i^a)
$$

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ETH in CFT

- quantization of CFT on a cylinder $\mathbb{S}^1\times R$
- \bullet thermodynamic limit (while c is fixed)

$$
E/\ell
$$
 – fixed, $\ell \to \infty$

- **•** spectrum of H is highly degenerate, hence diagonal ensemble may not have full predictive power
- ETH in 2d CFT statement about special KdV basis

$$
\langle E|\mathcal{O}|E\rangle = f_{\mathcal{O}}(Q_{2k-1})
$$

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Generalized Eigenstate Thermalization in 2d CFTs

working in the "quasi-classical" regime

$$
\frac{q_{2k-1}}{q_1^k} - 1 = O(1/c), \quad Q_{2k-1} = \frac{q_{2k-1}}{\ell} - \text{fixed}, \quad \ell \to \infty
$$

expectation values of quasi-primaries from the vacuum block in KdV eigenstates $|\Psi\rangle = |n_k, \Delta\rangle$ depend on q_{2k-1} for example, at level 8

$$
\mathcal{O} = (\partial^2 T \partial^2 T) - \frac{10}{9} (\partial T \partial^3 T) + \frac{10}{63} (T \partial^4 T) - \frac{13}{2268} \partial^6 T
$$

$$
\langle \Psi | \mathcal{O} | \Psi \rangle = \frac{63}{143} \frac{180}{c^2} \left(q_7 - q_1^4 - 4q_1(q_5 - q_1^3) + 6q_1(q_3 - q_1^2) \right) + O(c^0)
$$

Generalized Eigenstate Thermalization in 2d CFTs at second order in $1/c$

we introduce $\delta q_{2k-1} \equiv q_1^{-k} q_{2k-1} - 1 \sim O(1/c)$, $\delta = \Delta/c$

• level 6 operator

$$
\mathcal{O}_6^{(2)} = (\partial T \partial T) - \frac{4}{5}(T \partial^2 T) + \frac{23}{210} \partial^4 T.
$$

eigenstate expectation value

$$
\frac{5}{18} \langle \Psi | \mathcal{O}_6^{(2)} | \Psi \rangle = q_1^3 \left[\frac{6}{c} (\delta q_5 - 3 \delta q_3) - \frac{1}{c^2} (84 \delta q_5 + 180 \delta q_3) \right] - \sum_k n_k^2 \left(\frac{1}{2} k^4 + 72 k^2 \delta + 864 \delta^2 \right) + O(1/c)
$$

• at $1/c$ order strong ETH even without taking $\ell \to \infty$ limit

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Weak ETH

typicality of thermal eigenstates in thermodynamic limit

order one contribution in the thermodynamic limit

$$
\frac{1}{\ell^{2p}}\sum_{k}n_{k}k^{2p-1}\to \int d\,\mathbf{k}\,\mathbf{k}^{2p-1}\,n(\mathbf{k}),\qquad \mathbf{k}=\frac{k}{\ell}
$$

suppressed contribution in the thermodynamic limit

$$
\frac{1}{\ell^{2p}} \sum_{k} n_k^2 k^{2p-2} \to \frac{1}{\ell} \int d \, \mathbf{k} \, \mathbf{k}^{2p-2} \, n^2(\mathbf{k})
$$

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Matching Eigenstates to GGE

• regular systems with non-degenerate spectrum satisfying generalized ETH are expected to thermalize to Generalized Gibbs Ensemble

$$
\rho = \exp\left\{-\sum_{k=1}^{\infty} \mu_{2k-1} Q_{2k-1}\right\}/Z
$$

• in 2d CFTs generalized ETH guarantees equivalence of KdV-eigenstate and GGE ensembles, provided there are μ_{2k-1} for the given set of q_{2k-1} , but can not unambiguously predict emergence of GGE at late times

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Matching Eigenstates to GGE

• matching GGE to KdV eigenstates

$$
\frac{q_{2k-1}}{q_1^k} - 1 = \frac{24k}{c} \int_0^\infty \frac{d\kappa \kappa \left[(2k-1) \cdot 2F_1(1, 1-k, 3/2, -\kappa^2) - 1 \right]}{e^{2\pi \kappa \gamma} - 1},
$$

$$
\gamma = \sum_{k=1}^\infty t_{2k-1} k(2k-1) \sigma^{k-1/2} \cdot 2F_1(1, 1-k, 3/2, -\kappa^2) \ge 1
$$

- **•** primary states singular GGE ensemble
- when $q_3/q_1^2-1\geq \frac{22}{5c}$ $\frac{22}{5c}$ some of the chemical potentials have to be negative
- at leading order in c , dual black hole is the conventional BTZ solution controlled by q_1, \bar{q}_1

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Main results

- qKdV hierarchy defines preferred basis of CFT eigenstates
	- − control over eigenvalues/eigenvectors of qKdV charges at first two orders in $1/c$
	- − GGE/generalized partition function
	- − modular invariance?. . .
- Generalized Eigenstate Thermalization in 2d CFTs
	- − analytic result establishing (weak) ETH
	- − completeness of qKdV charges to describe KdV eigenstate

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− late time equilibrium? . . .