Multiparticle production in $\lambda\phi^4$ theory: method of singular solutions and numerical results

based on 2212.03268 [hep-ph]

S.V. Demidov, **B.R. Farkhtdinov**[1,2], D.G. Levkov

[1]Institute for Nuclear Research of the Russian Academy of Sciences

[2]Moscow Institute of Physics and Technology

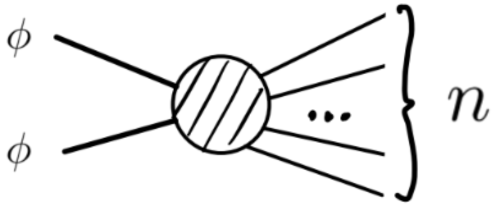
Motivation

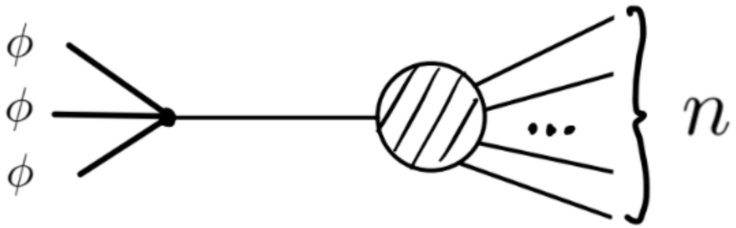
- Multiparticle probabilities
- Perturbation theory

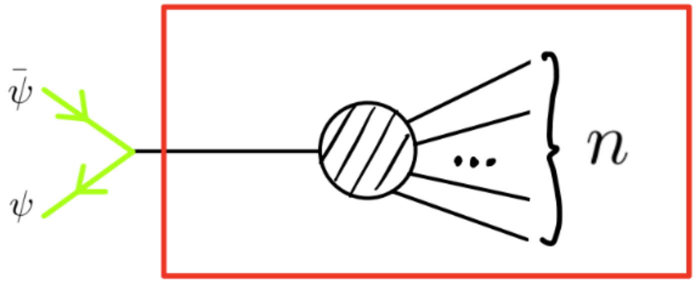
Multiparticle processes

Processes with $n \gg 1$ bosons in the final state and $n_i \ll n$ in the initial

Examples





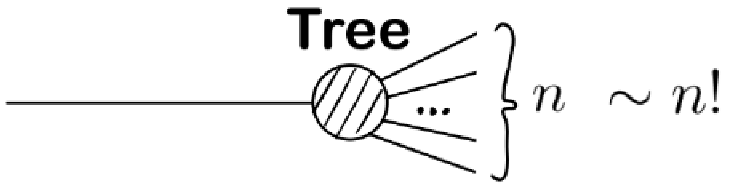


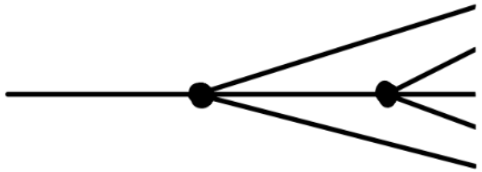
Growing number of diagrams

In theory $S[\phi] = \frac{1}{2}\int d^4x(-\phi\Box\phi - m^2\phi^2 - \lambda\phi^4/2)$

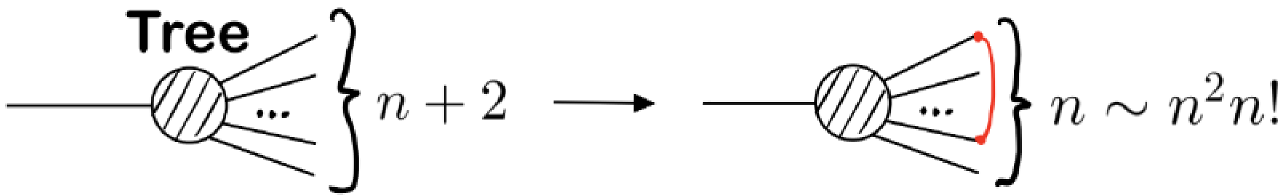
$$A_{1\to n} = \langle n, E = nm|\,\hat{\phi}\,(0)\,|0\rangle$$

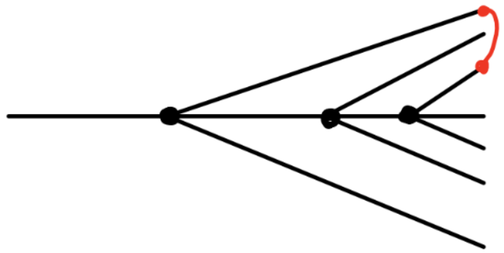
The number of tree diagrams





The number of one-loop diagrams





Tree-level and one-loop

Tree − level amplitudes are [Brown, 1992]

$$A^{tree}_{1\to n} = n!\left(\frac{\lambda}{8m^2}\right)^{\frac{n-1}{2}}$$

Indeed grows factorially

One-loop correction [Voloshin, 1992]

$$A^{1-loop}_{1\to n} = A^{tree}_{1\to n}\, B\lambda(n^2 + O(n))$$

$$B = const$$

Indeed has $n!\, n^2$

Failure of perturbation theory

Perturbative series for $A_{1\to\mathrm{n}}$ [Argyres, 1993]

$$A_{1\to n} = n!\left(\frac{\lambda}{8m^2}\right)^{\frac{n-1}{2}}\left[1 + \#\lambda(n^2 + \cdots) + \#\lambda^2(n^4 + \cdots) + \cdots\right]$$

At L loops leading contribution $\propto n!\,(\lambda n^2)^L$

Blow up at $n \gtrsim \lambda^{-1}$

Series resummation

Resummation of leading contributions at large n gives

$$A_{1\to n}^{resummed} = A_{1\to n}^{tree} \exp(B\lambda n^2)(\dots)$$

B is the same as in the one-loop $\propto n^2$ term[Libanov et al., 1994]

Using Stirling formula

$$A_{1\to n}^{tree} \sim \exp\left(n \ln \lambda^{1/2} n - n\right)$$

Then

$$A_{1\to n}^{resummed} \sim \exp(F_A/\lambda)$$

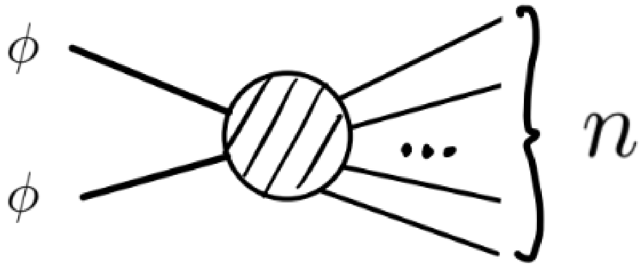
Exponential form $\stackrel{?}{=}$ Semiclassical treatment

The exponent

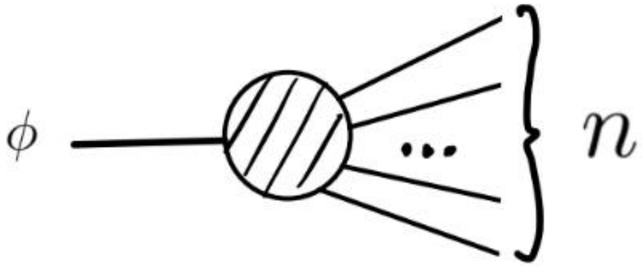
Conjectures [Libanov et al., 1994]

1. $A_{few \to n} \sim e^{F_A(\lambda n)/\lambda}, n \gg 1, \lambda n = const.$ $F_A(\lambda n)$— "holy grail" function
2. $F_A(\lambda n)$ do not depend on the initial state if $n \gg n_i$

Semiclassical limit: $\lambda \to 0, \lambda n = const$



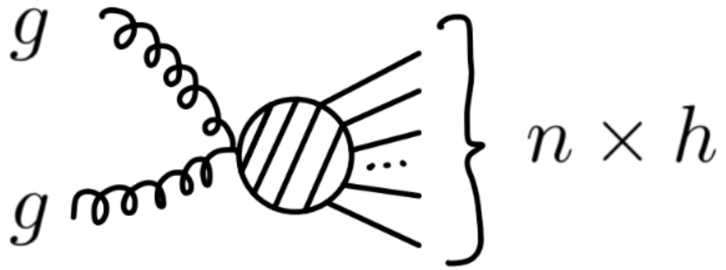
or



etc.

"Higgsplosion"

It was suggested [Khoze, 2017] that at high energies



Probability also has exponential form $\propto e^{F_{Higgsplosion}/\lambda}$

$$F_{Higgsplosion} = \lambda n \ln\left(\frac{\lambda n}{4}\right) + \frac{3}{2}\lambda n \ln\frac{\varepsilon}{3\pi m} + \frac{\lambda n}{2} + 0.845(\lambda n)^{3/2}, \quad n \le n_*, E = n(m+\varepsilon)$$

$F_{Higgsplosion} = 0$ at n_* and grows for $n > n_*$ (unitarity?)

Result was obtained semiclassically with additional assumptions

Consistency check?

Method of singular solutions

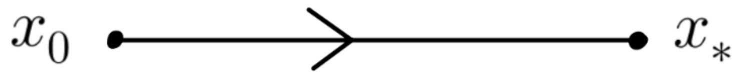
- Formulation
- Numerical implementation

[Son, 1995]

Landau method in QM

<u>In QM one can consider</u>

$$\langle E' | \hat{O} | E \rangle \sim e^f, \quad f = -\mathrm{Im}\left[\int^{x_*} [2m(E'-V)]^{1/2} dx - \int^{x_*} [2m(E-V)]^{1/2} dx \right]$$



$E > V(x_0)$ $\qquad$ $V(x_*) = \infty$

- $\hat{O}$ can be $\hat{x}, \hat{x}^2, \hat{p}$, etc. — answer is insensible
- x_* is a singular point of $V \Rightarrow$ singular solutions in path integral
- We need only exponential accuracy

Parametrization

- In $(3+1)$ we consider theory

$$S[\phi] = \frac{1}{2\lambda} \int d^4x(-\phi\Box\phi - m^2\phi^2 - \phi^4/2)$$

- At $\lambda \to 0$ action becomes large $\Rightarrow$ semiclassical description
- Consider processes $n_i \to n$ with $n \gg 1$ and $n \gg n_i$
- $n \propto \lambda^{-1} \Rightarrow E = n(m+\varepsilon) \propto \lambda^{-1}$

Probability

Our aim is

$$P_n(E) \equiv \sum_f \left| \langle f; E, n| \hat{S}\hat{O} |0\rangle \right|^2 = \int D[f,\phi] e^{W/\lambda} \sim \mathrm{e}^{F(\lambda n, \varepsilon)/\lambda},$$

where

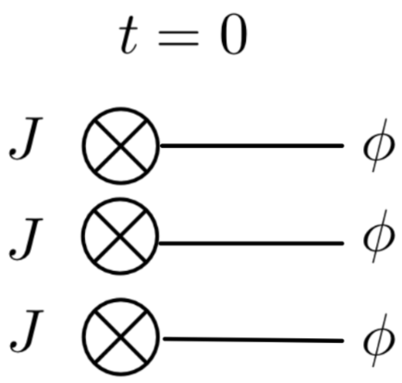
$$\varepsilon = E/n - m$$

$\hat{O}|0\rangle$ — initial state with $n_i \ll n$ particles

Initial state

$$\hat{O}_J|0\rangle = \exp\left(-\frac{1}{\lambda}\int d^3\mathbf{x}\, J(\mathbf{x})\hat{\phi}(0,\mathbf{x})\right)|0\rangle$$

1. Creates $n_J \propto J^2/\lambda$ particles
2. $n_J \ll \lambda^{-1}$ or $J \to 0 \Rightarrow$ universality
3. At $1 \ll n_J \ll \lambda^{-1}$ we can
   - Calculate semiclasically
   - Use universality



Probability as a path integral

We compute

$$P_n^J(E) \equiv \sum_f \left|\langle f; E, n|\, \hat{S}\hat{O}_J |0\rangle\right|^2 = \int D[f,\phi] e^{W_J/\lambda},$$

Landau conjecture: $P_n(E) = \lim_{J\to 0} P_n^J(E)$

In the path integral representation

$$P_n^J(E) = \int \mathbf{D}f\, |A_J|^2 \qquad A_J = \int \mathbf{D}\phi_{i,j}\, \langle f; E, n|\, \phi_f\rangle \langle \phi_f|\, \hat{S}\hat{O}_J\, |\phi_i\rangle\, \langle \phi_i|\, 0\rangle$$

$e^{B_f(\phi_f,\, E,\, n)}$ $\quad$ $e^{iS[\phi] - \int J\phi}$ $\quad$ $e^{B_i(\phi_i)}$

Saddle point

Use saddle-point approximation at $\lambda \to 0$

$$\frac{\delta W_J}{\delta \phi_{\text{cl}}} = 0$$

$$P_n^J(E) \sim e^{F_J(\lambda n,\, \varepsilon)/\lambda}$$

$$F_J(\lambda n, \varepsilon) = W_J[\phi_{\text{cl}}]$$

ϕ_{cl} obeys

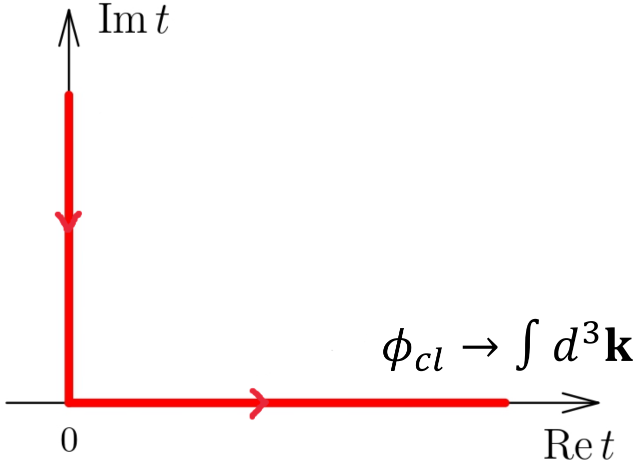
$$\Box\phi_{cl}(x) + m^2\phi_{cl}(x) + \phi_{cl}^3(x) = iJ(\mathbf{x})\delta(t)$$

We consider only spherically-symmetrical ϕ_{cl}

Boundary conditions

Solution is calculated on the complex time contour

$\phi_{cl} \to 0$ as $t \to +i\infty$
(vacuum)



$\phi_{cl} \to \int d^3\mathbf{k}\, \left(a_\mathbf{k} \mathrm{e}^{-i\omega_\mathbf{k} t} + b^*_\mathbf{k} \mathrm{e}^{i\omega_\mathbf{k} t}\right)$ as $t \to +\infty$

$$a_\mathbf{k} = e^{-\theta + 2T\omega_\mathbf{k}} b_\mathbf{k}$$

T, θ — Lagrange multipliers (E, n fixing)

Limit $J \to 0$

After ϕ_{cl} was found, we calculate

$$F_J = 2\lambda ET - \lambda n\theta - 2\lambda \mathrm{Im} S[\phi_{cl}] - 2\mathrm{Re} \int d^3\mathbf{x}\, J(\mathbf{x})\phi_{cl}(0, \mathbf{x})$$

Then perform the limit

$$F(\lambda n, \varepsilon) = \lim_{J \to 0} F_J(\lambda n, \varepsilon)$$

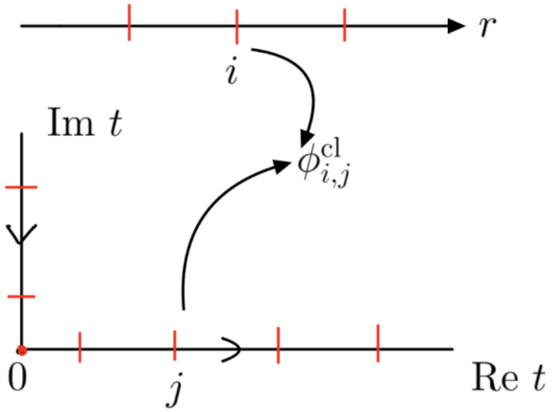
Solutions become singular in the limit

- $E_i = 0$, $E_f = E \Rightarrow$ discontinuity:
- $iJ(\mathbf{x})\delta(t) \Rightarrow$ energy change at $t = 0$
- $J = 0 \Rightarrow$ energy conservation conflicts with BC

Numerical implementation

To solve the saddle-point boundary value problem numerically we

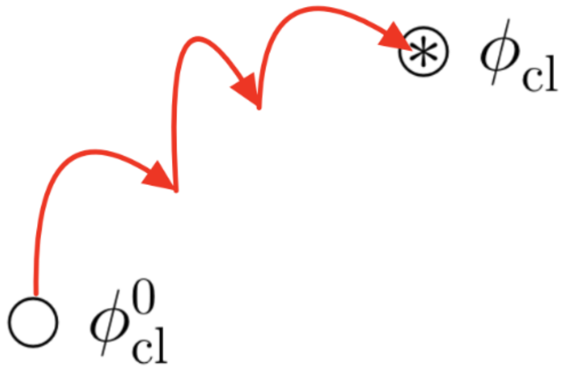
- Use $J(\mathbf{x}) = j_0 e^{-\mathbf{x}^2/2\sigma^2}$
- Discretize:



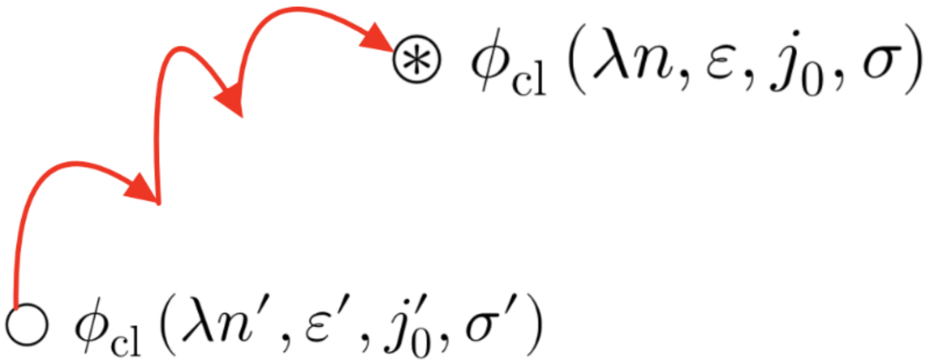
- Solve $2 \times N_r \times N_t + 2$ real non-linear equations

Solving the equations

- We used Newton-Raphson method



- Solution with required parameters is obtained by walking in the parameter space



Need ϕ_0 to start procedure

Source-dominated ϕ_{cl}^0

When $\lambda n \ll 1; \sigma, \lambda E = const$ and $\lambda n \propto j_0^2$

only source produces particles

$$\Box\phi_{cl} + m^2\phi_{cl} + \phi_{cl}^3 = i j_0 e^{-\mathbf{x}^2/2\sigma^2}$$

$$\phi_{cl} \sim j_0 \qquad m^2\phi_{cl} \sim j_0 \qquad \phi_{cl}^3 \sim j_0^3$$

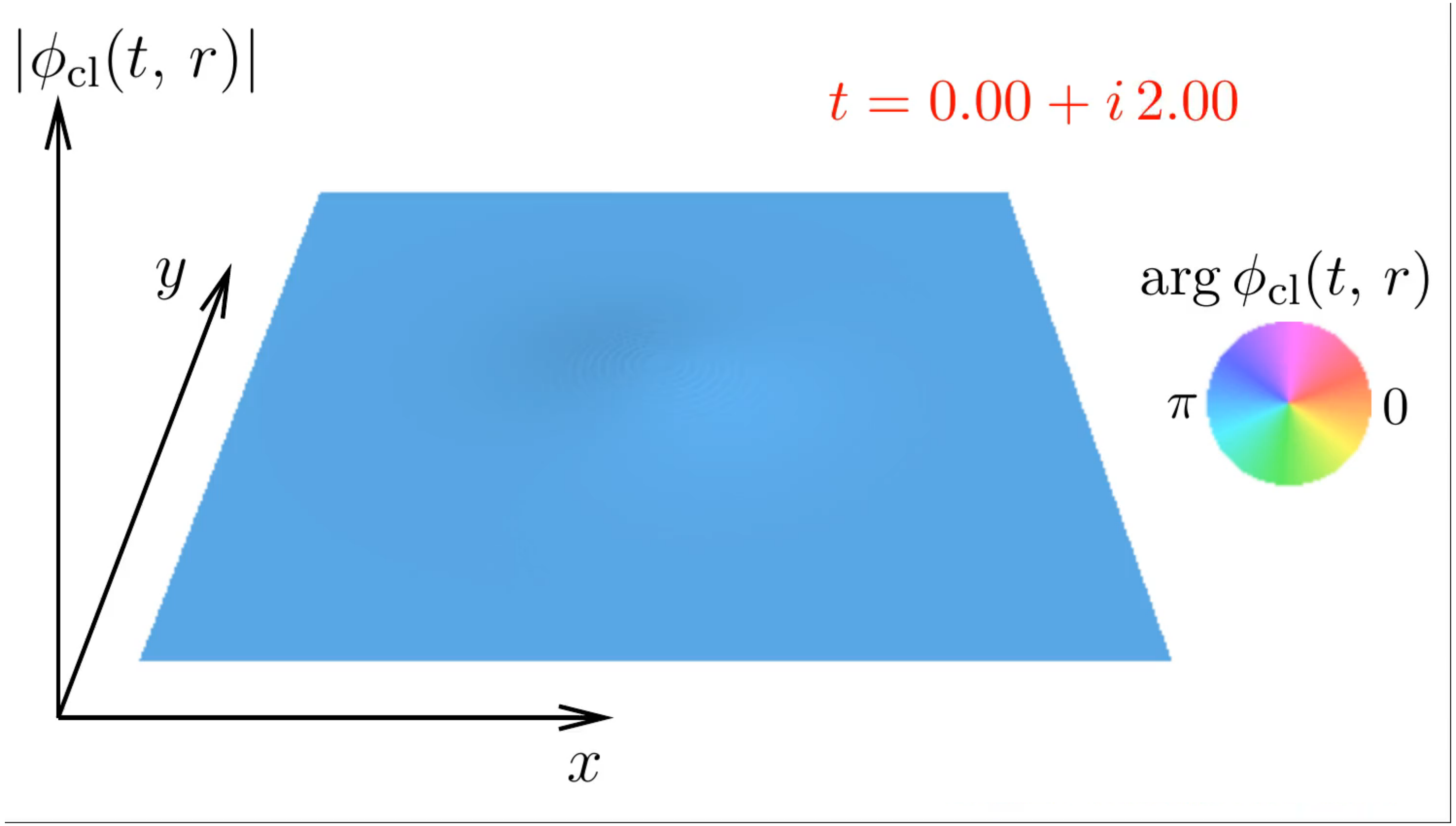
Can be analytically solved!

We use the solution as ϕ_{cl}^0

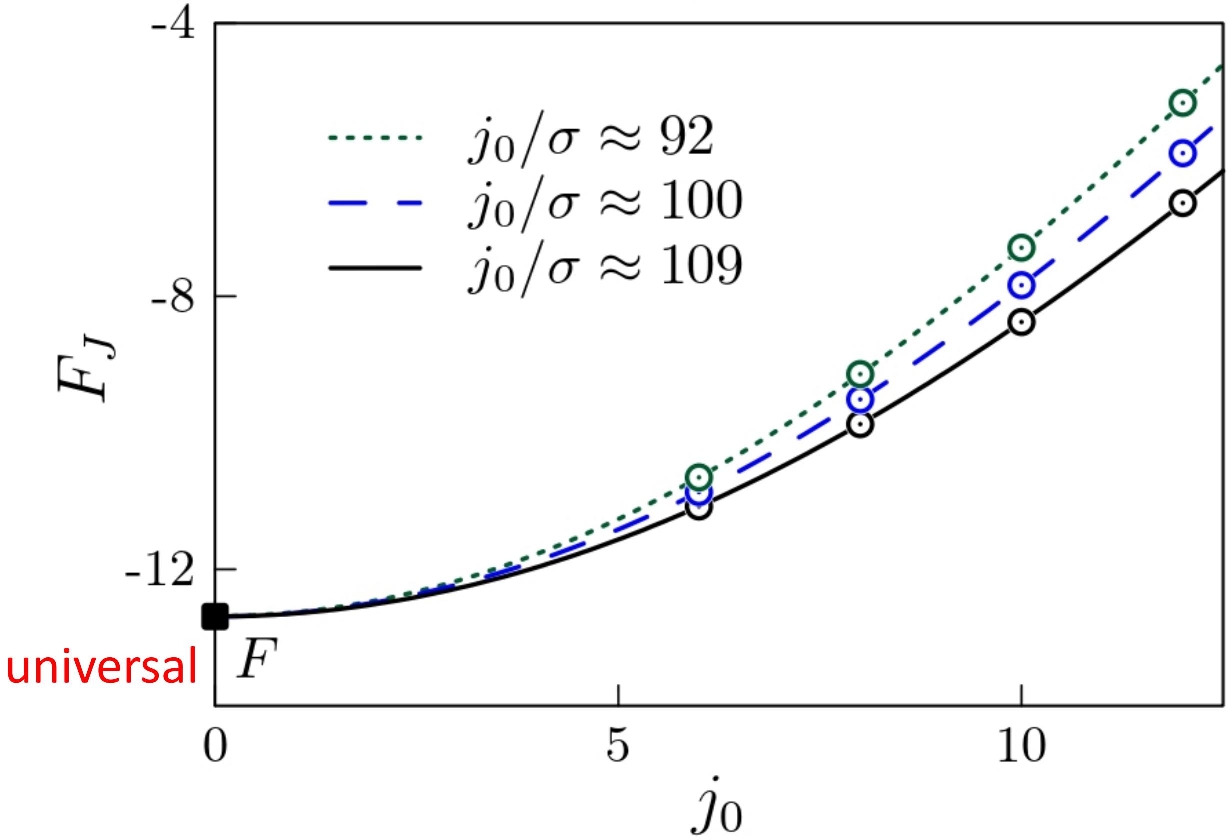
Numerical data

- Suppression exponent and amplitudes
- Limit $\lambda n \gg 1$
- Limit $\varepsilon \gg m$

Example of a solution



Extrapolation $J \to 0$



- Use $J(\mathbf{x}) = j_0 e^{-\mathbf{x}^2/2\sigma^2}$
- Consider $j_0 \to 0,\ j_0/\sigma = const$ — weak narrow source
- Use $F_J = F + F_2 j_0^2 + F_4 j_0^4 + \cdots$
- Compute $j_0 \to 0$ with different j_0/σ

F at $\lambda n \ll 1$

Reminder: $P_n(E) \sim e^{F/\lambda}$

When $n \ll \lambda^{-1}$(tree-level)

$$F(\lambda n, \varepsilon) = \lambda n \ln\left(\frac{\lambda n}{16}\right) - \lambda n + \lambda n\, f(\varepsilon) + O(\lambda n)^2$$

- For $\varepsilon \ll m$ function $f(\varepsilon)$ is known up to $O(\varepsilon^2)$
- For larger ε it can be evaluated numerically [Bezrukov, 1998]

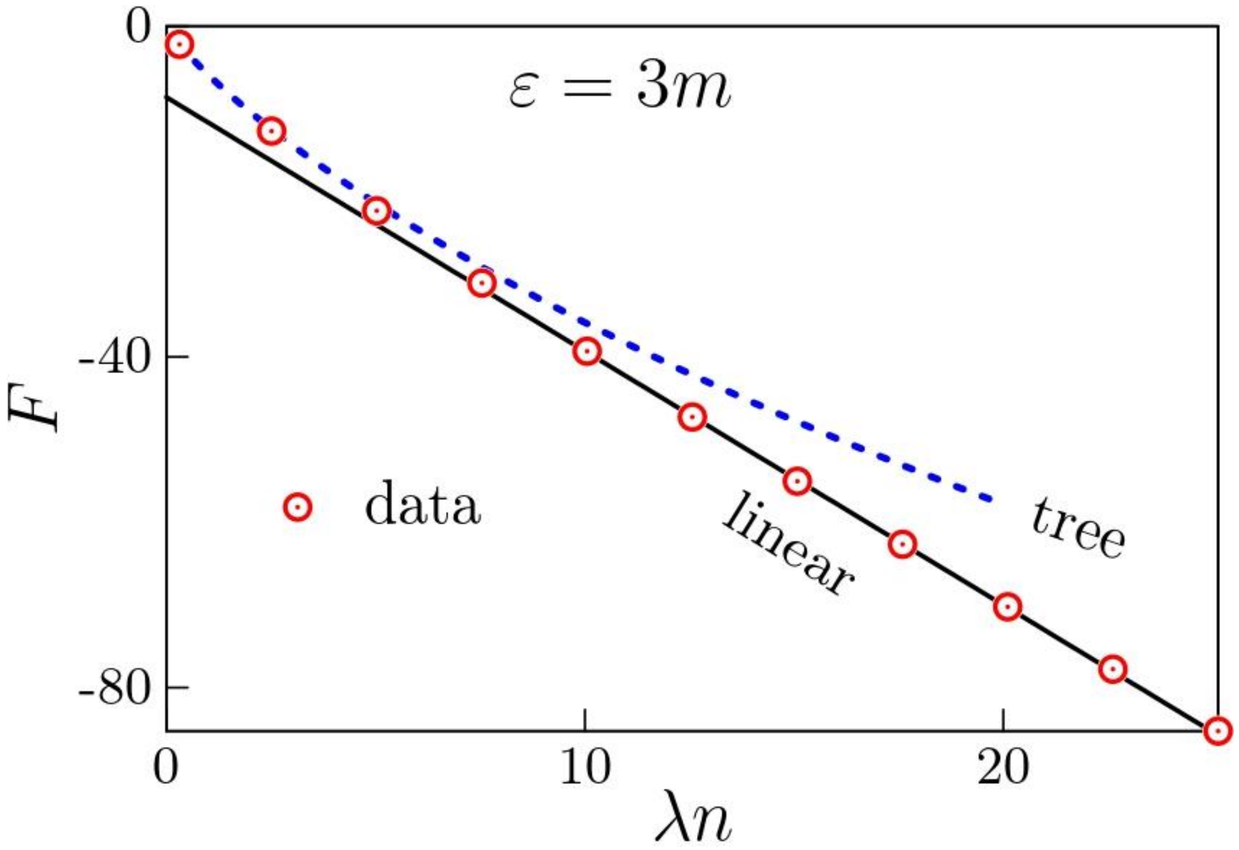
F at $\lambda n \gg 1$

In the limit $n \gg \lambda^{-1}$

1. $F(\lambda n, \varepsilon) \xrightarrow[\lambda n \to +\infty]{} f_\infty(\varepsilon)\lambda n + g_\infty(\varepsilon)$
2. $f_\infty, g_\infty < 0$ for all ε

$$P_n(E) \sim e^{f_\infty(\varepsilon)n + g_\infty(\varepsilon)/\lambda}$$

Example of typical behavior



- Tree:

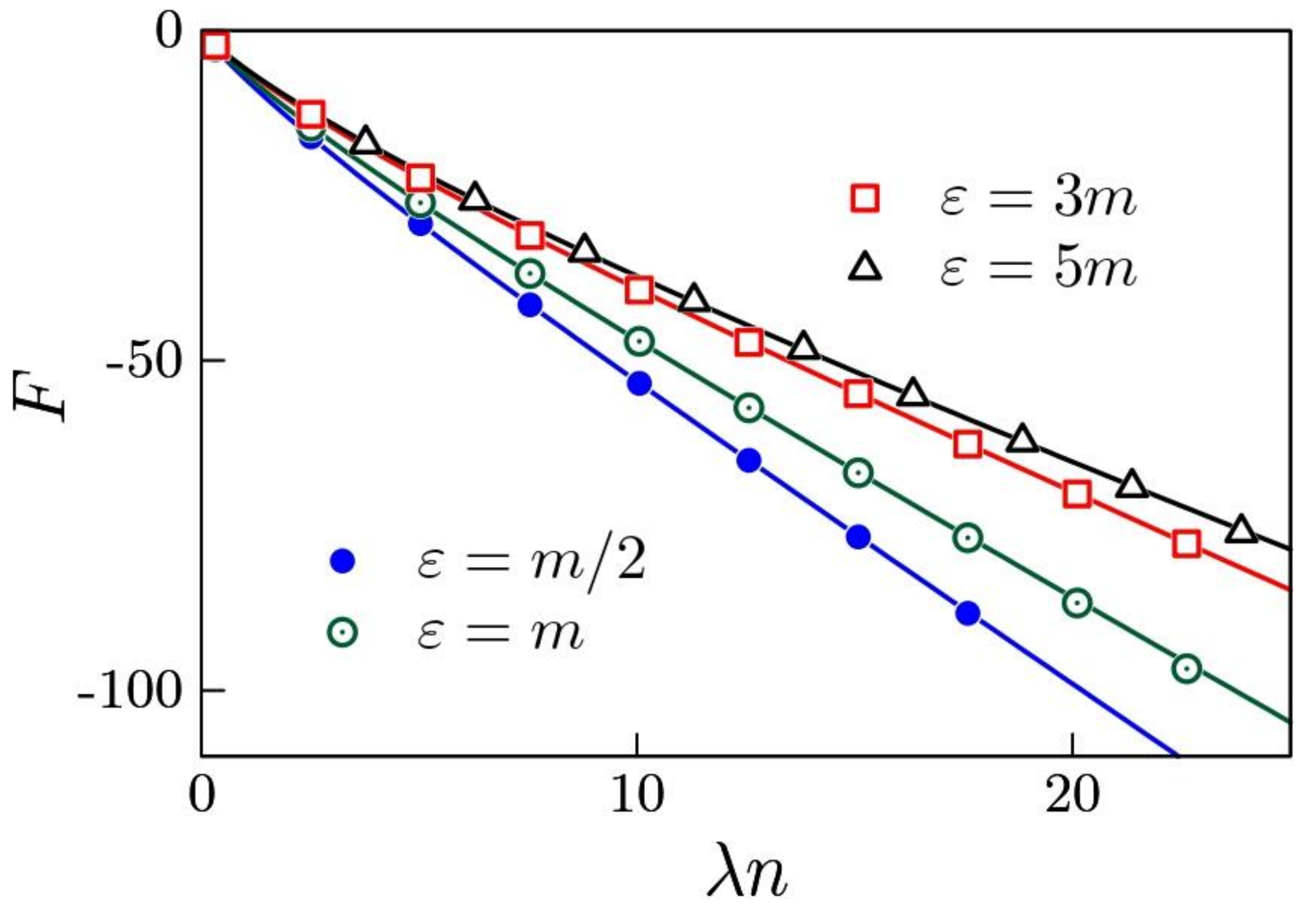
$$F = \lambda n \ln\left(\frac{\lambda n}{16}\right) - \lambda n + \lambda n\, f(\varepsilon) + O(\lambda n)^2$$

- Linear:

$$F = f_\infty(\varepsilon)\lambda n + g_\infty(\varepsilon)$$

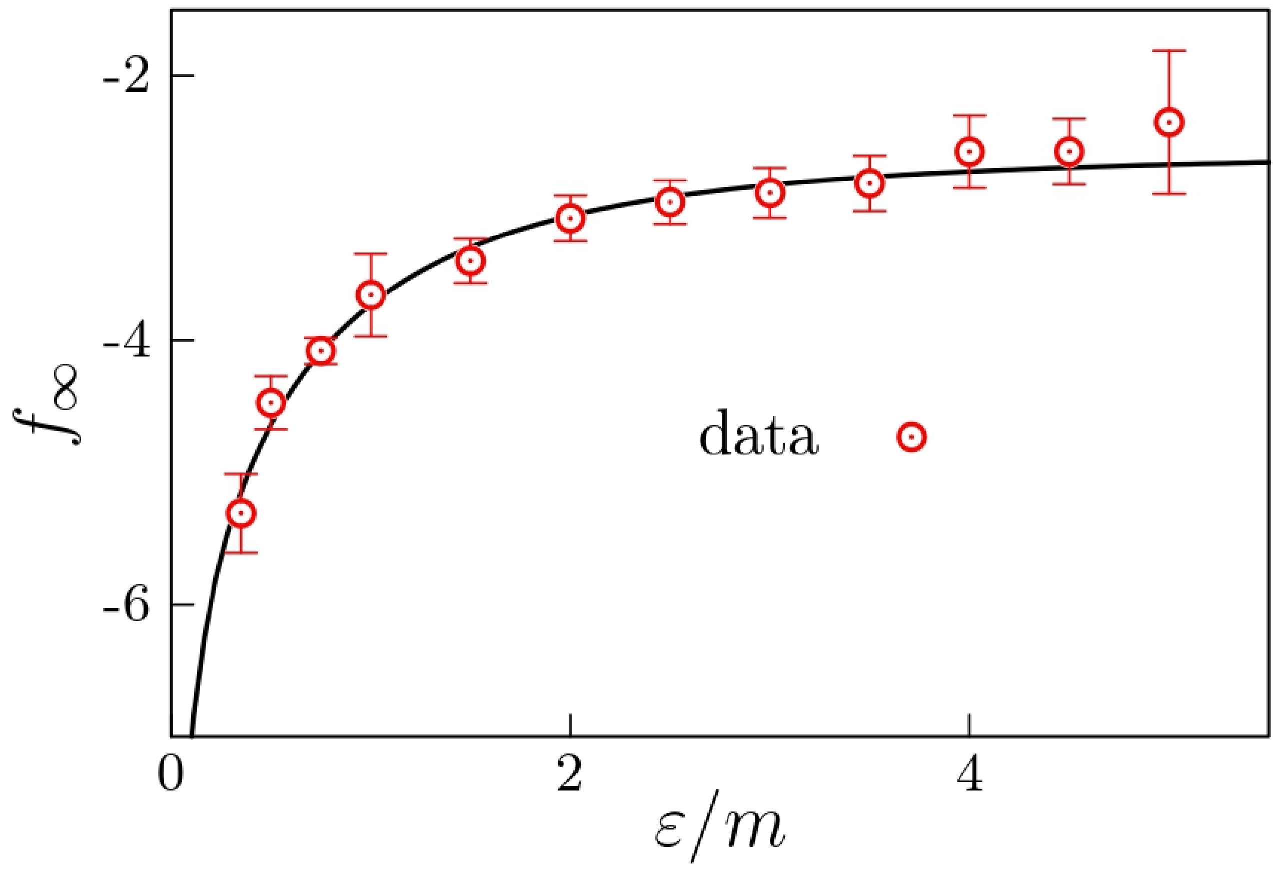
- $\varepsilon = \frac{E}{n} - m$
- $P_n(E) \sim e^{F/\lambda}$

Numerical suppression exponents



- $\varepsilon = \frac{E}{n} - m$
- F become closer, when ε grows

f_∞ as a function of ε



- $F \rightarrow f_\infty(\varepsilon)\lambda n + g_\infty(\varepsilon)$ for $\lambda n \gg 1$
- f_∞ grows to -2.57 ± 0.06 for $\varepsilon \rightarrow \infty$
- $\varepsilon = \frac{E}{n} - m$

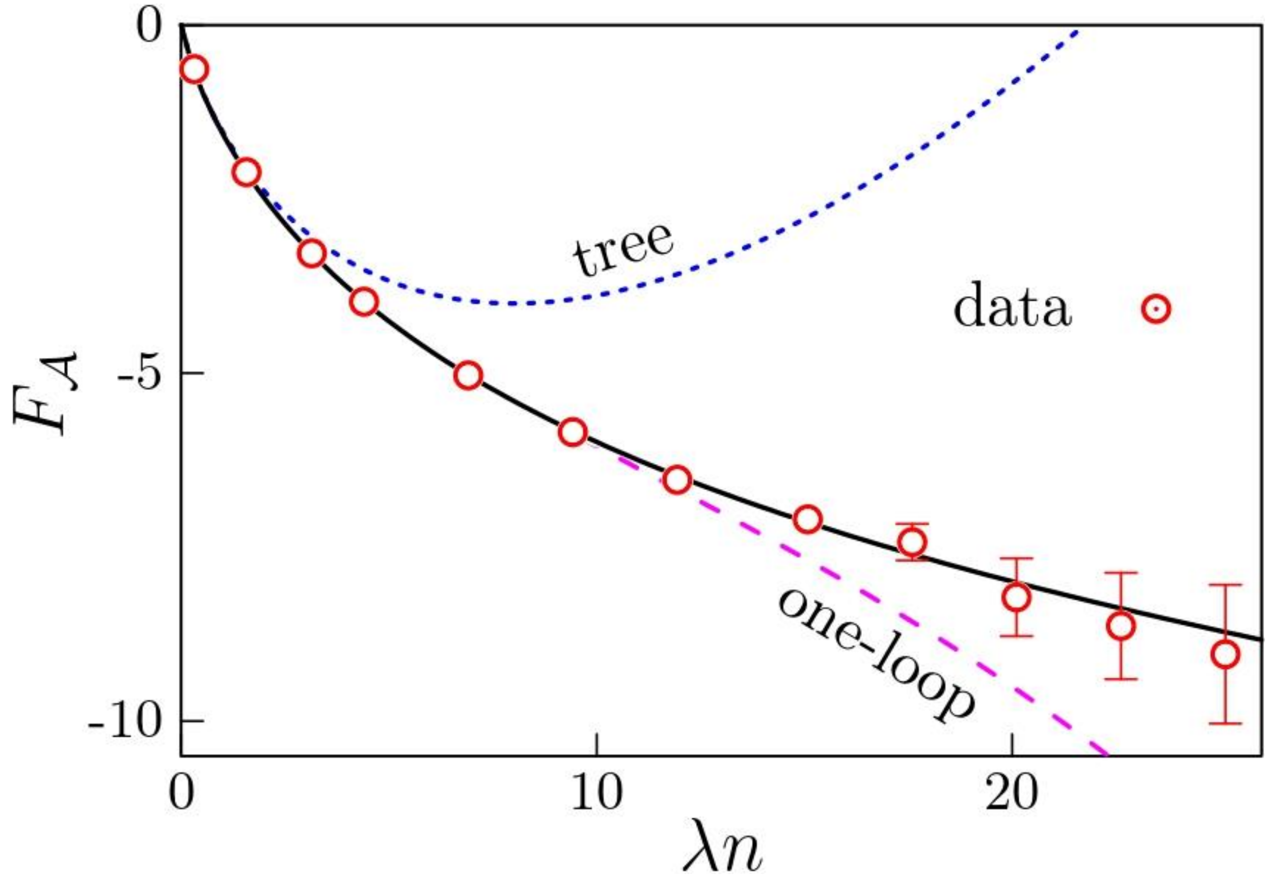
Amplitudes at threshold ($\varepsilon \to 0$)

In the limit $\varepsilon \to 0$ one can estimate

$$P_n(E \to mn) \approx |A_n|^2 \times \text{phase volume} \approx e^{F(\lambda n, \varepsilon \leq m)}$$

Can get $|A_n| = \exp(F_A/\lambda)$ from $F(\lambda n, \varepsilon \leq m)$ via extrapolation $\varepsilon \to 0$

Fitting of $F_A(\lambda n)$



- $F_A = \frac{\lambda}{2} \lim_{E \to nm} \ln \frac{P_n(E) m^{4-2n}}{\text{phase vol.}}$
- Tree:

$$F_A^{\text{tree}} = \frac{\lambda n}{2} \ln\left(\frac{\lambda n}{8}\right) - \frac{\lambda n}{2}$$

- One-loop:

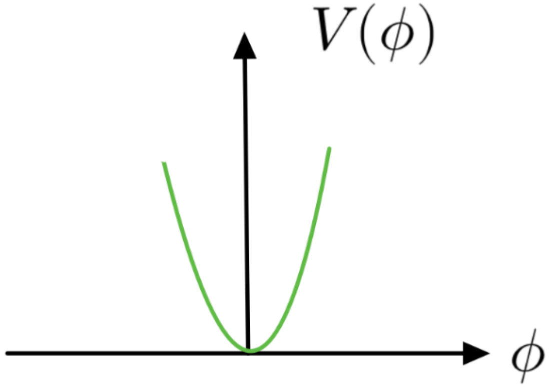
$$F_A^{1-\text{loop}} = F_A^{\text{tree}} + \frac{(\lambda n)^2 3^{3/2}}{32\pi^2} \ln(2+\sqrt{3})$$

Conclusions

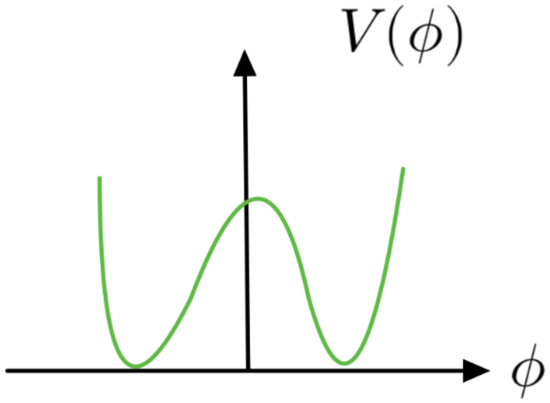
We calculated "holy grail" function $F(\lambda n, \varepsilon)$

$$P_n(E) \xrightarrow[\lambda n \to +\infty]{} e^{n f_\infty + g_\infty/\lambda}$$

Generic?



Suppression



Higgsplosion?

Thank you!