Quantum KdV hierarchy in 2d CFTs

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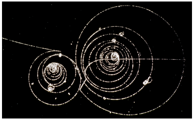
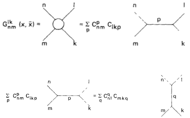
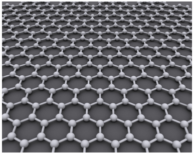
What is Conformal Field Theory?



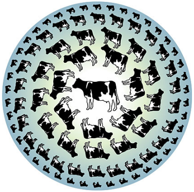
- relativistic theory of massless quantum fields



- algebra of operators



- continuous limit of lattice systems



- theory of quantum gravity

Motivation to study thermalization in CFTs

- Thermalization in 1D systems after a quantum quench
- CFT thermal physics $\Leftrightarrow$ black hole physics
- Eigenstate Thermalization Hypothesis in CFT
  - new property of CFT correlation functions
  - fundamental difference between small and large c theories?

Conformal symmetry & KdV hierarchy

- Conformal symmetry controls many aspects of 2d CFT dynamics
- Holographically, many aspects of classical gravity emerge from conformal symmetry
- Solving CFTs in 2d requires understanding KdV hierarchy
  - ETH = property of "correct" qKdV basis of CFT descendants
- Modular invariance may mix symmetry protected and unprotected data

Origin of qKdV charges

- Verma module (free action of L_n)

$$L_{-m_1}\ldots L_{-m_k}|\Delta\rangle, \qquad\qquad m_1 \geq m_2 \geq \ldots m_k \geq 0$$

- quantum KdV charges Q_{2k-1}

$$H \equiv Q_1 = \int_0^\ell du\, T(u), \quad Q_3 = \int_0^\ell du\, T(u)^2, \quad \ldots$$

- Q_{2k-1} commute and act within the Verma module

Classical vs quantum

- co-adjoint orbit of Virasoro algebra

$$-\psi'' + u\,\psi = 0 \quad \Rightarrow \quad -\tilde{\psi}'' + \tilde{u}\,\tilde{\psi} = 0$$

$$\tilde{u}(\theta) = \left(\frac{d\theta}{d\varphi}\right)^{-2}\left(u + \frac{1}{2}\{\theta, \varphi\}\right)$$

$$\frac{c}{6\pi}\{u(\varphi), u(\varphi')\} = 4u(\phi)\delta'(\varphi-\varphi') + 2u'(\varphi)\delta(\varphi-\varphi') - \delta'''(\varphi-\varphi')$$

- stress-tensor in 2d conformal field theory

$$\tilde{T}(w) = \left(\frac{dw}{dz}\right)^{-2}\left(T + \frac{c}{12}\{w, z\}\right)$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m}$$

What are KdV charges?

- classical KdV hierarchy

$$Q_1 = \int d\varphi\, u(\varphi), \quad Q_3 = \int d\varphi\, u(\varphi)^2, \quad Q_5 = \int d\varphi\, u(\varphi)^3 + 2u'(\varphi)^2,$$

charges generate Hamiltonian dynamics

$$\dot{u} = \{Q_1, u\} = \partial u, \quad \dot{u} = \{Q_3, u\} = 6u\,\partial u - \partial^3 u$$

- quantum KdV hierarchy

$$Q_1 = \int d\varphi\, T, \quad Q_3 = \int d\varphi\, T^2, \quad Q_5 = \int d\varphi\, T^3 + \frac{c+2}{12} T'^2,$$

$$\dot{T} = [Q_1, T] = \partial_\varphi T, \quad \dot{T} = [Q_3, T] = -3\partial(TT) - \frac{c-1}{6}\partial^3 T$$

Quantum KdV charges of 2d CFTs

- infinite tower of charges in involution

$$H \equiv Q_1 = \int_0^\ell d\varphi\, T(\varphi), \quad Q_3 = \int_0^\ell d\varphi\, T(\varphi)^2, \quad \ldots$$

$$\begin{aligned}
\ell^1 Q_1 &= L_0 - \frac{c}{24} \\
\ell^3 Q_3 &= L_0^2 - \frac{c+2}{12} L_0 + \frac{c(5c+22)}{2880} + 2\sum_{n=1}^{\infty} L_{-n} L_n \\
\ell^5 Q_5 &= L_0^3 - \frac{c+4}{8} L_0^2 + \frac{(c+2)(3c+20)}{576} L_0 - \frac{c(3c+14)(7c+68)}{290304} + \\
&\sum : L_{n_1} L_{n_2} L_{n_3} : + \sum_{n=1}^{\infty} \left(\frac{(c+11)}{6} n^2 - \frac{4+c}{4} \right) L_{-n} L_n + \frac{3}{2} \sum_{r=1}^{+\infty} L_{1-2r} L_{2r-1}
\end{aligned}$$

Quantum KdV charges in 2d CFTs

- infinite tower of charges Q_{2k-1} in involution
- spectrum of $Q_1 = L_0 - c/24$ is highly degenerate, spectrum of all other charges is not
- quantum KdV hierarchy defines preferred eigenbasis on the Verma module

$$|\Psi\rangle = L_{-m_1} \dots L_{-m_k} |\Delta\rangle + \dots , \qquad Q_{2k-1}|\Psi\rangle = \lambda_{2k-1}|\Psi\rangle$$

  and hence on the space of descendants of 2d CFT

- a priory quantum numbers to index $|\Psi\rangle$ are not known

Spectrum of Q_3

- spectrum of Q_3 can be parametrized by Young diagrams

$$\ell^3 Q_3 |m_i, \Delta\rangle = \lambda_3 |m_i, \Delta\rangle, \qquad |m_i, \Delta\rangle = L_{-m_1} \dots L_{-m_r} |\Delta\rangle + \dots$$

$$\lambda_3 = P_2(\Delta + n) + 4\Delta \sum_i m_i + \frac{c}{6} \sum_i \left(m_i^3 - m_i\right) + \dots$$

- free boson representation,

$$\sum_i (m_i)^p = \sum_k n_k \, k^p$$

- free shift of $\Delta' = \Delta - c/24$ (inspired by holography)

Spectrum of Q_3

- spectrum of Q_3 in terms of "boundary gravitons"

$$\ell^3 Q_3 |n_k, \Delta\rangle = \lambda |n_k, \Delta\rangle$$

$$\lambda_3 = \Delta'^2 + \Delta' \left(\sum_k n_k \, 6k - \frac{1}{6} \right) + c \left(\sum_k n_k \frac{k^3}{6} + \frac{1}{1440} \right) +$$
$$\sum_k n_k \left(\frac{(5k^3 - 9k^2)}{6} + 36(2k-1)\frac{\Delta'}{c} \right) -$$
$$\sum_k n_k^2 \left(3\frac{k^2}{2} + 36\frac{\Delta'}{c} \right) + 2 \left(\sum_k n_k \, k \right)^2 + O(1/c)$$

qKdV hierarchy at large c

- Spectrum of Q_{2k-1} at first two orders in $1/c$ was calculated in terms of "boundary gravitons"

  guesswork supported by quasi-classical quantization

- Spectrum of Q_3, Q_5 is known up to $1/c^2$

- Generalized partition function in cylinder limit is known including leading $1/c$ corrections

Generalized Eigenstate Thermalization in 2d CFTs

Eigenstate Thermalization Hypothesis

- Eigenstate Thermalization Hypothesis - ansatz for the matrix elements in the energy eigenbasis

$$\langle E_i|\mathcal{O}|E_j\rangle = \delta_{ij} f_{\mathcal{O}}(E_i) + e^{-S/2} g_{\mathcal{O}}(E_i, E_j) R_{ij}$$

- ETH predicts emergence of thermal equilibrium under the assumption that the latter is equal to the diagonal ensemble

$$\langle\psi|\mathcal{O}(t)|\psi\rangle = \sum_i |C_i|^2 \langle E_i|\mathcal{O}|E_i\rangle + \sum_{i\neq j} C_i^* C_j \langle E_i|\mathcal{O}|E_j\rangle e^{-i(E_i - E_j)t}$$

Generalized Eigenstate Thermalization

what is the minimal set of physical quantities characterizing an equilibrium state?

- Eigenstate Thermalization - energy is the only thermodynamically-relevant quantity

$$\langle E_i|\mathcal{O}|E_i\rangle = f_{\mathcal{O}}(E_i)$$

- Generalized Eigenstate Thermalization - infinite tower of thermodynamically-relevant quantities

$$\langle E_i|\mathcal{O}|E_i\rangle = f_{\mathcal{O}}(E_i, \mathcal{I}_i^a)$$

ETH in CFT

- quantization of CFT on a cylinder $\mathbb{S}^1 \times R$
- thermodynamic limit (while c is fixed)

$$E/\ell - \text{fixed}, \quad \ell \to \infty$$

- spectrum of H is highly degenerate, hence diagonal ensemble may not have full predictive power
- ETH in 2d CFT – statement about special KdV basis

$$\langle E|\mathcal{O}|E\rangle = f_{\mathcal{O}}(Q_{2k-1})$$

Generalized Eigenstate Thermalization in 2d CFTs

- working in the "quasi-classical" regime

$$\frac{q_{2k-1}}{q_1^k} - 1 = O(1/c), \quad Q_{2k-1} = \frac{q_{2k-1}}{\ell} - \text{fixed}, \quad \ell \to \infty$$

- expectation values of quasi-primaries from the vacuum block in KdV eigenstates $|\Psi\rangle = |n_k, \Delta\rangle$ depend on q_{2k-1}

  for example, at level 8

$$\mathcal{O} = (\partial^2 T \partial^2 T) - \frac{10}{9}(\partial T \partial^3 T) + \frac{10}{63}(T \partial^4 T) - \frac{13}{2268}\partial^6 T$$

$$\langle \Psi | \mathcal{O} | \Psi \rangle = \frac{63}{143}\frac{180}{c^2}\left(q_7 - q_1^4 - 4q_1(q_5 - q_1^3) + 6q_1(q_3 - q_1^2)\right) + O(c^0)$$

Generalized Eigenstate Thermalization in 2d CFTs at second order in $1/c$

- we introduce $\delta \mathrm{q}_{2k-1} \equiv q_1^{-k} q_{2k-1} - 1 \sim O(1/c)$, $\quad \delta = \Delta/c$
- level 6 operator

$$\mathcal{O}_6^{(2)} \quad = \quad (\partial T \partial T) - \frac{4}{5}(T\partial^2 T) + \frac{23}{210}\partial^4 T.$$

eigenstate expectation value

$$\frac{5}{18}\langle\Psi|\mathcal{O}_6^{(2)}|\Psi\rangle \quad = \quad q_1^3\left[\frac{6}{c}\left(\delta q_5 - 3\delta q_3\right) - \frac{1}{c^2}\left(84\delta q_5 + 180\delta q_3\right)\right] - \sum_k n_k^2\left(\frac{1}{2}k^4 + 72k^2\delta + 864\delta^2\right) + O(1/c)$$

- at $1/c$ order strong ETH even without taking $\ell \to \infty$ limit

Weak ETH

- typicality of thermal eigenstates in thermodynamic limit

order one contribution in the thermodynamic limit

$$\frac{1}{\ell^{2p}} \sum_k n_k \, k^{2p-1} \to \int d\,\mathrm{k}\,\mathrm{k}^{2p-1}\, n(\mathrm{k}), \qquad \mathrm{k} = \frac{k}{\ell}$$

suppressed contribution in the thermodynamic limit

$$\frac{1}{\ell^{2p}} \sum_k n_k^2 \, k^{2p-2} \to \frac{1}{\ell} \int d\,\mathrm{k}\,\mathrm{k}^{2p-2}\, n^2(\mathrm{k})$$

Matching Eigenstates to GGE

- regular systems with non-degenerate spectrum satisfying generalized ETH are expected to thermalize to Generalized Gibbs Ensemble

$$\rho = \exp\left\{-\sum_{k=1}^{\infty} \mu_{2k-1} Q_{2k-1}\right\}/Z$$

- in 2d CFTs generalized ETH guarantees equivalence of KdV-eigenstate and GGE ensembles, provided there are μ_{2k-1} for the given set of q_{2k-1}, but can not unambiguously predict emergence of GGE at late times

Matching Eigenstates to GGE

- matching GGE to KdV eigenstates

$$\frac{q_{2k-1}}{q_1^k} - 1 = \frac{24k}{c} \int_0^\infty \frac{d\kappa \kappa \left[(2k-1) {}_2F_1(1, 1-k, 3/2, -\kappa^2) - 1\right]}{e^{2\pi\kappa\gamma} - 1},$$

$$\gamma = \sum_{k=1}^{\infty} t_{2k-1} k(2k-1) \sigma^{k-1/2} {}_2F_1(1, 1-k, 3/2, -\kappa^2) \geq 1$$

- primary states - singular GGE ensemble
- when $q_3/q_1^2 - 1 \geq \frac{22}{5c}$ some of the chemical potentials have to be negative
- at leading order in c, dual black hole is the conventional BTZ solution controlled by $q_1, \bar{q}_1$

Main results

- qKdV hierarchy defines preferred basis of CFT eigenstates
  - control over eigenvalues/eigenvectors of qKdV charges at first two orders in $1/c$
  - GGE/generalized partition function
  - modular invariance?...
- Generalized Eigenstate Thermalization in 2d CFTs
  - analytic result establishing (weak) ETH
  - completeness of qKdV charges to describe KdV eigenstate
  - late time equilibrium? ...