# Special Kähler geometry, Localization and Mirror symmetry

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- Superstring theory is considered as a possible approach for unifying the Standard model and Quantum gravity.
- If we wish to obtain 4d theory with Space-Time supersymmetry (which is needed for the phemenological reasons) we have to compactify 6 of 10 dimensions of Superstring theory on Calabi-Yau manifolds $X$.
- In result we obtain the 4-d low-energy effective theory Supergravity theory. The Lagrangian of the effective theory is defined by so called Special Kähler geometry which appears on the moduli space of CY manifold $X$.
- Indeed the moduli space of CY manifold $X$ is a product of two factors: Moduli space $M_k(X)$ of the Kähler structure defomations and Moduli space of the complex structure deformations $M_c(X)$.
- Therefore for finding the Effective low energy theory we have to compute the Special Kähler geometry on the both Moduli spaces of CY manifolds.

- Recently we suggest an efficient approach for computing Special geometry on the complex moduli space. Our approach is based on the isomorphism between the cohomologies on CY and Chiral ring defined by the polynomial $W_X$ whose zero locus is the CY hypersurface $X$ in the weighted projective space.

- On the other hand it was suggested recently a conjecture for the explicit expressions the Kähler potential the Kähler moduli spaces [Jockers et al]. This conjecture (JKLMR conjecture) is the equality

$$e^{-K_k(Y)} = Z_{S^2}(Y)$$

where $K_k(Y)$ is the Kähler potential of the Special geometry on the Kähler moduli space of CY as the hypersurface $Y$ in a toric variety.

- Here $Z_{S^2}(Y)$ is the partition function of the Witten gauged linear sigma model (GLSM) on $S^2$ which was exactly computed by Supersymmetric localization [Benini et al, Doroud et al ].
CY manifold $Y$ is the manifold of the supersymmetric vacua of the GLSM.

- Since we want to know Special Kähler geometry of the both Moduli spaces for each given family Calabi-Yau manifolds, we need to find the connection between these two computations.

- To reach this goul we use the Mirror symmetry and the duality of the Batyrev's reflexive polytopes.

- The Mirror symmetry , if $Y$ is the mirror manifold to $X$ together with JKLMR conjecture predict the relation

$$e^{-K_c(X)} = Z_{S^2}(Y) = e^{-K_k(Y)}.$$

So the problem is to find $Y$ that is the mirror counterpart for $X$.
We do this with help the Batyrev construction.

- The main idea of Batyrev is to interpret the monomials of the homogenious polynom $W_X$ which defines CY hypersurface $X$ as the lattice points of the Polytope defining the enveloping weighted projective space.

- These lattice points are used for constructing Fan which defines $Y$, which is mirror to $X$, as a hypersurface in the toric manifold built by the Fan.

- Knowing the Fan we also find the corresponding GLSM and the values of the electric charges of the chiral fields as the coefficients in the linear relations between the vectors of the Fan.

- In result we obtain the explicit expressions for the Special geometry of the both moduli spaces.

# Covariantly constant spinor and $3,0$ form $\Omega$

The requirement for the compact 6d manifold $X$ to be CY manifold arises as follows.

Since after the compactification the background has to be invariant with respect to d=4 Super-Poincaré algebra, then the supersymmetry variations of the gravitini have to vanish

$$\langle \delta_\epsilon \psi_{\mu,\alpha} \rangle = \langle \nabla_\mu \epsilon_\alpha \rangle = 0.$$

It means that $X$ admits the covariantly constant spinor field.
The existence of the covariantly constant spinor is one of the few equivalent definitions of Calabi-Yau manifold.

Another important propperty of three-dimensional Calabi-Yau manifold is existence on $X$ the holomorphic nonvanishing $3,0$ form.
We will denote it $\Omega$.

The Kähler potential of the Special geometry on the complex moduli space of $X$ can be expressed in terms of the form $\Omega$.

The Kähler potential of this geometry is given by the logarithm of the holomorphic volume of Calabi-Yau manifold $X_\phi$
($\phi$ denote the parameters of the complex structure of CY):

$$G(\phi)_{a\bar{b}} = \partial_a \overline{\partial_b}\, K(\phi, \bar{\phi})$$
$$e^{-K(\phi)} = \int_{X_\phi} \Omega \wedge \overline{\Omega}.$$

This can be rewritten in terms of periods of $\Omega$ as:

$$\omega_\mu(\phi) = \int_{q_\mu} \Omega, \quad q_\mu \in H_3(X, \mathbb{R}).$$
$$e^{-K} = \omega_\mu(\phi) C_{\mu\nu}\, \overline{\omega_\nu(\phi)},$$

where $C_{\mu\nu} = [q_\mu] \cap [q_\nu]$ is an intersection matrix of 3-cycles.

# Example. Hypersurfaces in weighted projective spaces

Consider the 4-d weighted projective space

$$\mathbb{P}^4_{(k_1:\ldots:k_5)} := \mathbb{C}^5\backslash\{0\} = /\mathbb{C}^* = \{(x_1 : \ldots : x_5) \mid x_i \sim \lambda^{k_i} x_i,\ \bar{x} \neq 0\}.$$

When $k_i = 1$ we have an ordinary projective space. Each variable has integral degree (or U(1) charge) $k_i$.

$W(x)$ is weighted homogeneous $\iff W(\lambda^{k_i} x_i) = \lambda^d W(x_i) \implies$ its zero locus $\mathcal{X} = \{W = 0\} \subset \mathbb{P}^4_{\vec{k}}$ is well-defined.

$W(x)$ is non-degenerate if $\mathrm{d}W(x) = 0, W = 0$ takes place only when $x = 0$. In this case $X$ is not singularic manifold.

$W(x)$ defines a Calabi-Yau manifold $\iff \sum_{i=1}^5 k_i = d$. We consider the family of of Calabi-Yau manifolds defined as the zero locus of the polynomals

$$W(x, \phi) = W_0(x) + \sum_{s=1}^{h} \phi_s e_s(x).$$

such that manifolds with different $\phi$ have different complex structures.

The holomorphic volume form is explicitly

$$\Omega = \frac{x_5 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3}{\partial W(x, \phi)/\partial x_4} = \oint_{x_5=0} \oint_{W=0} \frac{\mathrm{d}^5 x}{W(x, \phi)}.$$

The periods of such a form are

$$\omega_i(\phi) := \int_{q_i} \Omega = \int_{Q_i} \frac{\mathrm{d}^5 x}{W(x,\phi)}.$$

A good example is the quintic threefold in the projective space $\mathbb{P}^4$:

$$X = \{(x_1 : \cdots : x_5) \in \mathbb{P}^4 \mid W(x,\phi) = 0\},$$

$$W(x,\phi) = W_0(x) + \sum_{t=0}^{100} \phi_t e_t(x), \; W_0(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$$

and $e_t(x)$ are the degree 5 monomials such that each variable has the power that is a non-negative integer less then four.
The chiral ring defined as

$$\mathcal{R}^Q = \frac{\mathbb{C}[x_1,\dots,x_5]^Q}{(\partial_1 W,\dots,\partial_5 W)}.$$

invariant under the discrete gauge symmetry $Q \, : \, X_i \to e^{2\pi i k_i/d} X_i$ and decomposes as

$$\mathcal{R}^Q = \langle 1 \rangle \oplus (\mathcal{R}^Q)^1 \oplus (\mathcal{R}^Q)^2 \oplus \langle \mathrm{Hess}\, W \rangle.$$

Let $e_a(x)$a are elements of a basis of the chiral ring.

Kähler potential for the Special geometry can be written in terms of the oscilatory integrals as

$$e^{-K} = C^{ij} \int_{Q^i_+} e^{-W(x,\phi)} \mathrm{d}^5 x \, \overline{\int_{Q^i_-} e^{W(x,\phi)} \mathrm{d}^5 x}.$$

It can be derived from the equality

$$\omega_\mu(\phi) = \int_{q_\mu} \Omega = \int_{Q_i} \frac{\mathrm{d}^5 x}{W(x,\phi)} = \int_{Q_i^+} e^{-W(x,\phi)} \mathrm{d}^5 x$$

.

The key point for computing the periods is the Stokes formula for oscillatory integrals which implies

$$\int e^{-W} D_{-}\alpha := \int e^{-W}(\mathrm{d}\alpha - \mathrm{d}W \wedge \alpha) = 0.$$

Therefore the oscillatory integrands $e_a(x)\,\mathrm{d}^5x$ form a cohomology group $H^5_{D_-}(\mathbb{C}^5)^Q$ which is dual to steepest descent contours $H_5(\mathbb{C}^5, \mathrm{Re}(W) \gg 0)^Q$.

Define a basis of cycles by duality formula

$$\langle \Gamma^a_+, e_b(x)\,\mathrm{d}^5x \rangle = \int_{\Gamma^a_+} e^{-W_0} e_b(x)\,\mathrm{d}^5x = \delta^a_b.$$

The cycles $\Gamma^a_+$ are not actual geometric cycles but complex linear combinations of such cycles.

Using the duality it is very easy to find that the intersection matrix of cycles $\Gamma^i_+ \cap \Gamma^j_- = (\eta^{-1})^{ij}$, where $\eta^{ij}$ is a residue pairing for the $\mathcal{R}^Q$

$$\eta^{ij} = Res\,\frac{e_i(x)\,e_j(x)\,\mathrm{d}^5x}{\partial_1 W_0 \cdots \partial_5 W_0}.$$

# Main working formula

We use the formula for the Kähler potential in the basis of cycles $\Gamma^i_+$:

$$e^{-K} = \eta^{ij} \int_{\Gamma^i_+} e^{-W(x,\phi)} \mathrm{d}^5 x \int_{\overline{\Gamma^i_-}} \overline{e^{W(x,\phi)} \mathrm{d}^5 x},$$

where the last conjugation is due to the fact that $\Gamma^i_\pm$ are linear combinations of cycles with complex coefficients.

We denote

$$\sigma_i(\phi) := \int_{\Gamma^i_+} e^{-W(x,\phi)} \mathrm{d}^5 x, \quad \overline{\Gamma^j_-} = \mathbf{M}^k_j \Gamma^k_-$$

for a matrix $\mathbf{M}^k_j$ which is called the real structure matrix. $\mathbf{M}\bar{\mathbf{M}} = 1$.

Our main formula becomes

$$e^{-K} = \sigma_i(\phi)\, \eta^{ik} \mathbf{M}^j_k\, \overline{\sigma_j(\phi)}.$$

Quintic CY manifold $X$ be given as a solution of the equation

$$W(x,\phi)=\sum_{i=1}^{5}x_i^5+\sum_{\mathbf{l}=\mathbf{1}}^{101}\phi_{\mathbf{l}}\prod_{i=1}^{5}x_i^{s_{li}}=0$$

$0\le s_i\le 3$, $\deg(\mathbf{s}):=\sum_{i=1}^{5}s_i=5$.
The complex structures Kähler potential in this case is

$$e^{-K(\phi)}=\sum_{\mu=0}^{203}(-1)^{\deg(\mu)/5}\prod_{i=1}^{5}\gamma\left(\frac{\mu_i+1}{5}\right)|\sigma_\mu(\phi)|^2,$$

$$\sigma_\mu(\phi)=\sum_{n_1,\dots,n_5\ge 0}\prod_{i=1}^{5}\frac{\Gamma(\frac{\mu_i+1}{5}+n_i)}{\Gamma(\frac{\mu_i+1}{5})}\sum_{m_1,\dots,m_{101}\in\Sigma_n}\prod_{l=1}^{101}\frac{\phi_l^{m_{\mathbf{l}}}}{m_l!},$$

$\mu=(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)$, $0\le\mu_i\le 3$, $\quad\deg(\mu)=\sum_{i=1}^{5}\mu_i=0,5,10,15.$

$$\gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)},\qquad \Sigma_n=\{m_{\mathbf{l}}\mid\sum_{l=1}^{101}m_{\mathbf{l}}s_{li}=5n_i+\mu_i\}$$

The Fermat hypersurfaces (around 100 threefolds) are given by

$$W(x,\phi)=\sum_{i=1}^{5}x_i^{d/k_i}+\sum_{l=1}^{h}\phi_l\prod_{i=1}^{5}x_i^{s_{li}}=0$$

$0\le s_{li}\le d/k_i-1$ and $\sum_{i=1}^{5}k_i s_{li}=\sum_{i=1}^{5}k_i:=d$.
The complex structures Kähler potential in this case is

$$e^{-K(\phi)}=\sum_{\mu=0}^{2h+1}(-1)^{\deg(\mu)/d}\prod\gamma\left(\frac{k_i(\mu_i+1)}{d}\right)|\sigma_\mu(\phi)|^2,$$

$$\sigma_\mu(\phi)=\sum_{n_1,\dots,n_5\ge 0}\prod_{i=1}^{5}\frac{\Gamma(\frac{k_i(\mu_i+1)}{d}+n_i)}{\Gamma(\frac{k_i(\mu_i+1)}{d})}\sum_{m_l\in\Sigma_n}\prod_l\frac{\phi_s^{m_l}}{m_l!},$$

$\mu=(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)$, $0\le\mu_i\le d/k_i-1$, $\quad\sum_{i=1}^{5}\mu_i=0,d,2d,3d$.

$$\Sigma_n=\{m_l\mid\sum_l m_l s_{li}k_i=dn_i+k_i\mu_i\}$$

The 2d N=(2,2) supersymmetric GLSM have superspace Lagrangians of the type

$$L = \int \mathrm{d}^4\theta \left( \sum_{a=1}^{N} \overline{\Phi_a} e^{Q_{al} V_l} \Phi_a - \sum_{l} \frac{1}{2e_l^2} \overline{\Sigma_l} \Sigma_l \right) + \\ + \frac{1}{2} \left( - \int \mathrm{d}^2\tilde{\theta} \sum_{l=1}^{k} t_l \Sigma_l + \int \mathrm{d}^2\theta W(\Phi) + \text{h.c.} \right),$$

where $V_l$ is 2d vector multiplets, $\Phi_a$ are 2d chiral multiplets which are charged with respect to the $U_l(1)$ gauge group with the charges $Q_{al}$. $W(\Phi)$ is the superpotential which is gauge invariant.
The parameters $t_l = r_l + i\theta_l$ are complexified Fayet-Iliopoulos terms. The theory has the potential energy for the scalar fields

$$U = \sum_{l=1}^{k} \frac{e_l^2}{2} \left( \sum_{a=1}^{N} Q_{al} |\phi_a|^2 - r_l \right)^2 + \sum_{a=1}^{k} \left| \frac{\partial W}{\partial \phi_a} \right|^2 .$$

Depending on $r_l$ the vacuum manifold can be either a nontrivial manifold or a point $\phi = 0$. In the first case the theory flows to a nonlinear sigma model in the infrared.

The nonlinear sigma model case the vacuum manifold is a Hamliltonian reduction

$$Y_r = \left\{ (\phi_1, \ldots, \phi_N) \in \mathbb{C}^N \;\middle|\; \sum_{a=1}^{N} Q_{al} |\phi_a|^2 = r_l, \; l = 1, \ldots, k, \; \frac{\partial W}{\partial \phi_a} = 0 \right\} / U(1)^k.$$

This manifold is isomorphic to a hypersurface $\mathrm{d}W = 0$ in a toric variety

$$\mathbb{C}^N // (\mathbb{C}^*)^k,$$

where the action of $(\mathbb{C}^*)^k$ is defined by the $N \times k$ charge matrix $Q_{al}$ as

$$\phi_a => \lambda^{Q_{al}} \phi_a$$

.

The classical way to describe a toric variety is a Fan. The vectors $v_a$, whose components are $\{v_{ai}\}_{a \leq N, i \leq 5}$, form the edges of the Fan.
The vectors $v_a$ satisfy the linear relations. The integral basis of these relations can be witten as $\sum_{a=1}^{N} Q_{al} v_a = 0$.
Then namely these integral numbers $Q_{al}$ are the weights which define the toric variety $\mathbb{C}^N // (\mathbb{C}^*)^k$.

In the recent years the partition function of GLSM was computed in a supersymmetric background on $S^2$ using the Supersymmetric localization (Benini et al, Doroud et al):

$$Z_{S^2} = \sum_{m_l} \int_{C_l} \left( \prod_{l \leq k} \frac{\mathrm{d}\sigma_l}{2\pi} \right) Z_{class}(\sigma, m) \prod_{a \leq N} Z_{\Phi_a}(\sigma, m),$$

where the classical action is

$$Z_{class} = \prod_l e^{-4\pi i r_l \sigma_l - i\theta_l m_l}$$

and the one-loop determinant of a chiral field $\Phi_a$ is

$$Z_{\Phi_a} = \frac{\Gamma(q_a/2 - i \sum_l (Q_{al}\sigma_l - m_l/2))}{\Gamma(1 - q_a/2 - i \sum_l (Q_{al}\sigma_l + m_l/2))}.$$

# Mirror version of JKLMR conjecture

Shortly after this Localization computation has been proposed by Jockers et al (JKLMR ) a conjecture that

$$Z_{S^2}(Y_r) = e^{-K_k(Y_r)},$$

where $K_k(Y_r)$ is the Kähler potential of the special geometry on the Kähler moduli space of the vacuum manifold $Y_r$.

The Mirror symmetry relates the special geometry on the moduli space of the Kähler structures $M_k(Y_r)$ and the special geometry on the moduli space of the complex structures $M_c(X_\phi)$ of two different families of Calabi-Yau manifolds $Y_r$ and $X\phi$ through a Mirror map $r = r(\phi)$.

We want to verify the Mirror version of the JKLMR conjecture by the direct computations in the cases where we are able to compute special geometry using our method.

The Mirror version is the statement that

$$Z_{S^2}(Y_r) = e^{-K_c(X_\phi)}$$

under a suitable mirror map.

We use a version of the Batyrev mirror construction for hypersurfaces in the toric varieties.
The family $X$ of Calabi-Yau varieties defined by the equation (for example the quintic)

$$W_X(x,\phi) = \sum_{i=1}^{5} x_i^5 + \sum_{l=1}^{101} \phi_l e_l(x) = \sum_{a=1}^{106} C_a(\phi) \prod_{j=1}^{5} x_i^{v_{ai}},$$

expressed in terms the exponent matrix $v_{ai}$. Vectors $v_a$ define the lattice points of the reflexive polytope in $\mathbb{R}^5$.

Following Batyrev construction for finding the mirror to $X$ manifold we take the Fan whose edges are the vectors $v_a$ with the components $v_{ai}$ and construct the toric variety with this Fan as explained above.
Then hypersurface $Y$ defined by zero locus of a quasihomogenious polynomial $W_Y$ inside this toric variety will be the mirror partner to the quintic $X$ in the projective space.

For the quintic the vectors of the fan are

$$v_{ai} = \begin{cases} 5\delta_{a,i}, & 1 \le a \le 5, \\ s_{a-5,i}, & 6 \le a \le 106. \end{cases}$$

We build a GLSM whose vacuum manifold is a mirror quintic. We easily reconstruct the charge matrix $Q_{al}$

$$Q_{al} = \begin{cases} s_{la}, & 1 \le i \le 5, \\ -5\delta_{a-5,l}, & 6 \le a \le 106. \end{cases}$$

such that

$$\sum_{a \le 106} Q_{al} v_a = 0.$$

Elements $Q_{al}$ form a basis in linear relations among $v_a$.

To write the superpotential of the GLSM it is convenient to separate the chiral fields as

$$\Phi_a = \begin{cases} S_a, & 1 \leq i \leq 5, \\ P_{a-5}, & 6 \leq a \leq 106. \end{cases}$$

It can be shown that the quasihomogenious polynomial (superpotential) $W_Y$ is of the form

$$W_Y := P_1\, G(S_1, \ldots, S_5; P_2, \ldots, P_{101}).$$

Therefore the potential for the scalars whose zeroes define the mirror for the quintic is

$$U(\phi) = \sum_{l=1}^{101} \frac{e_l^2}{2} \left( \sum_{i=1}^{5} s_{li} |S_a|^2 - 5|P_l|^2 - r_l \right)^2 + \frac{1}{4} |G(S_1, \ldots, S_5; P_2, \ldots, P_{101})|^2 + \\ + \frac{1}{4} |P_1|^2 \sum_{i=1}^{5} \left| \frac{\partial G}{\partial S_i} \right|^2 + \frac{1}{4} |P_1|^2 \sum_{l=2}^{101} \left| \frac{\partial G}{\partial P_l} \right|^2 .$$

The partition function of the GLSM in this case is given by a 101-fold contour integral

$$Z_{S^2} = \sum_{m_l \in V} \int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_{101}} \prod_{l=1}^{101} \frac{d\tau_l}{(2\pi i)} \left( z_l^{-\tau_l + \frac{m_l}{2}} \bar{z}_l^{-\tau_l - \frac{m_l}{2}} \right) \times$$

$$\times \frac{\Gamma\left(1 - 5(\tau_1 - \frac{m_1}{2})\right)}{\Gamma\left(5(\tau_1 + \frac{m_1}{2})\right)} \prod_{i=1}^{5} \frac{\Gamma\left(\sum_l s_{li}(\tau_l - \frac{m_l}{2})\right)}{\Gamma\left(1 - \sum_l s_{li}(\tau_l + \frac{m_l}{2})\right)} \prod_{l=2}^{101} \frac{\Gamma\left(-5(\tau_l - \frac{m_l}{2})\right)}{\Gamma\left(1 + 5(\tau_l + \frac{m_l}{2})\right)},$$

where

$$z_l := e^{-(2\pi r_l + i\theta_l)},$$

and summation is over $m_l$ such that $\sum_a m_a Q_{ai} \in \mathbb{Z}$ for all i.
To connect with our previous computations we compute the integral at $r_a << 0$ that is at $|z_a| >> 0$. The contours can be deformed to the right picking up the residues at

$$5\left(\tau_l - \frac{m_l}{2}\right) - 1 = p_1,\ 5\left(\tau_l - \frac{m_l}{2}\right) = p_l;$$

$$p_1 = 1, 2, \ldots,\ p_l = 0, 1, \ldots \quad \text{so that} \quad p_l + 5m_l > 0.$$

# Partition function for the mirror quintic

After computing the residues the partition function reduces to

$$Z_{S^2} = \pi^{-5} \sum_{p_1>0, p_l\geq 0} \sum_{\bar{p}_l \in \Sigma_p} \prod_l \frac{(-1)^{p_l}}{p_l! \bar{p}_l!} z_l^{-\frac{p_l}{5}} \bar{z}_l^{-\frac{\bar{p}_l}{5}}$$
$$\prod_{i=1}^{5} \Gamma\left(\frac{1}{5}\sum_{l=1}^{h} s_{li} p_l\right) \Gamma\left(\frac{1}{5}\sum_{l=1}^{h} s_{li} \bar{p}_l\right) \sin\left(\frac{\pi}{5}\sum_{l=1}^{h} s_{li} \bar{p}_l\right),$$

where the set $\Sigma_p$ - is a set of all $\{\bar{p}_l\}$ such that $\sum_a (\bar{p}_a - p_a) Q_{al}/5 = \sum_a m_a Q_{al} \in \mathbb{Z}$.
After a rearrangement this formula becomes

$$Z_{S^2} = \sum_{\boldsymbol{\mu}} (-1)^{|\boldsymbol{\mu}|} \prod_{i=1}^{5} \frac{\Gamma\left(\frac{\mu_i}{5}\right)}{\Gamma\left(1 - \frac{\mu_i}{5}\right)} |\sigma_{\boldsymbol{\mu}}(\mathbf{z})|^2,$$

where

$$\sigma_{\boldsymbol{\mu}}(\mathbf{z}) = \sum_{n_i \geq 0} \prod_{i=1}^{5} \frac{\Gamma\left(\frac{\mu_i}{5} + n_i\right)}{\Gamma\left(\frac{\mu_i}{5}\right)} \sum_{\mathbf{p} \in S_{\boldsymbol{\mu},\mathbf{n}}} \prod_{l=1}^{101} \frac{(-1)^{p_l} z_l^{-\frac{p_l}{5}}}{p_l!}.$$

The formula for partition function on $S^2$ coincides with the special geometry on the moduli space of the quintic itself after a simple mirror map

$$z_I = -\phi_I^{-5}.$$

Thus we constructed an explicit correspondence between a family of Calabi-Yau manifolds $X_\phi$ and the Gauge Linear Sigma Model whose vacuum manifold $Y_r$ is a mirror of $X_\phi$.

The GLSM partition function presentation also gives an useful analytic continutation of the Special geometry on the complex moduli space $X_\phi$.

Thank you for your attention!