Efficient Rules for All Conformal Blocks: A Dream Come True

Valentina Prilepina

Département de Physique, de Génie Physique et d'Optique
Université Laval, Québec, QC G1V 0A6, Canada

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based on

arXiv:2002.09007 [hep-ph]

Also see: arXiv:1905.00036 [hep-ph], arXiv:1906.12349 [hep-ph]
arXiv:1907.08599 [hep-ph], arXiv:1907.10506 [hep-ph]

with Jean-François Fortin, Wen-Jie Ma, and Witold Skiba

Why Study Conformal Field Theories (CFTs)?

CFTs describe universal physics of scale invariant critical points:

- continuous phase transitions in condensed matter and statistical systems
- fixed points of RG flows

Provide a handle on

- Universal structure of the landscape of QFTs
- Quantum gravity via the AdS/CFT correspondence and holography
- String theory
- Black holes

The Conformal Bootstrap

The conformal bootstrap program seeks to systematically apply

- conformal symmetry
- crossing symmetry
- unitarity/reflection positivity

conditions to map out and solve the space of allowed CFTs

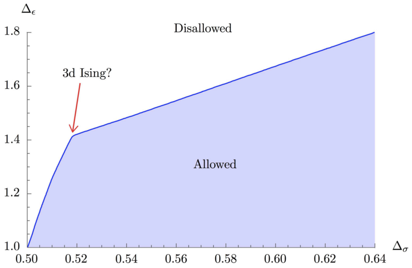


Figure: Allowed region for 3D Ising Model [El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi, '12; '14]

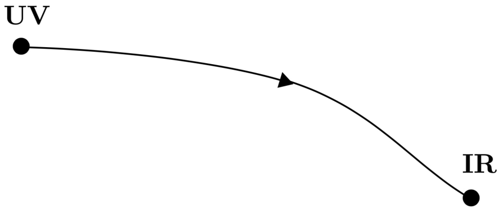
The Ultimate Dream

- Tremendous progress both on the numerical and analytic fronts! e.g. Ferrara et al. (1971, 1973), Dobrev et al. (1976, 1977), Polyakov (1974), Dolan & Osborn (2001, 2004, 2011), Poland et al. (2012), Simmons-Duffin (2014), El-Showk et al. (2014), Kos et al. (2014, 2015, 2016), Costa & Hansen (2015), Rejon-Barrera & Robbins (2016), Echeverri et al. (2016), Costa et al. (2016), Fortin & Skiba (2016, 2019), Karateev et al. (2017), Poland & Simmons-Duffin (2019)
- Dream: to classify and solve the entire landscape of CFTs and predict their observables

CFTs are signposts in the landscape of QFTs!

The Ultimate Dream (cont.)

QFTs: Renormalization group flows from UV to IR fixed points



- Large classes of QFTs as relevant deformations of small subset of CFTs

Outline: Part I

Part I: Setting the Stage

- A Little Bit of Background on CFTs
- Goal: Efficient Rules for Arbitrary Conformal Blocks
- Embedding Space OPE Formalism
- Three- and Four-Point Functions
- Bases of Tensor Structures

Outline: Part II

Part II: The Rules

- Tensor Structures for Towers of Exchanged Operators
- Projection Operators to Exchanged Representations
- Diagrammatic Notation
- Rule for Rotation Matrices
- Rule for Conformal Blocks
- Examples
- Conclusions and Outlook

What is a CFT?

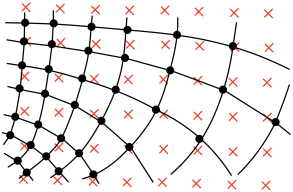
A special quantum field theory invariant under the conformal transformations:

$$g'_{\mu\nu}(x') = c(x)\delta_{\mu\nu}$$

Jacobian:

$$J = \frac{\partial x'^{\mu}}{\partial x^{\nu}} = b(x)M^{\mu}{}_{\nu}(x), \quad M \in SO(d)$$

- Preserve angles
- Locally look like a rotation followed by a scale transformation $x \to \lambda x$



The Spectrum of Operators

Two kinds of operators in CFTs:

- quasi-primaries $[K_\mu, \mathcal{O}^{(x)}(0)] = 0$: transform simply under conformal transformations, e.g.

$$x \to x', \quad \mathcal{O}^{(x)}(x) \to \tilde{\mathcal{O}}^{(x)}(x') = b(x)^{-\Delta} \mathcal{O}^{(x)}(x)$$

- descendants: don't!

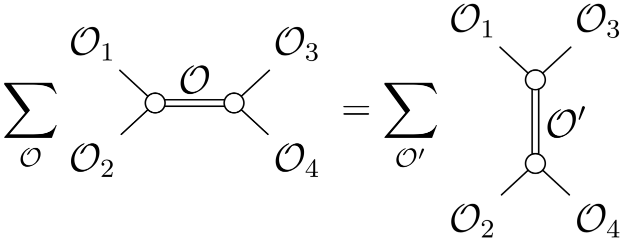
- Complete spectrum of operators: primaries+infinite towers of descendants
- Organic observables in CFTs: M-point correlation functions of operators, $\langle \mathcal{O}^{(x)}(x_1) \ldots \mathcal{O}^{(x)}(x_M) \rangle$

What are Conformal Blocks?

- Well-defined objects appearing in expansion of the four-point functions
- Capture contributions of particular exchanged operators in the OPE
- Similar to an expansion in spherical harmonics Y_ℓ^m but for CFTs

Conformal Bootstrap

Impose **crossing symmetry**



Interchanging $x_1 \leftrightarrow x_3$ gives the crossing symmetry condition:

$$\sum_{\Delta,\ell} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(u,v) = \sum_{\Delta,\ell} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(v,u)$$

Our Goal

Goal: Efficient Rules for Arbitrary Conformal Blocks

- Approach based on embedding space OPE formalism given in Fortin, VP, Skiba (2019)
- Conformal blocks expressed as specific linear combinations of Gegenbauer polynomials in a special variable, with a unique substitution rule ascribed to each polynomial piece
- Applying each rule term-by-term directly generates the complete conformal block in terms of a four-point tensorial generalization of the Exton G-function ($\propto$ scalar exchange block in $\langle SSSS \rangle$)

Our Goal (cont.)

Procedure for determining a given block:

1. Writing down the relevant group theoretic input data: the projection operators and tensor structures

2. Identifying the specific linear combination of Gegenbauer polynomials along with the associated substitution rules for each piece

In this work: Wish to make this approach systematic $\Rightarrow$ Derive a set of general rules

Embedding Space OPE

Replace the product of two local quasi-primary operators by an infinite sum of operators at some point on the lightcone:

$$\mathcal{O}_i(\eta_1)\mathcal{O}_j(\eta_2) = (\mathcal{T}_{12}^{\boldsymbol{N}_i}\Gamma)(\mathcal{T}_{21}^{\boldsymbol{N}_j}\Gamma) \cdot \sum_k \sum_{a=1}^{N_{ijk}} \frac{{}_a c_{ij}{}^k \, {}_a t_{ij}^{12k}}{(\eta_1 \cdot \eta_2)^{p_{ijk}}}$$

$$\cdot \mathcal{D}_{12}^{(d,h_{ijk}-n_a/2,n_a)}(\mathcal{T}_{12\boldsymbol{N}_k}\Gamma) * \mathcal{O}_k(\eta_2)$$

where

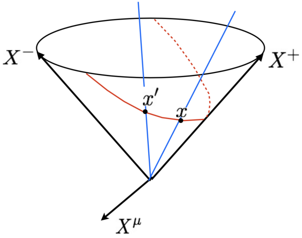
$$p_{ijk} = \frac{1}{2}(\tau_i + \tau_j - \tau_k), \qquad h_{ijk} = -\frac{1}{2}(\chi_i - \chi_j + \chi_k),$$

$$\tau_{\mathcal{O}} = \Delta_{\mathcal{O}} - S_{\mathcal{O}}, \qquad \chi_{\mathcal{O}} = \Delta_{\mathcal{O}} - \xi_{\mathcal{O}}, \qquad \xi_{\mathcal{O}} = S_{\mathcal{O}} - \lfloor S_{\mathcal{O}} \rfloor$$

- Most convenient form for computing M-point correlation functions

The Embedding Space

Embedding space $\mathcal{M}^{d+2}$:



- A natural habitat for the conformal group:
  $(d+2)$-dimensional hypercone where operators live

$$\eta^2 \equiv g_{AB}\eta^A\eta^B = 0$$

- Light rays in one-to-one correspondence with position space points

The Embedding Space (cont.)

Coordinates on the hypercone:

$$\eta^A = (\eta^\mu, \eta^{d+1}, \eta^{d+2})$$

- $\lambda\eta^A$ identified with η^A for $\lambda > 0$

Connection to position space:

$$x^\mu = \frac{\eta^\mu}{-\eta^{d+1} + \eta^{d+2}}$$

In the embedding space,

- Conformal transformations act linearly: Conformal group becomes like Lorentz group!
- All operators in d-dimensional CFT need to somehow be lifted to $\mathcal{M}^{d+2}$.

Essential Ingredients of the Formalism

- OPE differential operator $\mathcal{D}_{12}^{(d,h_{ijk}-n_a/2,n_a)}$
- Projection operators $\hat{\mathcal{P}}_{ij}^{\mathbf{N}}$
- Half-projection operators $\mathcal{T}_{12}^{\boldsymbol{N}_i}\Gamma$
- Tensor structures ${}_at_{ij}^{12k}$
- Special metric $\mathcal{A}_{ij}^{AB}$

OPE differential operator

OPE differential operator $\mathcal{D}_{12}^{(d,h_{ijk}-n_a/2,n_a)}$ given by

$$\mathcal{D}_{ij}^{(d,h,n)A_1\cdots A_n} = \frac{1}{(\eta_1\cdot\eta_2)^{\frac{n}{2}}}\mathcal{D}_{ij}^{2(h+n)}\eta_j^{A_1}\cdots\eta_j^{A_n},$$
$$\mathcal{D}_{ij}^2 = (\eta_i\cdot\eta_j)\partial_j^2 - (d+2\eta_j\cdot\partial_j)\eta_i\cdot\partial_j$$

- Explicit action of this operator known for any relevant quantity!
- Consequence: Its action can be accounted for by simple substitution rules on specific quantities
- Useful for computation of conformal blocks

Ferrara et al. (1971, 1973), Fortin, Skiba (2019)

Projection Operators

- Projection operators: $\hat{\mathcal{P}}^{\boldsymbol{N}}_{ij}$ in place to restrict operators to the proper representations

Operators satisfy the essential properties:

1. the projection property $\hat{\mathcal{P}}^{\boldsymbol{N}} \cdot \hat{\mathcal{P}}^{\boldsymbol{N}'} = \delta_{\boldsymbol{N}'\boldsymbol{N}} \hat{\mathcal{P}}^{\boldsymbol{N}}$,
2. the completeness relation $\sum_{\boldsymbol{N}|n_v \text{ fixed}} \hat{\mathcal{P}}^{\boldsymbol{N}} = \mathbb{1} - \text{traces}$,
3. the tracelessness condition $g \cdot \hat{\mathcal{P}}^{\boldsymbol{N}} = \gamma \cdot \hat{\mathcal{P}}^{\boldsymbol{N}} = \hat{\mathcal{P}}^{\boldsymbol{N}} \cdot g = \hat{\mathcal{P}}^{\boldsymbol{N}} \cdot \gamma = 0$

with n_v the total number of vector indices

Half-Projection Operators

- Half-projection operators $(\mathcal{T}^{\boldsymbol{N}})^{\mu_1\cdots\mu_{n_v}\delta}_{\alpha_1\cdots\alpha_n}$ to general irreps $\boldsymbol{N}$

$$n = 2S = 2\sum_{i=1}^{r-1} N_i + N_r , \quad n_v = \sum_{i=1}^{r-1} iN_i + r\lfloor N_r/2 \rfloor$$

- δ spinor index only present for odd N_r in odd d
- Encode transformation properties of operators $\mathcal{O}^{\boldsymbol{N}}$, $\mathcal{O}^{\boldsymbol{N}} \sim \mathcal{T}^{\boldsymbol{N}}$

$$\mathcal{O}^{\boldsymbol{N}}_{\alpha_1\cdots\alpha_n} = (\mathcal{T}^{\boldsymbol{N}})^{\delta\mu_{n_v}\cdots\mu_1}_{\alpha_1\cdots\alpha_n}\mathcal{O}^{\boldsymbol{N}}_{\mu_1\cdots\mu_{n_v}\delta},$$

$$\mathcal{O}^{\boldsymbol{N}}_{\mu_1\cdots\mu_{n_v}\delta} = (\mathcal{T}_{\boldsymbol{N}})^{\alpha_n\cdots\alpha_1}_{\mu_1\cdots\mu_{n_v}\delta}\mathcal{O}^{\boldsymbol{N}}_{\alpha_1\cdots\alpha_n}$$

Half-Projection Operators (cont.)

- Essentially square roots of projection operators:

$$\mathcal{T}_{\boldsymbol{N}} * \mathcal{T}^{\boldsymbol{N}} = \hat{\mathcal{P}}^{\boldsymbol{N}}$$

- Are transverse objects to match the transversality of operators
- Serve to translate the spinor indices carried by each operator to "dummy" vector and spinor indices

Tensor Structures

Tensor structures ${}_a t^{12}_{ijk}$ are

- Determined by three irreps of operators in 3-point function $\langle \mathcal{O}^{\boldsymbol{N}_i} \mathcal{O}^{\boldsymbol{N}_j} \mathcal{O}^{\boldsymbol{N}_k} \rangle$
- Serve to intertwine $\boldsymbol{N}_i$, $\boldsymbol{N}_j$, and $\boldsymbol{N}_k$ into a symmetric traceless representation
- Number N_{ijk} of symmetric irreducible representations appearing in $\boldsymbol{N}_i \otimes \boldsymbol{N}_j \otimes \boldsymbol{N}_k$ matches number of OPE coefficients
- Set of all ${}_a t^{12}_{ijk}$ forms basis for a vector space

Embedding Space Metric

For general irreps of the Lorentz group, necessary to properly remove traces!

- For this, require a new embedding space metric:

$$\mathcal{A}_{ij}^{AB} = g^{AB} - \frac{\eta_i^A \eta_j^B}{(\eta_i \cdot \eta_j)} - \frac{\eta_i^B \eta_j^A}{(\eta_i \cdot \eta_j)}$$

Embedding Space Metric (cont.)

Special metric is **doubly-transverse** and **symmetric**:

$$\mathcal{A}_{ij}^{AB} = \mathcal{A}_{ij}^{BA} = \mathcal{A}_{ji}^{AB} = \mathcal{A}_{ji}^{BA},$$

$$\eta_{iA}\mathcal{A}_{ij}^{AB} = \eta_{jA}\mathcal{A}_{ij}^{AB} = 0,$$

$$\mathcal{A}_{ij}^{AC}\mathcal{A}_{ijC}{}^{B} = \mathcal{A}_{ij}^{AB},$$

Same trace as in position space:

$$\mathcal{A}_{ijA}{}^{A} = d$$

From Position Space to Embedding Space

Building blocks:

- $g^{\mu\nu}$
- $\epsilon^{\mu_1\cdots\mu_d}$
- $\gamma^{\mu_1\cdots\mu_n}$

Relationship between position-space and embedding space quantities:

$$g^{\mu\nu} \to \mathcal{A}_{12}^{AB} = g^{AB} - \frac{\eta_1^A\eta_2^B}{(\eta_1\cdot\eta_2)} - \frac{\eta_1^B\eta_2^A}{(\eta_1\cdot\eta_2)},$$

$$\epsilon^{\mu_1\cdots\mu_d} \to \epsilon_{12}^{A_1\cdots A_d} = \frac{1}{(\eta_1\cdot\eta_2)}\eta_{1A_0'}\epsilon^{A_0'A_1'\cdots A_d'A_{d+1}'}\eta_{2A_{d+1}'}\mathcal{A}_{12A_d'}{}^{A_d}\cdots\mathcal{A}_{12A_1'}{}^{A_1},$$

$$\gamma^{\mu_1\cdots\mu_n} \to \Gamma_{12}^{A_1\cdots A_n} = \Gamma^{A_1'\cdots A_n'}\mathcal{A}_{12A_n'}{}^{A_n}\cdots\mathcal{A}_{12A_1'}{}^{A_1} \qquad \forall\, n\in\{0,\ldots,r\}.$$

Three-Point Correlation Functions

Most general embedding space 3-point function:

$$\langle \mathcal{O}_i(\eta_1)\mathcal{O}_j(\eta_2)\mathcal{O}_m(\eta_3)\rangle = \frac{(\mathcal{T}_{12}^{\boldsymbol{N}_i}\Gamma)^{\{Aa\}}(\mathcal{T}_{21}^{\boldsymbol{N}_j}\Gamma)^{\{Bb\}}(\mathcal{T}_{31}^{\boldsymbol{N}_m}\Gamma)^{\{Ee\}}}{(\eta_1\cdot\eta_2)^{\frac{1}{2}(\tau_i+\tau_j-\chi_m)}(\eta_1\cdot\eta_3)^{\frac{1}{2}(\chi_i-\chi_j+\tau_m)}(\eta_2\cdot\eta_3)^{\frac{1}{2}(-\chi_i+\chi_j+\chi_m)}} \cdot \sum_{a=1}^{N_{ijm}} {}_a c_{ijm} (\mathscr{G}_{(a|}^{ij|m})_{\{aA\}\{bB\}\{eE\}}$$

- $(\mathscr{G}_{(a|}^{ij|m})_{\{aA\}\{bB\}\{eE\}}$ - "3-point" conformal blocks

Four-Point Correlation Functions

Most general embedding space 4-point function:

$$\langle \mathcal{O}_i(\eta_1)\mathcal{O}_j(\eta_2)\mathcal{O}_k(\eta_3)\mathcal{O}_l(\eta_4)\rangle =$$
$$\frac{(\mathcal{T}_{12}^{\boldsymbol{N}_i}\Gamma)^{\{Aa\}}(\mathcal{T}_{21}^{\boldsymbol{N}_j}\Gamma)^{\{Bb\}}(\mathcal{T}_{34}^{\boldsymbol{N}_k}\Gamma)^{\{Cc\}}(\mathcal{T}_{43}^{\boldsymbol{N}_l}\Gamma)^{\{Dd\}}}{(\eta_1\cdot\eta_2)^{\frac{1}{2}\alpha_{12}}(\eta_1\cdot\eta_3)^{\frac{1}{2}\alpha_{13}}(\eta_1\cdot\eta_4)^{\frac{1}{2}\alpha_{14}}(\eta_3\cdot\eta_4)^{\frac{1}{2}\alpha_{34}}}$$
$$\cdot\sum_m\sum_{a=1}^{N_{ijm}}\sum_{b=1}^{N_{klm}}{}_ac_{ij}{}^m{}_b\alpha_{klm}(\mathscr{G}_{(a|b]}^{ij|m|kl})_{\{aA\}\{bB\}\{cC\}\{dD\}}$$

with

$$\alpha_{12}=(\tau_i-\chi_i+\tau_j+\chi_j),\quad \alpha_{13}=(\chi_i-\chi_j+\chi_k-\chi_l),$$
$$\alpha_{14}=(\chi_i-\chi_j-\chi_k+\chi_l),\quad \alpha_{34}=(-\chi_i+\chi_j+\tau_k+\tau_l)$$

- $(\mathscr{G}_{(a|b]}^{ij|m|kl})_{\{aA\}\{bB\}\{cC\}\{dD\}}$ - “4-point” conformal blocks

Bases of Tensor Structures

Two kinds of bases arise naturally in the context of the formalism:

1. OPE basis $(a$
2. Three-point basis $[a$

Three-point blocks in the two bases related via rotation matrices

$$\mathscr{G}_{(a|}^{ij|m} = \sum_{a'=1}^{N_{ijm}} (R_{ijm}^{-1})_{aa'} \mathscr{G}_{[a'|}^{ij|m}, \quad {}_a c_{ijm} = \sum_{a'=1}^{N_{ijm}} {}_{a'}\alpha_{ijm} (R_{ijm})_{a'a}$$

where ${}_a\alpha_{ijm}$ are the associated 3-point function coefficients, implying

$$\sum_{a=1}^{N_{ijm}} {}_a c_{ijm} \mathscr{G}_{(a|}^{ij|m} = \sum_{a=1}^{N_{ijm}} {}_a\alpha_{ijm} \mathscr{G}_{[a|}^{ij|m}$$

Bases of Tensor Structures (cont.)

- Three-point basis $[a$ is the natural one for 3-point functions!

- 3-point conformal blocks in this basis:

$$\mathscr{G}_{[a|}^{ij|m} = \bar{\eta}_3 \cdot \Gamma \, {}_aF_{ijm}^{12}(\mathcal{A}_{12}, \Gamma_{12}, \epsilon_{12}; \mathcal{A}_{12} \cdot \bar{\eta}_3)$$

- $\bar{\eta}_3 \cdot \Gamma$ appears only if $\xi_k = \frac{1}{2}$, *i.e.* the exchanged quasi-primary operator is fermionic

Arbitrary 3-point functions simply obtained by enumerating basis $\{{}_aF_{ijm}^{12}\}$ made from

- $\mathcal{A}_{12}$'s
- Γ_{12}'s
- ϵ_{12}'s
- $\mathcal{A}_{12} \cdot \bar{\eta}_3$'s

Bases of Tensor Structures (cont.)

- Conformal blocks feature simplest form in mixed OPE-three-point basis: $\mathscr{G}^{ij|m|kl}_{(a|b]}$ (Fortin, VP, Skiba (2019))
- For the conformal bootstrap: most convenient to work in the pure three-point basis
- Pure three-point blocks obtained from mixed ones via

$$\mathscr{G}^{ij|m|kl}_{[a|b]} = \sum_{a'=1}^{N_{ijm}} (R_{ijm})_{aa'} \mathscr{G}^{ij|m|kl}_{(a'|b]}$$

So, strategy is to determine

1. Mixed basis blocks $\mathscr{G}^{ij|m|kl}_{(a|b]}$
2. Rotation matrices $(R_{ijm})_{aa'}$

Tensor Structures for Towers of Exchanged Operators

- Consider tensor structures for exchanged towers of quasi-primary operators $\boldsymbol{N}_m + \ell \boldsymbol{e}_1$

- If seed irrep $\boldsymbol{N}_m + \ell_{min} \boldsymbol{e}_1$ can be exchanged, so can $\boldsymbol{N}_m + \ell \boldsymbol{e}_1$ for any $\ell \geq \ell_{min}$

Idea:

1. Take ℓ-dependence into account once and for all (fixed)
2. Just compute seed part $\boldsymbol{N}_m + \ell_{min} \boldsymbol{e}_1$ (varies)

- Both $\boldsymbol{N}_m$ and ℓ_{min} depend on the irreps of the operators of interest

Tensor Structures for Towers of Exchanged Operators (cont.)

- Therefore, for exchanged quasi-primary operators in $\boldsymbol{N}_m + \ell \boldsymbol{e}_1$, three-point basis can be separated as

$$ {}_bF^{34}_{kl,m+\ell} = {}_bF^{34}_{kl,m+i_b}(\mathcal{A}_{34} \cdot \bar{\bar{\eta}}_2)^{\ell-i_b}, $$

$$ {}_aF^{12}_{ij,m+\ell} = {}_aF^{12}_{ij,m+i_a}(\mathcal{A}_{12} \cdot \bar{\eta}_3)^{\ell-i_a} \to {}_at^{12}_{ij,m+\ell} = {}_at^{12}_{ij,m+i_a}(\mathcal{A}_{12})^{\ell-i_a} $$

  with

- $(\mathcal{A}_{34} \cdot \bar{\bar{\eta}}_2)_{E''_{i_b+1}} \cdots (\mathcal{A}_{34} \cdot \bar{\bar{\eta}}_2)_{E''_\ell}$
- $(\mathcal{A}_{12} \cdot \bar{\eta}_3)_{E_{i_a+1}} \cdots (\mathcal{A}_{12} \cdot \bar{\eta}_3)_{E_\ell}$

the symmetrized ℓ-dependent parts of the respective tensor structures

Tensor Structures for Towers of Exchanged Operators (cont.)

- "Special" parts of tensor structures ${}_a t^{12}_{ij,m+i_a}$ and ${}_b F^{34}_{kl,m+i_b}$ fixed by knowledge of the specific irreps in question

- **OPE basis** obtained from **three-point basis** by replacing $\mathcal{A}_{12} \cdot \bar{\eta}_3 \to \mathcal{A}_{12}$

- with the extra F index contracting with the OPE differential operator

For example,

$$(\mathcal{A}_{12} \cdot \bar{\eta}_3)_{E_{i_a+1}} \cdots (\mathcal{A}_{12} \cdot \bar{\eta}_3)_{E_\ell} \to \mathcal{A}_{12E'_{i_a+1}F_{i_a+1}} \cdots \mathcal{A}_{12E'_\ell F_\ell}$$

Tensor Structures: An Example

Case of symmetric traceless $\ell \boldsymbol{e}_1$ exchange in $\langle SVSV \rangle$: Tensor structures are

$$b=1: \quad ({}_bF^{34}_{kl,m+\ell})_{\{cC\}\{dD\}\{e''E''\}} = (\mathcal{A}_{34}\cdot\bar{\bar{\eta}}_2)_D[(\mathcal{A}_{34}\cdot\bar{\bar{\eta}}_2)_{E''}]^\ell$$
$$\to ({}_bt^{34}_{kl,m+\ell})_{\{cC\}\{dD\}\{e''E''\}\{F''\}} = \mathcal{A}_{34DF''}(\mathcal{A}_{34E''F''})^\ell,$$
$$b=2: \quad ({}_bF^{34}_{kl,m+\ell})_{\{cC\}\{dD\}\{e''E''\}} = \mathcal{A}_{34DE''_1}[(\mathcal{A}_{34}\cdot\bar{\bar{\eta}}_2)_{E''}]^{\ell-1}$$
$$\to ({}_bt^{34}_{kl,m+\ell})_{\{cC\}\{dD\}\{e''E''\}\{F''\}} = \mathcal{A}_{34DE''_1}(\mathcal{A}_{34E''F''})^{\ell-1},$$
$$a=1: \quad ({}_aF^{12}_{ij,m+\ell})_{\{aA\}\{bB\}\{eE\}} = (\mathcal{A}_{12}\cdot\bar{\eta}_3)_B[(\mathcal{A}_{12}\cdot\bar{\eta}_3)_E]^\ell$$
$$\to ({}_at^{12,m+\ell}_{ij})_{\{aA\}\{bB\}}{}^{\{Ee\}\{F\}} = \mathcal{A}_{12B}{}^F(\mathcal{A}_{12}^{EF})^\ell,$$
$$a=2: \quad ({}_aF^{12}_{ij,m+\ell})_{\{aA\}\{bB\}\{eE\}} = \mathcal{A}_{12BE_1}[(\mathcal{A}_{12}\cdot\bar{\eta}_3)_E]^{\ell-1}$$
$$\to ({}_at^{12,m+\ell}_{ij})_{\{aA\}\{bB\}}{}^{\{Ee\}\{F\}} = \mathcal{A}_{12B}{}^{E_1}(\mathcal{A}_{12}^{EF})^{\ell-1}$$

Tensor Structures: An Example (cont.)

Only interested in information about the special parts

$$
\begin{aligned}
b = 1: &\qquad n_b = \ell + 1, &\qquad i_b = 0, &\qquad ({}_b t^{34}_{klm})_{DF''} = \mathcal{A}_{34DF''},\\
b = 2: &\qquad n_b = \ell - 1, &\qquad i_b = 1, &\qquad ({}_b t^{34}_{kl,m+1})_{DE_1''} = \mathcal{A}_{34DE_1''},\\
a = 1: &\qquad n_a = \ell + 1, &\qquad i_a = 0, &\qquad ({}_a t^{12m}_{ij})_B{}^F = \mathcal{A}_{12B}{}^F,\\
a = 2: &\qquad n_a = \ell - 1, &\qquad i_a = 1, &\qquad ({}_a t^{12,m+1}_{ij})_B{}^{E_1} = \mathcal{A}_{12B}{}^{E_1}.
\end{aligned}
$$

Tensor Structures: An Example

Case of $\boldsymbol{e}_r + \ell \boldsymbol{e}_1$ exchange in $\langle SFSF \rangle$:

$$
\begin{aligned}
b = 1: \quad & ({}_bF^{34}_{kl,m+\ell})_{\{cC\}\{dD\}\{e''E''\}} = (C_\Gamma^{-1})_{de''}[(\mathcal{A}_{34}\cdot\bar{\bar{\eta}}_2)_{E''}]^\ell \\
& \to ({}_bt^{34}_{kl,m+\ell})_{\{cC\}\{dD\}\{e''E''\}\{F''\}} = (C_\Gamma^{-1})_{de''}(\mathcal{A}_{34E''F''})^\ell, \\
b = 2: \quad & ({}_bF^{34}_{kl,m+\ell})_{\{cC\}\{dD\}\{e''E''\}} = (\bar{\bar{\eta}}_2\cdot\Gamma_{34}C_\Gamma^{-1})_{de''}[(\mathcal{A}_{34}\cdot\bar{\bar{\eta}}_2)_{E''}]^\ell \\
& \to ({}_bt^{34}_{kl,m+\ell})_{\{cC\}\{dD\}\{e''E''\}\{F''\}} = (\Gamma_{34F''}C_\Gamma^{-1})_{de''}(\mathcal{A}_{34E''F''})^\ell \\
a = 1: \quad & ({}_aF^{12}_{ij,m+\ell})_{\{aA\}\{bB\}\{eE\}} = (C_\Gamma^{-1})_{be}[(\mathcal{A}_{12}\cdot\bar{\eta}_3)_E]^\ell \\
& \to ({}_at^{12,m+\ell}_{ij})_{\{aA\}\{bB\}}{}^{\{Ee\}\{F\}} = \delta_b{}^e(\mathcal{A}^{EF}_{12})^\ell, \\
a = 2: \quad & ({}_aF^{12}_{ij,m+\ell})_{\{aA\}\{bB\}\{eE\}} = (\bar{\eta}_3\cdot\Gamma_{12}C_\Gamma^{-1})_{be}[(\mathcal{A}_{12}\cdot\bar{\eta}_3)_E]^\ell \\
& \to ({}_at^{12,m+\ell}_{ij})_{\{aA\}\{bB\}}{}^{\{Ee\}\{F\}} = (\Gamma^F_{12})_b{}^e(\mathcal{A}^{EF}_{12})^\ell
\end{aligned}
$$

Tensor Structures: An Example (cont.)

Only interested in information about the special parts

$$
\begin{aligned}
b = 1: &\quad n_b = \ell, \quad i_b = 0, \quad ({}_b t^{34}_{klm})_{de''} = (C_\Gamma^{-1})_{de''}, \\
b = 2: &\quad n_b = \ell + 1, \quad i_b = 0, \quad ({}_b t^{34}_{klm})_{de''F''} = (\Gamma_{34F''} C_\Gamma^{-1})_{de''}, \\
a = 1: &\quad n_a = \ell, \quad i_a = 0, \quad ({}_a t^{12m}_{ij})_b{}^e = \delta_b{}^e, \\
a = 2: &\quad n_a = \ell + 1, \quad i_a = 0, \quad ({}_a t^{12m}_{ij})_b{}^{eF} = (\Gamma^F_{12})_b{}^e.
\end{aligned}
$$

Projection Operators to Exchanged Representations

- Projection operator to exchanged irreps appears explicitly in conformal block
- Need $\hat{\mathcal{P}}_{13}^{\boldsymbol{N}_m+\ell\boldsymbol{e}_1}$

Useful to decompose operators as

$$\hat{\mathcal{P}}_{13}^{\boldsymbol{N}_m+\ell\boldsymbol{e}_1}=\sum_t \mathscr{A}_t(d,\ell)\hat{\mathcal{Q}}_{13|t}^{\boldsymbol{N}_m+\ell_t\boldsymbol{e}_1}\hat{\mathcal{P}}_{13|d+d_t}^{(\ell-\ell_t)\boldsymbol{e}_1}$$

- Coefficients $\mathscr{A}_t(d,\ell)$ are constants
- Sum is finite and ℓ-independent
- Number of terms depends on irrep $\boldsymbol{N}_m$

Projection Operators to Exchanged Representations (cont.)

- Tensor quantities $\hat{\mathcal{Q}}_{13|t}^{\boldsymbol{N}_m+\ell_t\boldsymbol{e}_1}$ encode information about the special parts of the irrep $\boldsymbol{N}_m+\ell_t\boldsymbol{e}_1$
- $\mathscr{A}_t(d,\ell)$ and $\hat{\mathcal{Q}}_{13|t}^{\boldsymbol{N}_m+\ell_t\boldsymbol{e}_1}$ fixed by details of specific exchanged irrep
- Remaining indices carried by shifted projection operators for some d' and ℓ'

$$(\hat{\mathcal{P}}_{13|d'}^{\ell'\boldsymbol{e}_1})_{E_\ell'\cdots E_1'}{}^{E_1''\cdots E_\ell''}=\sum_{i=0}^{\lfloor \ell'/2\rfloor}\frac{(-\ell')_{2i}}{2^{2i}i!(-\ell'+2-d'/2)_i}$$

$$\times\mathcal{A}_{13(E_1'E_2'}\mathcal{A}_{13}^{(E_1''E_2''}\cdots\mathcal{A}_{13E_{2i-1}'E_{2i}'}\mathcal{A}_{13}^{E_{2i-1}''E_{2i}''}\mathcal{A}_{13E_{2i+1}'}{}^{E_{2i+1}''}\cdots\mathcal{A}_{13E_\ell')}{}^{E_{\ell'}'')}$$

- Shifted projectors **not** traceless when $d_t\neq 0$
- Special indices in special parts need to be extracted
- For this, derived general index separation result for $\hat{\mathcal{P}}_{13|d'}^{\ell'\boldsymbol{e}_1}$

Projection Operators: An Example

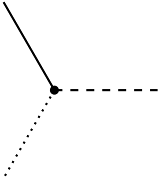
The projection operator to $\boldsymbol{e}_2 + \ell \boldsymbol{e}_1$ can be decomposed in terms of shifted projectors as (in position space)

t	(d_t, ℓ_t)	$\mathscr{A}_t(d,\ell)$	$\hat{\mathcal{Q}}_t$
1	$(2,0)$	$\frac{2}{\ell+2}$	$g_{[\nu_1}{}^{\nu_1'} \cdots g_{\nu_2]}{}^{\nu_2'}$
2	$(4,1)$	$\frac{2\ell}{\ell+2}$	$g_{[\nu_1}{}^{[\nu_1'} g_{\nu_2]}{}^{\mu'} g_{\mu}{}^{\nu_2']}$
3	$(4,1)$	$\frac{2\ell}{\ell+2}$	$g_{[\nu_1}{}^{[\nu_1'} g_{\nu_2]\mu} g^{\nu_2']\mu'}$
4	$(2,1)$	$-\frac{2\ell(-\ell-d/2)(d+\ell-1)}{(\ell+2)(-\ell+1-d/2)(d+\ell-2)}$	$g_{[\nu_1}{}^{[\nu_1'} g_{\nu_2]\mu} g^{\nu_2']\mu'}$
5	$(4,2)$	$-\frac{2\ell(\ell-1)(-\ell-d/2)}{(\ell+2)(-\ell+1-d/2)(d+\ell-2)}$	$g_{[\nu_1\mu} g^{[\nu_1'\mu'} g_{\nu_2]}{}^{\mu'} g_{\mu}{}^{\nu_2']}$
6	$(4,2)$	$-\frac{2\ell(\ell-1)}{2(\ell+2)(-\ell+1-d/2)}$	$\left(g_{[\nu_1\mu} g_{\nu_2]}{}^{[\nu_1'} g_{\mu}{}^{\nu_2']} g^{\mu'\mu'} + g^{[\nu_1'\mu'} g_{[\nu_1}{}^{\nu_2']} g_{\nu_2]}{}^{\mu'} g_{\mu\mu} \right)$

Diagrammatic Notation

Introduce convenient diagrammatic notation for index separation:

- We symbolize shifted projection operator by the vertex

$$(\hat{\mathcal{P}}^{\ell \boldsymbol{e}_1}_{13|d})_{\{E'\}}{}^{\{E''\}} =$$ 

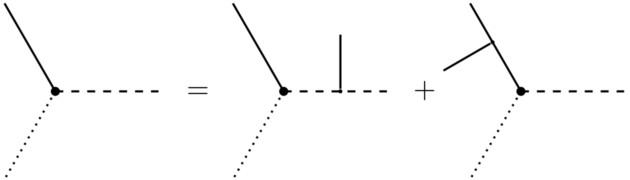
- Solid, dotted, dashed lines represent metrics of the form $\mathcal{A}_{13E'E'}$, $\mathcal{A}_{13}^{E''E''}$, and $\mathcal{A}_{13E'}{}^{E''}$, respectively
- A line is associated to metrics with one special index, a loop to metrics with two special indices

Diagrammatic Notation for Separation of Special Indices

For example, the index separation identity

$$(\hat{\mathcal{P}}^{\ell \boldsymbol{e}_1}_{13|d})_{\{E'\}}{}^{\{E''\}} = \mathcal{A}_{13E'_s}{}^{(E''}(\hat{\mathcal{P}}^{(\ell-1)\boldsymbol{e}_1}_{13|d+2})_{\{E'\}}{}^{\{E''\})}$$
$$+ \frac{\ell-1}{2(-\ell+2-d/2)}\mathcal{A}_{13E'_s(E'}\mathcal{A}_{13}^{(E''E''}(\hat{\mathcal{P}}^{(\ell-2)\boldsymbol{e}_1}_{13|d+2})_{\{E'\})}{}^{\{E''\})}$$

is represented as



Rule for the Rotation Matrix

Can determine the rotation matrix from the relation

$$\mathscr{G}^{ij|m+\ell}_{(a|} = \sum_{a'=1}^{N_{ij,m+\ell}} (R^{-1}_{ij,m+\ell})_{aa'}\, \bar{\eta}_3 \cdot \Gamma_{\ a'} F^{12}_{ij,m+\ell}(\mathcal{A}_{12}, \Gamma_{12}, \epsilon_{12}; \mathcal{A}_{12} \cdot \bar{\eta}_3)$$

$$= \sum_{a'=1}^{N_{ij,m+\ell}} (R^{-1}_{ij,m+\ell})_{aa'}\, \bar{\eta}_3 \cdot \Gamma_{\ a'} F^{12}_{ij,m+i_{a'}}(\mathcal{A}_{12}, \Gamma_{12}, \epsilon_{12}; \mathcal{A}_{12} \cdot \bar{\eta}_3)(\mathcal{A}_{12} \cdot \bar{\eta}_3)^{\ell - i_{a'}},$$

using the symmetry properties of the irreps of the three quasi-primary operators in question

Rule for the Rotation Matrix (cont.)

Rotation matrix determined from

$$\sum_{a'=1}^{N_{ij,m+\ell}} (R^{-1}_{ij,m+\ell})_{aa'} ({}_{a'}F^{12}_{ij,m+i_{a'}})_{\{aA\}\{bB\}\{eE\}\{F\}} (\mathcal{A}_{12}\cdot\bar{\eta}_3)^{\ell-i_{a'}}$$

$$= (-1)^{2\xi_m(r+1)} ({}_a t^{12}_{ij,m+i_a})_{\{aA\}\{bB\}\{e'E'\}\{F\}} (C_\Gamma \Gamma_F C_\Gamma^{-1})^{e'}{}_e \times$$

$$\sum_{r_0,r_3,s_0,s_3,t\geq 0}\ \sum_{q_0,q_1,q_2,q_3\geq 0} {}_a\mathcal{K}^{ij|m+\ell}_{(\boldsymbol{q},r_0,r_3,s_0,s_3,t)}$$

$$\times \sum_\sigma g_{E_{\sigma(1)}}{}^{E'_{\sigma(1)}} \cdots g_{E_{\sigma(r_0)}}{}^{E'_{\sigma(r_0)}} \bar{\eta}_3^{E'_{\sigma(r_0+1)}} \cdots \bar{\eta}_3^{E'_{\sigma(r_0+r_3)}} (g_E^{(Z})^{s_0} (\bar{\eta}_3^{Z})^{s_3}$$

$$\times S_{(q_0,q_1,q_2,q_3)}{}^{Z^{n_v^{m+i_a}+2\xi_m+n_a-\ell_a-r_0-r_3-s_0-s_3})}{}_{E_\sigma^{n_v^{m+i_a}-r_0}} (-\bar{\eta}_{2E})^{\ell_a-s_0}$$

Main Elements

- Totally symmetric S-tensor: structure built from g's, $\bar{\eta}_1$s, $\bar{\eta}_2$s, $\bar{\eta}_3$s

$$
\begin{aligned}
S^{A_1\cdots A_{\bar{q}}}_{(q_0,q_1,q_2,q_3)} &= g^{(A_1A_2}\cdots g^{A_{2q_0-1}A_{2q_0}}\bar{\eta}_1^{A_{2q_0+1}}\cdots\bar{\eta}_1^{A_{2q_0+q_1}} \\
&\quad\times\bar{\eta}_2^{A_{2q_0+q_1+1}}\cdots\bar{\eta}_2^{A_{2q_0+q_1+q_2}}\bar{\eta}_3^{A_{2q_0+q_1+q_2+1}}\cdots\bar{\eta}_3^{A_{\bar{q}})}, \\
\bar{q} &= 2q_0+q_1+q_2+q_3
\end{aligned}
$$

- Z indices $Z\in\{E'_{\sigma(r_0+r_3+1)},\ldots,E'_{\sigma(n_v^{m+i_a})},F^{n_a-\ell_a+2\xi_m}\}$
- ${}_a\kappa^{ij|m+\ell}_{(\boldsymbol{q},r_0,r_3,s_0,s_3,t)}$ coefficients comprised from various Pochhammer symbols, e.g. $(\Delta_{m+\ell}+n_v^m+\xi_m+\ell-r_0)_{h_{ij,m+\ell}+n_a/2-\ell+i_a-s_0+t-q_0-q_1}$

Rule for Conformal Blocks in the Mixed Basis

Conformal blocks in the mixed basis $(a|b]$ given by

$$
\begin{aligned}
(\mathscr{G}_{(a|b]}^{ij|m+\ell|kl})_{\{aA\}\{bB\}\{cC\}\{dD\}} &= \sum_t \mathscr{A}_t(d,\ell) \sum_{j_a,j_b\geq 0} \binom{i_a}{j_a}\binom{i_b}{j_b}\times \\
&\frac{(-\ell_t)_{i_a-j_a}(-\ell_t)_{i_b-j_b}(-\ell+\ell_t)_{j_a}(-\ell+\ell_t)_{j_b}}{(-\ell)_{i_a}(-\ell)_{i_b}} \\
&\times \sum_{\substack{r,\boldsymbol{r}',\boldsymbol{r}''\geq 0\\ r+2r_0'+r_1'+r_2'=j_a\\ r+2r_0''+r_1''+r_2''=j_b\\ r_0'+r_1'+r_3'=r_0''+r_1''+r_3''}} (-1)^{\ell-\ell'-i_a+r_1'+r_2'}\frac{(-2)^{r_3'+r_3''}\ell'!}{(d'/2-1)_{\ell'}} \\
&\times \mathscr{C}_{j_a,j_b}^{(d+d_t,\ell-\ell_t)}(r,\boldsymbol{r}',\boldsymbol{r}'')\left(C_{\ell'}^{(d'/2-1)}(X)\right)_{s_{(a|b)}^{ij|m+\ell|kl}(t,j_a,j_b,r,\boldsymbol{r}',\boldsymbol{r}'')}
\end{aligned}
$$

Rule for Conformal Blocks in the Mixed Basis (cont.)

With the associated substitution rule

$$
s^{ij|m+\ell|kl}_{(a|b)}(t, j_a, j_b, r, \boldsymbol{r}', \boldsymbol{r}'') : \alpha_2^{s_2}\alpha_3^{s_3}\alpha_4^{s_4}x_3^{r_3}x_4^{r_4} \to \mathrm{Sym}_{\{E_s'\},\{E_s''\}}
$$

$$
(-1)^{2\xi_m}\left({}_a t^{12,m+i_a}_{ij}\right)_{\{aA\}\{bB\}}{}^{\{Ee\}\{F\}}(g_{E_sE_s})^{r_0'}(\mathcal{S}_{E_s}{}^{E_s''})^{r}[(\mathcal{S}\cdot\bar{\eta}_4)_{E_s}]^{r_2'}
$$

$$
\left(G^{ij|m+\ell|kl}_{(\ell'-\ell+2r_3',n_a-\ell,n_3',n_4',n_5')}\right)_{F^{n_a-\ell+i_a}E_s^{r_1'}}{}^{E_s''^{r_1''}F''^{4\xi_m}F''^{n_b-\ell+i_b}F''^{\ell_t-i_b+j_b}}(\bar{\eta}_2^{E})^{\ell_t-i_a+j_a}
$$

$$
(\Gamma_{F''}\bar{\eta}_3\cdot\Gamma\,\mathcal{S}^{n_v^m+\ell_t}\hat{\mathcal{Q}}^{\boldsymbol{N}_m+\ell_t\boldsymbol{e}_1}_{13|t}\,\Gamma_{F''})_{eE^{n_v^m}(E^{\ell_t-i_a+j_a}E_s^{i_a-j_a})}{}^{(E_s''^{i_b-j_b}E''^{\ell_t-i_b+j_b})E''^{n_v^m}e''}
$$

$$
[(\bar{\eta}_2\cdot\mathcal{S})^{E_s''}]^{r_2''}(g^{E_s''E_s''})^{r_0''}\left({}_b t^{34}_{kl,m+i_b}\right)_{\{cC\}\{dD\}\{e''E''\}\{F''\}}(\mathcal{A}_{34E''F''})^{\ell_t-i_b+j_b}
$$

Main Elements

- Gegenbauer polynomials $C_n^{(\lambda)}(X)$ in the special variable

$$X = \frac{(\alpha_4 - \alpha_2)x_4 - (\alpha_3 - \alpha_2)x_3}{2}, \qquad x_3 = \frac{u}{v}, \quad x_4 = u$$

- $G^{ij|m+\ell|kl}_{(n_1,n_2,n_3,n_4,n_5)A_1\cdots A_n}$ – related to tensorial generalization of Exton G function that appears in $\langle SSSS \rangle$ scalar exchange blocks
- $G^{ij|m+\ell|kl}_{(n_1,n_2,n_3,n_4,n_5)A_1\cdots A_n}$ totally symmetric in all of its indices

Main Elements (cont.)

- Special combination of Gs is ubiquitous:

$$\mathcal{S}_A{}^B = g_A{}^B G^{ij|m+\ell|kl}_{(0,0,0,0,0)} - G^{ij|m+\ell|kl}_{(0,0,2,0,0)A}\bar{\eta}_1^B$$
$$-\bar{\eta}_{3A}(G^{ij|m+\ell|kl}_{(0,0,2,2,0)})^B + (G^{ij|m+\ell|kl}_{(0,0,4,2,0)})_A{}^B$$

- Appears either by itself or via contractions with embedding space coordinates $\bar{\eta}_2$, $\bar{\eta}_4$, e.g. as in $(\bar{\eta}_2 \cdot \mathcal{S})^{E_s''}$, $(\mathcal{S} \cdot \bar{\eta}_4)_{E_s}$
- $\Gamma_{F''}$s, $\bar{\eta}_3 \cdot \Gamma$ present only if exchanged operator is fermionic, $\xi_m = \frac{1}{2}$
- Special part $\hat{\mathcal{Q}}_{13|t}^{\boldsymbol{N}_m+\ell_t\boldsymbol{e}_1}$ of projector $\hat{\mathcal{P}}_{13}^{\boldsymbol{N}_m+\ell\boldsymbol{e}_1}$ contracts with Gs, $\mathcal{S}$s, and tensor structures

Properties of G

Substitution rules necessitate the multiplication of several G's according to

$$G^{ij|m+\ell|kl}_{(n_1,n_2,n_3,n_4,n_5)A^n} G^{ij|m+\ell|kl}_{(m_1,m_2,m_3,m_4,m_5)B^m}$$
$$= G^{ij|m+\ell|kl}_{(n_1+m_1,n_2+m_2,n_3+m_3,n_4+m_4,n_5+m_5)A^n B^m}$$

Moreover, G satisfies the contiguous relations

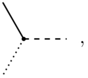
$$g \cdot G^{ij|m+\ell|kl}_{(n_1,n_2,n_3,n_4,n_5)} = 0,$$
$$\bar{\eta}_1 \cdot G^{ij|m+\ell|kl}_{(n_1,n_2,n_3,n_4,n_5)} = G^{ij|m+\ell|kl}_{(n_1,n_2,n_3-2,n_4,n_5)},$$
$$\bar{\eta}_2 \cdot G^{ij|m+\ell|kl}_{(n_1,n_2,n_3,n_4,n_5)} = G^{ij|m+\ell|kl}_{(n_1+2,n_2,n_3,n_4,n_5)},$$
$$\bar{\eta}_3 \cdot G^{ij|m+\ell|kl}_{(n_1,n_2,n_3,n_4,n_5)} = G^{ij|m+\ell|kl}_{(n_1,n_2,n_3-2,n_4-2,n_5)},$$
$$\bar{\eta}_4 \cdot G^{ij|m+\ell|kl}_{(n_1,n_2,n_3,n_4,n_5)} = G^{ij|m+\ell|kl}_{(n_1,n_2,n_3-2,n_4,n_5+2)}$$

- Useful for facilitating contractions

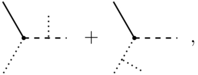
Example: ℓe_1 Exchange in $\langle SVSV \rangle$

With the aid of diagrams, it is easy to extract indices as needed

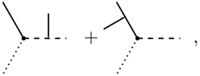
$a = 1, b = 1: \quad \hat{\mathcal{P}}^{\ell e_1}_{13|d} =$



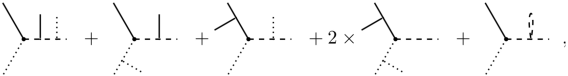
$a = 1, b = 2: \quad \hat{\mathcal{P}}^{\ell e_1}_{13|d} =$



$a = 2, b = 1: \quad \hat{\mathcal{P}}^{\ell e_1}_{13|d} =$



$a = 2, b = 2: \quad \hat{\mathcal{P}}^{\ell e_1}_{13|d} =$



Example: ℓe_1 Exchange in $\langle SVSV \rangle$ (cont.)

These directly lead to four conformal blocks expressed in terms of Gegenbauer polynomials

$$\mathscr{G}_{(1|1]}^{ij|m+\ell|kl} = \frac{\ell!}{(d/2-1)_\ell}\left(C_\ell^{(d/2-1)}(X)\right)_{s^1_{(1|1)}},$$

$$\mathscr{G}_{(1|2]}^{ij|m+\ell|kl} = -\frac{(\ell-1)!}{(d/2)_{\ell-1}}\left(C_{\ell-1}^{(d/2)}(X)\right)_{s^1_{(1|2)}} + \frac{(\ell-1)!}{(d/2)_{\ell-1}}\left(C_{\ell-2}^{(d/2)}(X)\right)_{s^2_{(1|2)}},$$

$$\mathscr{G}_{(2|1]}^{ij|m+\ell|kl} = -\frac{(\ell-1)!}{(d/2)_{\ell-1}}\left(C_{\ell-1}^{(d/2)}(X)\right)_{s^1_{(2|1)}} + \frac{(\ell-1)!}{(d/2)_{\ell-1}}\left(C_{\ell-2}^{(d/2)}(X)\right)_{s^2_{(2|1)}},$$

$$\begin{aligned}\mathscr{G}_{(2|2]}^{ij|m+\ell|kl} = {} & \frac{(\ell-1)!}{\ell(d/2+1)_{\ell-2}}\left(C_{\ell-2}^{(d/2+1)}(X)\right)_{s^1_{(2|2)}} - \frac{(\ell-1)!}{\ell(d/2+1)_{\ell-2}}\left(C_{\ell-3}^{(d/2+1)}(X)\right)_{s^2_{(2|2)}} \\ & - \frac{(\ell-1)!}{\ell(d/2+1)_{\ell-2}}\left(C_{\ell-3}^{(d/2+1)}(X)\right)_{s^3_{(2|2)}} - \frac{(\ell-1)!}{\ell(d/2)_{\ell-1}}\left(C_{\ell-2}^{(d/2)}(X)\right)_{s^4_{(2|2)}} \\ & + \frac{(\ell-1)!}{\ell(d/2+1)_{\ell-2}}\left(C_{\ell-4}^{(d/2+1)}(X)\right)_{s^5_{(2|2)}} + \frac{(\ell-1)!}{\ell(d/2)_{\ell-1}}\left(C_{\ell-1}^{(d/2)}(X)\right)_{s^6_{(2|2)}}.\end{aligned}$$

Example: ℓe_1 Exchange in $\langle SVSV \rangle$ (cont.)

A specific substitution rule is ascribed to each Gegenbauer term. For example,

$$s^1_{(1|1)} : \alpha_2^{s_2}\alpha_3^{s_3}\alpha_4^{s_4}x_3^{r_3}x_4^{r_4} \to G^{ij|m+\ell|kl}_{(0,1,3,1,-1)BD},$$

$$s^1_{(1|2)} : \alpha_2^{s_2}\alpha_3^{s_3}\alpha_4^{s_4}x_3^{r_3}x_4^{r_4} \to (\bar{\eta}_2 \cdot \mathcal{S})_D\, G^{ij|m+\ell|kl}_{(-1,1,0,0,0)B},$$

$$s^1_{(2|1)} : \alpha_2^{s_2}\alpha_3^{s_3}\alpha_4^{s_4}x_3^{r_3}x_4^{r_4} \to (\mathcal{S} \cdot \bar{\eta}_4)_B\, G^{ij|m+\ell|kl}_{(-1,-1,2,0,-2)D},$$

$$s^2_{(2|1)} : \alpha_2^{s_2}\alpha_3^{s_3}\alpha_4^{s_4}x_3^{r_3}x_4^{r_4} \to G^{ij|m+\ell|kl}_{(-2,-1,3,1,-1)BD}$$

Example: ℓe_1 Exchange in $\langle SVSV \rangle$ (cont.)

Rotation matrix for this case:

$$(R^{-1}_{ij,m+\ell})_{1,1} = {}_1\kappa^{ij|m+\ell}_{(0,0,0,0,0,0,0,1,0)} + {}_1\kappa^{ij|m+\ell}_{(0,0,0,1,0,0,0,0,0)},$$

$$(R^{-1}_{ij,m+\ell})_{1,2} = {}_1\kappa^{ij|m+\ell}_{(0,0,0,0,0,0,1,0,0)} + {}_1\kappa^{ij|m+\ell}_{(0,0,0,0,0,0,1,0,1)},$$

$$(R^{-1}_{ij,m+\ell})_{2,1} = -{}_2\kappa^{ij|m+\ell}_{(0,0,1,0,0,0,0,1,0)} - {}_2\kappa^{ij|m+\ell}_{(0,0,1,0,0,1,0,0,0)} - \frac{1}{2}{}_2\kappa^{ij|m+\ell}_{(0,0,1,1,0,0,0,0,0)},$$

$$(R^{-1}_{ij,m+\ell})_{2,2} = {}_2\kappa^{ij|m+\ell}_{(0,0,0,0,1,0,0,0,0)} - {}_2\kappa^{ij|m+\ell}_{(0,0,1,0,0,0,1,0,0)} - {}_2\kappa^{ij|m+\ell}_{(0,0,1,0,0,0,1,0,1)} + {}_2\kappa^{ij|m+\ell}_{(1,0,0,0,0,0,0,0,0)}$$

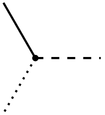
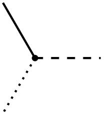
Example: $\boldsymbol{e}_r + \ell \boldsymbol{e}_1$ Exchange in $\langle SFSF \rangle$

Projection operator to $\boldsymbol{e}_r + \ell \boldsymbol{e}_1$:

t	(d_t, ℓ_t)	$\mathscr{A}_t(d, \ell)$	$\hat{\mathcal{Q}}_t$
1	$(2, 0)$	1	$\delta_\alpha{}^{\alpha'}$
2	$(2, 1)$	$\frac{\ell}{2(-\ell+1-d/2)}$	$(\gamma_\mu \gamma^{\mu'})_\alpha{}^{\alpha'}$

Since $i_a = i_b = 0$ for all tensor structures, no need to extract any indices

$$\forall\, a, b : \qquad \mathscr{A}_1 \hat{\mathcal{Q}}^{\boldsymbol{e}_r}_{13|1} \hat{\mathcal{P}}^{\ell \boldsymbol{e}_1}_{13|d+2} + \mathscr{A}_2 \hat{\mathcal{Q}}^{\boldsymbol{e}_r + \boldsymbol{e}_1}_{13|2} \hat{\mathcal{P}}^{(\ell-1)\boldsymbol{e}_1}_{13|d+2}$$

$$= \mathscr{A}_1 \hat{\mathcal{Q}}^{\boldsymbol{e}_r}_{13|1} \times$$  $$+ \mathscr{A}_2 \hat{\mathcal{Q}}^{\boldsymbol{e}_r + \boldsymbol{e}_1}_{13|2} \times$$ 

Example: $\boldsymbol{e}_r + \ell \boldsymbol{e}_1$ Exchange in $\langle SFSF \rangle$ (cont.)

All four different blocks have the same form:

$$\forall\, a, b: \ \mathscr{G}^{ij|m+\ell|kl}_{(a|b]} = \frac{\ell!}{(d/2)_\ell}\left(C^{(d/2)}_{\ell}(X)\right)_{s^1_{(a|b)}} + \frac{\ell!}{2(d/2)_\ell}\left(C^{(d/2)}_{\ell-1}(X)\right)_{s^2_{(a|b)}}$$

But different substitution rules due to the tensor structures and the different values of n_a and n_b, e.g.

$$s^1_{(1|1)}: \alpha_2^{s_2}\alpha_3^{s_3}\alpha_4^{s_4}x_3^{r_3}x_4^{r_4} \to -(\Gamma_{F''}\bar{\eta}_3 \cdot \Gamma\,\Gamma_{F''}C_\Gamma^{-T})_{bd}\left(G^{ij|m+\ell|kl}_{(0,0,4,3,-1)}\right)^{F''2} = -2(\Gamma_{F''}C_\Gamma^{-T})_{bd}\left(G^{ij|m+\ell|kl}_{(0,0,2,1,-1)}\right)^{F''}$$

Example: $\boldsymbol{e}_r + \ell \boldsymbol{e}_1$ Exchange in $\langle SFSF \rangle$ (cont.)

Rotation matrix for this case:

$$
\begin{aligned}
(R^{-1}_{ij,m+\ell})_{1,1} &= 0, \\
(R^{-1}_{ij,m+\ell})_{1,2} &= (-1)^r \, {}_1\kappa^{ij|m+\ell}_{(0,0,1,0,0,0,0,0,0)}, \\
(R^{-1}_{ij,m+\ell})_{2,1} &= (-1)^{r+1} \Big[{}_2\kappa^{ij|m+\ell}_{(0,0,1,0,0,0,0,1,0)} - \frac{1}{2} {}_2\kappa^{ij|m+\ell}_{(0,0,1,0,0,0,1,0,0)} \\
&\quad - \frac{1}{2} {}_2\kappa^{ij|m+\ell}_{(0,0,1,0,0,0,1,0,1)} + {}_2\kappa^{ij|m+\ell}_{(0,0,1,1,0,0,0,0,0)} \\
&\quad + d \, {}_2\kappa^{ij|m+\ell}_{(1,0,0,0,0,0,0,0,0)} \Big], \\
(R^{-1}_{ij,m+\ell})_{2,2} &= 0.
\end{aligned}
$$

where r is the rank of the Lorentz group

Conclusions and Outlook

- Established a set of efficient rules for determining all possible four-point conformal blocks in the context of embedding space OPE formalism
- Require knowledge of fundamental group theoretic quantities: projection operators of external and exchanged quasi-primary operators
- Projectors imply two tensor structures for left $({}_a t_{ij}^{12,m+\ell})_{\{aA\}\{bB\}}{}^{\{Ee\}\{F\}}$ and right $({}_b t^{34}_{kl,m+\ell})_{\{cC\}\{dD\}\{e''E''\}\{F''\}}$ OPE
- Input data: Projection operators and tensor structures
- Rules allow us to generate global conformal blocks for any exchanged Lorentz representation

Conclusions and Outlook (cont.)

- Conformal blocks given in terms of linear combinations of Gegenbauer polynomials in a specific variable X, coupled with associated substitution rules
- Introduced diagrammatic notation to easily determine appropriate linear combinations of Gegenbauer polynomials
- Blocks have simplest form in the mixed OPE-three-point basis
- For bootstrap, need to change to pure three-point basis $\Rightarrow$ rotation matrices
- In future: Use these rules to derive blocks for 4-point functions of conserved currents and energy-momentum tensors and other operators of theoretical interest

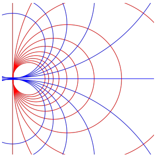
THANK YOU!

Backup Slides

What is a CFT?

A CFT is invariant under the conformal group $SO(1, d-1)$:

- Poincaré algebra
- dilatations
- special conformal transformations



Conformal algebra:

$$[M_{\mu\nu}, M_{\lambda\rho}] = -(s_{\mu\nu})_\lambda{}^\delta M_{\delta\rho} - (s_{\mu\nu})_\rho{}^\delta M_{\lambda\delta},$$

$$[M_{\mu\nu}, P_\lambda] = -(s_{\mu\nu})_\lambda{}^\rho P_\rho, \quad [M_{\mu\nu}, K_\lambda] = -(s_{\mu\nu})_\lambda{}^\rho K_\rho,$$

$$[P_\mu, D] = iP_\mu, \quad [K_\mu, D] = -iK_\mu, \quad [P_\mu, K_\nu] = 2i(g_{\mu\nu}D - M_{\mu\nu})$$

where

$$(s_{\mu\nu})^{\lambda\rho} = i(\delta_\mu{}^\lambda \delta_\nu{}^\rho - \delta_\mu{}^\rho \delta_\nu{}^\lambda), \quad [s_{\mu\nu}, s_{\lambda\rho}] = -(s_{\mu\nu})_\lambda{}^{\lambda'} s_{\lambda'\rho} - (s_{\mu\nu})_\rho{}^{\rho'} s_{\lambda\rho'}$$

The Spectrum of Operators

Two kinds of operators in CFTs:

- quasi-primaries $[K_\mu, \mathcal{O}^{(x)}(0)] = 0$: transform simply under conformal transformations, e.g.

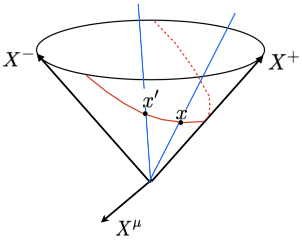
$$x \to x', \quad \mathcal{O}^{(x)}(x) \to \tilde{\mathcal{O}}^{(x)}(x') = b(x)^{-\Delta} \mathcal{O}^{(x)}(x)$$

- descendants: don't!

- Complete spectrum of operators: primaries+infinite towers of descendants
- Organic observables in CFTs: M-point correlation functions of operators, $\langle \mathcal{O}^{(x)}(x_1) \ldots \mathcal{O}^{(x)}(x_M) \rangle$

The Embedding Space

Embedding space $\mathcal{M}^{d+2}$:



- A natural habitat for the conformal group:
  $(d+2)$-dimensional hypercone where operators live

$$\eta^2 \equiv g_{AB}\eta^A\eta^B = 0$$

- Light rays in one-to-one correspondence with position space points

The Embedding Space (cont.)

Coordinates on the hypercone:

$$\eta^A = (\eta^\mu, \eta^{d+1}, \eta^{d+2})$$

- $\lambda\eta^A$ identified with η^A for $\lambda > 0$

Connection to position space:

$$x^\mu = \frac{\eta^\mu}{-\eta^{d+1} + \eta^{d+2}}$$

In the embedding space,

- Conformal transformations act linearly: Conformal group becomes like Lorentz group!
- All operators in d-dimensional CFT need to somehow be lifted to $\mathcal{M}^{d+2}$.

A New Uplift

Uplift based on quasi-primary operators with spinor indices only and standard projection operators (Fortin & Skiba (2019))

Idea:

- Start with a quasi-primary operator in position space $\mathcal{O}^{(x)}$ in a general irrep of $SO(1,d-1)$: $\mathbf{N}^{\mathcal{O}} = \{N_1^{\mathcal{O}}, \ldots, N_r^{\mathcal{O}}\}$
- Lift it to a quasi-primary $\mathcal{O}$ in the embedding space in an irrep of $SO(2,d)$: $\mathbf{N}_E^{\mathcal{O}} = \{0, N_1^{\mathcal{O}}, \ldots, N_r^{\mathcal{O}}\}$
- exact for the defining representations,
- true in general up to the removal of traces

A New Uplift (cont.)

With this,

- Scalars uplift to scalars, spinors to spinors, i-index antisymmetric tensors to (i+1)-index antisymmetric tensors
- Advantage: Approach treats fermions and bosons on an equal footing

From the perspective of the Dynkin indices, everything looks the same!

- Uplift makes universal treatment of all quasi-primary operators in arbitrary irreps of the Lorentz group possible.

The OPE Differential Operator: A Bit of Background

Seek most useful differential operator ${}_a\mathcal{D}_{ij}^{k}(\eta_1,\eta_2)$ for quasi-primary operators in general irreducible representations of the Lorentz group.

- What are our options?
- Only consistent **first order** operators:

$$\Theta = \eta^A \frac{\partial}{\partial \eta^A}, \qquad \mathcal{L}_{AB} = i\left(\eta_A \frac{\partial}{\partial \eta^B} - \eta_B \frac{\partial}{\partial \eta^A}\right)$$

- Unique consistent **second order** operator: Thomas-Todorov

$$\mathcal{K}_A = \left(\eta^B \frac{\partial}{\partial \eta^B} + \frac{d}{2}\right)\frac{\partial}{\partial \eta^A} - \frac{1}{2}\eta_A \frac{\partial}{\partial \eta_B}\frac{\partial}{\partial \eta^B}$$

A Brief History of the OPE Differential Operator

Θ doesn't work: Cannot generate descendants!

Left with:

- $\mathcal{L}_{AB}$
- $\mathcal{K}_A$
- $(\mathcal{L}^2)_{AB}$

- With two embedding space coordinates η_i and η_j, only one independent operator **well-defined** on the lightcone!

A Brief History of the OPE Differential Operator

Inspired by Ferrara et al. (1971, 1972, 1973),

Candidate OPE differential operator:

$$\mathcal{D}_{ij}^{A} \equiv \frac{1}{(\eta_i \cdot \eta_j)^{\frac{1}{2}}}[-i(\eta_i \cdot \mathcal{L}_j)^A - \eta_i^A \Theta_j] = (\eta_i \cdot \eta_j)^{\frac{1}{2}} \mathcal{A}_{ij}^{AB} \partial_{jB}$$

$$\begin{aligned}\mathcal{D}_{ij}^{2} \equiv \mathcal{D}_{ij}^{A}\mathcal{D}_{ijA} &= (\eta_i \cdot \eta_j)\partial_j^2 - \eta_i \cdot \partial_j (d_E - 4 + 2\Theta_j) \\ &= (\eta_i \cdot \eta_j)\partial_j^2 - (d_E - 2 + 2\Theta_j)\eta_i \cdot \partial_j\end{aligned}$$

A Brief History of the OPE Differential Operator

- Candidate has many nice properties!

Notably,

$$[\mathcal{D}_{ij}^A, \mathcal{D}_{ij}^{2h}] = \frac{2h}{(\eta_i \cdot \eta_j)^{\frac{1}{2}}} \eta_i^A \mathcal{D}_{ij}^{2h},$$
$$[\Theta_i, \mathcal{D}_{ij}^{2h}] = h\mathcal{D}_{ij}^{2h}, \quad [\Theta_j, \mathcal{D}_{ij}^{2h}] = -h\mathcal{D}_{ij}^{2h},$$
$$\mathcal{D}_{ij}^{2h}\eta_j^A - \eta_j^A \mathcal{D}_{ij}^{2h} = 2h(\eta_i \cdot \eta_j)^{\frac{1}{2}} \mathcal{D}_{ij}^A \mathcal{D}_{ij}^{2(h-1)} - h(d+2h-2)\eta_i^A \mathcal{D}_{ij}^{2(h-1)}$$

- But difficult to work with!

An OPE Differential Operator

Motivated by last commutation relation, instead find

- Winning candidate

$$\mathcal{D}_{ij|h}^{A} = \frac{\eta_j^A}{(\eta_i\cdot\eta_j)^{\frac{1}{2}}}\mathcal{D}_{ij}^2 + 2h\mathcal{D}_{ij}^A - h(d+2h-2)\frac{\eta_i^A}{(\eta_i\cdot\eta_j)^{\frac{1}{2}}}$$

- Embedding space OPE differential operator

$$\mathcal{D}_{ij}^{(d,h,n)A_1\cdots A_n} \equiv \mathcal{D}_{ij|h+n}^{A_n}\cdots\mathcal{D}_{ij|h+1}^{A_1}\mathcal{D}_{ij}^{2h} = \frac{1}{(\eta_i\cdot\eta_j)^{\frac{n}{2}}}\mathcal{D}_{ij}^{2(h+n)}\eta_j^{A_1}\cdots\eta_j^{A_n}$$

Fortin, Skiba (2019)

An OPE Differential Operator

Properties:

- well-defined on the lightcone
- fully symmetric and traceless with respect to embedding space metric g_{AB}
- satisfies simple contiguous relations
- very convenient, as evident from

$$\mathcal{D}_{ij}^{(d,h,n)A_1\cdots A_n}\eta_j^{A_{n+1}}\cdots\eta_j^{A_{n+k}} = (\eta_i\cdot\eta_j)^{\frac{k}{2}}\mathcal{D}_{ij}^{(d,h-k,n+k)A_1\cdots A_{n+k}}$$
$$= \frac{1}{(\eta_i\cdot\eta_j)^{\frac{n}{2}}}\mathcal{D}_{ij}^{(d,h+n,0)}\eta_j^{A_1}\cdots\eta_j^{A_{n+k}}.$$

Fractional derivative $\Rightarrow$ a sort of analytic continuation

A Special Tensorial Function

Enter a special tensorial object:

$$I_{ij}^{(d,h,n;\boldsymbol{p})A_1\cdots A_n} = \mathcal{D}_{ij}^{(d,h,n)A_1\cdots A_n} \prod_{a\neq i,j} \frac{1}{(\eta_j \cdot \eta_a)^{p_a}}$$

Naturally arises in computation of M-point correlation functions!

Fortin, Skiba (2019)

A Special Tensorial Function (cont.)

In terms of homogeneized coordinates $\bar{\eta}_i$:

$$\bar{I}_{ij;k\ell}^{(d,h,n;\boldsymbol{p})} = (-2)^h (\bar{p})_h (\bar{p}+1-d/2)_h x_m^{\bar{p}+h} \sum_{\substack{\{q_r\}\geq 0 \\ \bar{q}=n}} S_{(\boldsymbol{q})} x_m^{\bar{q}-q_0-q_i} K_{ij;k\ell;m}^{(d,h;\boldsymbol{p};\boldsymbol{q})}(x_m;\boldsymbol{y};\boldsymbol{z}),$$

where

$$S_{(\boldsymbol{q})}^{A_1\cdots A_{\bar{q}}} = g^{(A_1 A_2} \cdots g^{A_{2q_0-1}A_{2q_0}} \bar{\eta}_1^{A_{2q_0+1}} \cdots \bar{\eta}_1^{A_{2q_0+q_1}} \cdots \bar{\eta}_M^{A_{\bar{q}-q_M+1}} \cdots \bar{\eta}_M^{A_{\bar{q}})}$$

with $\bar{q} = 2q_0 + \sum_{r\geq 1} q_r$

It's totally symmetric in all of its indices!

It's all in $\bar{I}$!

$\bar{I}^{(d,h,n;\mathbf{p})}_{ij;k\ell}$ satisfies some convenient contiguous relations:

$$g \cdot \bar{I}^{(d,h,n;\boldsymbol{p})}_{ij;k\ell} = 0,$$

$$\bar{\eta}_i \cdot \bar{I}^{(d,h,n;\boldsymbol{p})}_{ij;k\ell} = \bar{I}^{(d,h+1,n-1;\boldsymbol{p})}_{ij;k\ell},$$

$$\bar{\eta}_j \cdot \bar{I}^{(d,h,n;\boldsymbol{p})}_{ij;k\ell} = (-2)(-h-n)(-h-n+1-d/2)\bar{I}^{(d,h,n-1;\boldsymbol{p})}_{ij;k\ell},$$

$$\bar{\eta}_a \cdot \bar{I}^{(d,h,n;\boldsymbol{p})}_{ij;k\ell} = \bar{I}^{(d,h+1,n-1;\boldsymbol{p}-\boldsymbol{e}_a)}_{ij;k\ell}$$

Upshot: We know the exact action of the embedding space differential operator for any quantity of interest!

Tensor Structures (cont.)

- Set of all ${}_a t^{12}_{ijk}$ forms basis for a vector space
- Equivalently, seen as intertwiners contracting four irreps into a singlet:

$$ {}_a t^{12}_{ijk} = (\hat{\mathcal{P}}^{\boldsymbol{N}_i}_{12})(\hat{\mathcal{P}}^{\boldsymbol{N}_j}_{21})(\hat{\mathcal{P}}^{\boldsymbol{N}_k}_{12})(\hat{\mathcal{P}}^{n_a \boldsymbol{e}_1}_{21}) \cdot {}_a t^{12}_{ijk} $$

- Fourth representation: Symmetric traceless
- Purpose: To restrict the ${}_a t^{12}_{ijk}$ onto the appropriate irreps $\boldsymbol{N}_i$, $\boldsymbol{N}_j$, $\boldsymbol{N}_k$, and $n_a \boldsymbol{e}_1$