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# Production of massive particles from the decay of a free massless field


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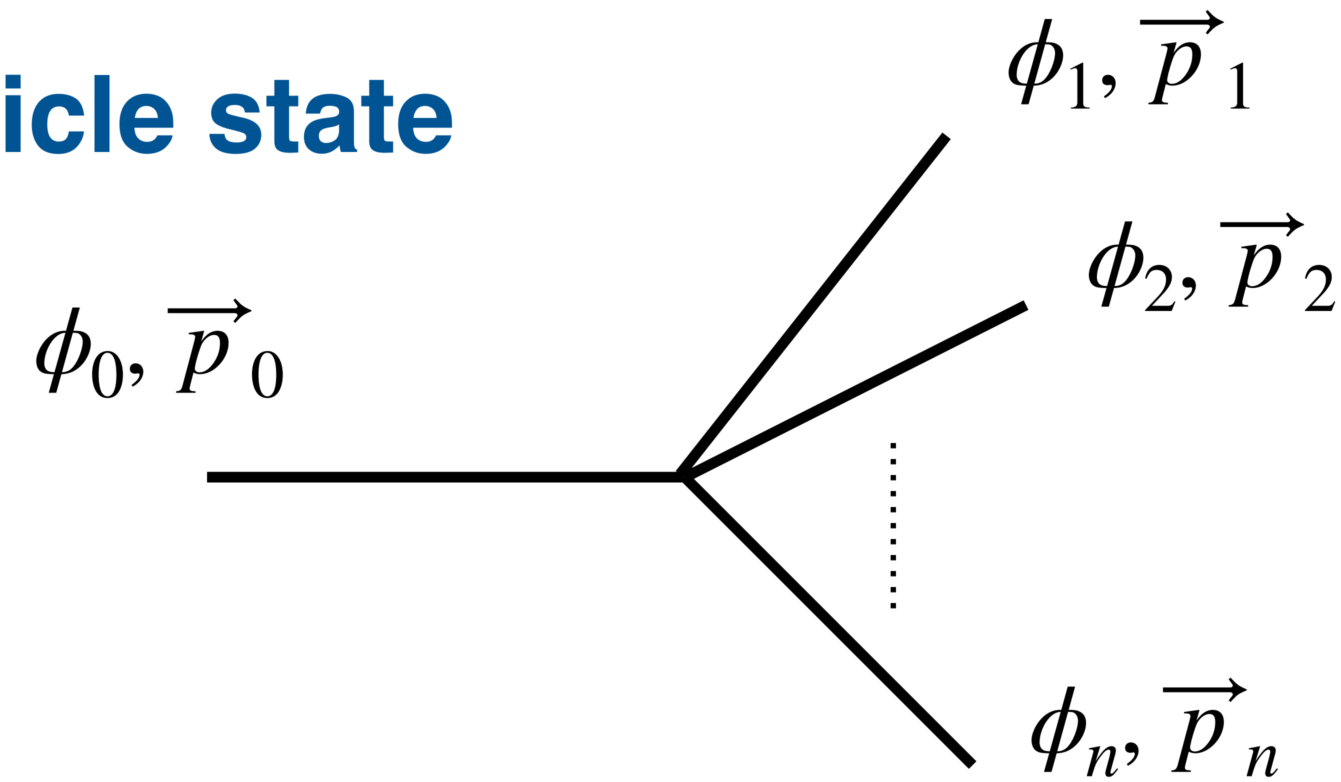
# Decay of a single particle state




$$\phi_0 \to \phi_1 + \phi_2 + \ldots + \phi_n$$

$$\omega_i = \sqrt{p_i^2 + m_i^2}$$

**Momentum conservation:** $\overrightarrow{p}_0 = \overrightarrow{p}_1 + \overrightarrow{p}_2 + \ldots + \overrightarrow{p}_n$

**Energy conservation:** $\omega_0 = \omega_1 + \omega_2 + \ldots + \omega_n$

$$\Gamma = \prod_{i=1}^{n} V \int \frac{d^3 k_i}{(2\pi)^3} \frac{d\,|S_{fi}|^2}{dt}$$

**As $\left| i \right\rangle$ and $\left| f \right\rangle$ are asymptotic free states**

$$S_{fi} = \frac{i(2\pi)^4}{\prod_{i=0}^{n} \sqrt{2\omega_i V}} \delta^4 \left( p_0 - \sum_{i=1}^{n} p_i \right) \mathscr{M}_{fi}$$

# Simple toy model

$$\mathscr{H}_I = g\phi\chi^2$$

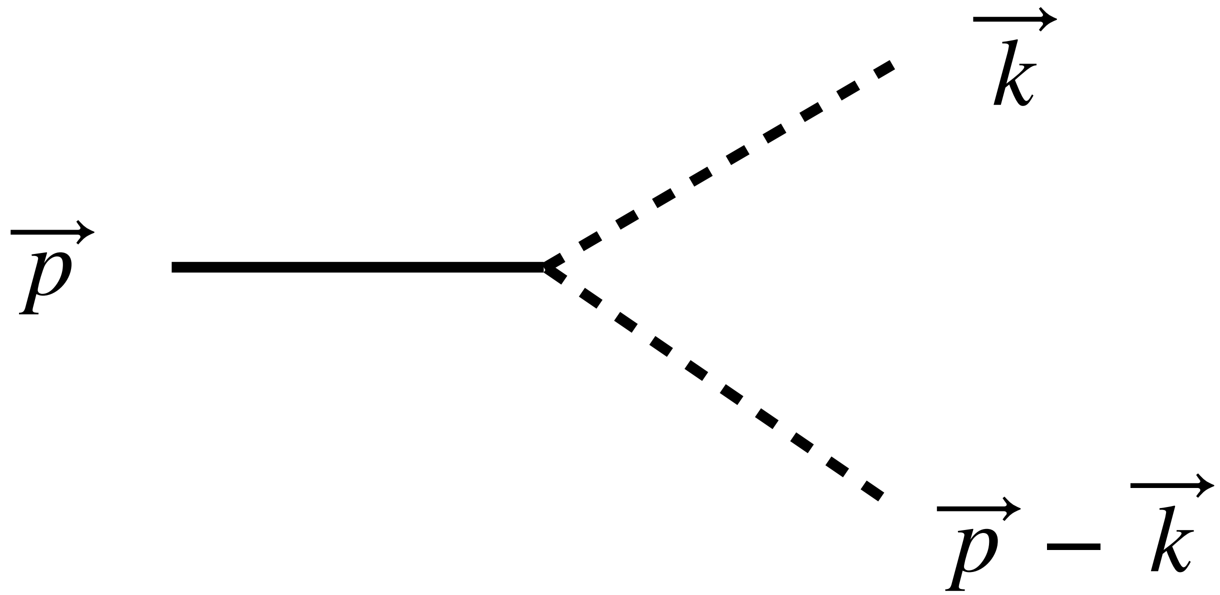
**We are interested in the process $\phi \to 2\chi$**

**We assume that $\phi$ has an initial free asymptotic state with momentum $\vec{p}$**

$m_\phi \longrightarrow$ **Mass of the scalar field $\phi$**

$m_\chi \longrightarrow$ **Mass of the scalar field $\chi$**

$$\omega_{\vec{k}} = \sqrt{k^2 + m_\phi^2} \qquad \Omega_{\vec{k}} = \sqrt{k^2 + m_\chi^2}$$




**we define** $$\epsilon_{\vec{k}} = \Omega_{\vec{k}} + \Omega_{\vec{p}-\vec{k}} - \omega_{\vec{p}}$$

## Two scenarios:

**(I)** $\phi$ **is a one particle state**

**(II)** $\phi$ **is highly occupied**

## Two opposite cases:

**(1) massive** $\phi$ **and massless** $\chi$

**(2) massless** $\phi$ **and massive** $\chi$

# Scenario (I)

**Here final states are asymptotically free, so $\Omega_{\overrightarrow{k}}$ and $\Omega_{\overrightarrow{p}-\overrightarrow{k}}$ are the energies of the final states**

$$S_{fi} = \frac{ig(2\pi)^4\delta^4\left(p_\phi - k_\chi - q_\chi\right)}{\sqrt{2\omega_{\overrightarrow{p}}V}\sqrt{2\Omega_{\overrightarrow{k}}V}\sqrt{2\Omega_{\overrightarrow{p}-\overrightarrow{k}}V}}$$

$\epsilon_{\overrightarrow{k}} = 0$ **is required**

## Case (1) $\epsilon_{\overrightarrow{k}}$ can be zero

**In the rest frame of the decaying particle**

$$k \equiv |\overrightarrow{k}| = \frac{m_\phi}{2}$$

$$\Gamma_{\phi\to 2\chi} = \frac{g^2}{8\pi m_\phi}$$

## Case (2) $\epsilon_{\overrightarrow{k}}$ can never be zero

$$\Gamma_{\phi\to 2\chi} = 0$$

# Scenario (II)

$$\phi = \frac{\sqrt{2\langle\rho\rangle}}{\omega_{\vec{p}}}\cos\left(\vec{p}\cdot\vec{x} - \omega_{\vec{p}}\,t\right)$$

$\langle\rho\rangle \longrightarrow$ **time averaged energy density**

**We assume the decaying field to be highly occupied and we are not considering back reactions on it**

$$\chi = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\Omega_{\vec{k}}}}\left(\chi_{\vec{k}}(t)\,e^{i\vec{k}\cdot\vec{x}} + \chi_{\vec{k}}(t)^{\dagger}\,e^{-i\vec{k}\cdot\vec{x}}\right)$$

$$\left[\chi_{\vec{k}},\chi^{\dagger}_{\vec{k}\,'}\right] = (2\pi)^3\delta^3\left(\vec{k} - \vec{k}\,'\right)$$

$$A_{\vec{k}} = \chi_{\vec{k}} + \chi^{\dagger}_{-\vec{k}} \qquad\qquad \alpha = \frac{g\sqrt{2\langle\rho\rangle}}{\omega^3_{\vec{p}}}$$

$$\left(\partial_t^2 + \Omega^2_{\vec{k}}\right)A_{\vec{k}} = -\,\omega^2_{\vec{p}}\,\alpha\left(\sqrt{\frac{\Omega_{\vec{k}}}{\Omega_{\vec{k}-\vec{p}}}}A_{\vec{k}-\vec{p}}e^{-i\omega_{\vec{p}}\,t} + \sqrt{\frac{\Omega_{\vec{k}}}{\Omega_{\vec{k}+\vec{p}}}}A_{\vec{k}+\vec{p}}e^{i\omega_{\vec{p}}\,t}\right)$$

# Rotating wave approximation (RWA)

$$\chi_{\vec{k}}(t) = a_{\vec{k}}(t)\, e^{-i\Omega_{\vec{k}} t}$$

$$\begin{aligned} H_\chi &= \int d^3x \left( \frac{1}{2}\left(\partial_t \chi\right)^2 + \frac{1}{2}\left(\vec{\nabla}\chi\right)^2 + \frac{1}{2} m_\chi^2 \chi^2 \right) \\ &= \int \frac{d^3k}{(2\pi)^3} \left( \Omega_{\vec{k}}\, a^\dagger_{\vec{k}}\, a_{\vec{k}} + R(t) \right) \end{aligned}$$

$$\begin{aligned} R(t) = &\frac{1}{2}\left( a^\dagger_{\vec{k}}\, \dot{a}_{\vec{k}} - \dot{a}^\dagger_{\vec{k}}\, a_{\vec{k}} \right) + \frac{1}{2\Omega_{\vec{k}}} \dot{a}^\dagger_{\vec{k}}\, \dot{a}_{\vec{k}} + \frac{e^{-2i\Omega_{\vec{k}} t}}{4\Omega_{\vec{k}}} \left( \dot{a}_{\vec{k}}\, \dot{a}_{-\vec{k}} - i\Omega_{\vec{k}} \left( \dot{a}_{\vec{k}}\, a_{-\vec{k}} + a_{\vec{k}}\, \dot{a}_{-\vec{k}} \right) \right) \\ &+ \frac{e^{2i\Omega_{\vec{k}} t}}{4\Omega_{\vec{k}}} \left( \dot{a}^\dagger_{\vec{k}}\, \dot{a}^\dagger_{-\vec{k}} + i\Omega_{\vec{k}} \left( \dot{a}^\dagger_{\vec{k}}\, a^\dagger_{-\vec{k}} + a^\dagger_{\vec{k}}\, \dot{a}^\dagger_{-\vec{k}} \right) \right) \end{aligned}$$

**If $a_{\vec{k}}$ variates much slower than $e^{\pm i\Omega_{\vec{k}} t}$**

**$R(t)$ can be neglected, so $a^\dagger_{\vec{k}}\, a_{\vec{k}}$ is interpreted as the particle number density**

$$\partial_t a_{\vec{k}} = \partial_t a^\dagger_{-\vec{k}} e^{2i\Omega_{\vec{k}} t} - i\sigma_{\vec{k}} \left( a_{\vec{k}-\vec{p}} e^{i\epsilon^{(1)}_{\vec{k}} t} + a^\dagger_{\vec{p}-\vec{k}} e^{i\epsilon^{(2)}_{\vec{k}} t} \right)$$

$$-i\sigma_{\vec{k}+\vec{p}} \left( a_{\vec{k}+\vec{p}} e^{i\epsilon^{(3)}_{\vec{k}} t} + a^\dagger_{-\vec{p}-\vec{k}} e^{i\epsilon^{(4)}_{\vec{k}} t} \right)$$

$$\sigma_{\vec{k}} = g\sqrt{\frac{\langle\rho\rangle/2}{\omega^2_{\vec{p}}\Omega_{\vec{p}-\vec{k}}\Omega_{\vec{k}}}} = \frac{\alpha\,\omega^2_{\vec{p}}}{2\sqrt{\Omega_{\vec{p}-\vec{k}}\Omega_{\vec{k}}}}$$

$\epsilon^{(1)}_{\vec{k}} = \Omega_{\vec{k}} - \Omega_{\vec{k}-\vec{p}} - \omega_{\vec{p}}$ $\Rightarrow$ $\phi_{\vec{p}} + \chi_{\vec{k}-\vec{p}} \leftrightarrow \chi_{\vec{k}}$

$\epsilon^{(2)}_{\vec{k}} = \Omega_{\vec{k}} + \Omega_{\vec{p}-\vec{k}} - \omega_{\vec{p}}$ $\Rightarrow$ $\phi_{\vec{p}} \leftrightarrow \chi_{\vec{k}} + \chi_{\vec{p}-\vec{k}}$

$\epsilon^{(3)}_{\vec{k}} = \Omega_{\vec{k}} - \Omega_{\vec{k}+\vec{p}} + \omega_{\vec{p}}$ $\Rightarrow$ $\phi_{\vec{p}} + \chi_{\vec{k}} \leftrightarrow \chi_{\vec{k}+\vec{p}}$

$\epsilon^{(4)}_{\vec{k}} = \Omega_{\vec{k}} + \Omega_{-\vec{k}-\vec{p}} + \omega_{\vec{p}}$ $\Rightarrow$ vacuum $\leftrightarrow \phi_{\vec{p}} + \chi_{\vec{k}} + \chi_{-\vec{k}-\vec{p}}$

**Processes with $\epsilon^{(i)}_{\vec{k}} \ll \Omega_{\vec{k}}$ are excited**

$t > \sim 0$ $$\phi_{\vec{p}} \leftrightarrow \chi_{\vec{k}} + \chi_{\vec{p}-\vec{k}}$$

$$\epsilon_{\vec{k}} \equiv \epsilon^{(2)}_{\vec{k}} \qquad \partial_t a_{\vec{k}} = -i\sigma_{\vec{k}}\, a^{\dagger}_{\vec{p}-\vec{k}}\, e^{i\epsilon_{\vec{k}} t}$$

$$a_{\vec{k}}(t) = e^{i\epsilon_{\vec{k}} t}\left[a_{\vec{k}}(0)\left(\cosh(s_{\vec{k}}\, t) - i\frac{\epsilon_{\vec{k}}}{2s_{\vec{k}}}\sinh(s_{\vec{k}}\, t)\right) - i\frac{\sigma_{\vec{k}}}{s_{\vec{k}}} a_{\vec{p}-\vec{k}}(0)^{\dagger}\sinh(s_{\vec{k}}\, t)\right]$$

$$s_{\vec{k}} = \sqrt{\sigma^2_{\vec{k}} - \epsilon^2_{\vec{k}}/4}$$

**RWA is valid for** $|s_{\vec{k}}| \ll \{\Omega_{\vec{k}}, \Omega_{\vec{p}-\vec{k}}\}$

**Instability condition:** $s_{\vec{k}}$ **must be real**

## We define dimensionless parameters

$$\vec{\kappa} = \frac{\vec{k}}{\omega_{\vec{p}}} \qquad \mu = \frac{m_\chi}{\omega_{\vec{p}}} \qquad \vec{v} = \frac{\vec{p}}{\omega_{\vec{p}}}$$

$$\beta_{\vec{\kappa}} = \sqrt{\kappa^2 + \mu^2}$$

$$\eta_{\vec{k}} = \frac{s_{\vec{k}}}{\omega_{\vec{p}}} = \sqrt{\frac{\alpha^2}{4\beta_{\vec{v}-\vec{\kappa}}\beta_{\vec{\kappa}}} - \frac{1}{4}(\beta_{\vec{\kappa}} + \beta_{\vec{v}-\vec{\kappa}} - 1)^2}$$

**RWA is valid for** $|\eta_{\vec{\kappa}}| \ll \{\beta_{\vec{\kappa}}, \beta_{\vec{v}-\vec{\kappa}}\}$

**Instability condition:** $\eta_{\vec{k}}$ **must be real**

## Case (1)

**we choose** $\overrightarrow{v} = 0$

**Instability condition:** $1/2 - \alpha < \kappa < 1/2 + \alpha$

$$\frac{m_\phi}{2} - \frac{g\sqrt{2\langle\rho\rangle}}{m_\phi^2} < k < \frac{m_\phi}{2} + \frac{g\sqrt{2\langle\rho\rangle}}{m_\phi^2}$$

$\epsilon_{\overrightarrow{k}} = 2k - m_\phi$ **it is 0 for** $k = m_\phi/2$

**What about energy conservation?**

# We go back to the hamiltonian

$$H_\chi = \int \frac{d^3k}{(2\pi)^3}\left(\Omega_{\vec{k}}\, a^\dagger_{\vec{k}}\, a_{\vec{k}} + R(t)\right)$$

$$R(t) = \frac{1}{2}\left(a^\dagger_{\vec{k}}\,\dot{a}_{\vec{k}} - \dot{a}^\dagger_{\vec{k}}\,a_{\vec{k}}\right) + \frac{1}{2\Omega_{\vec{k}}}\dot{a}^\dagger_{\vec{k}}\,\dot{a}_{\vec{k}} + \frac{e^{-2i\Omega_{\vec{k}}t}}{4\Omega_{\vec{k}}}\left(\dot{a}_{\vec{k}}\,\dot{a}_{-\vec{k}} - i\Omega_{\vec{k}}\left(\dot{a}_{\vec{k}}\,a_{-\vec{k}} + a_{\vec{k}}\,\dot{a}_{-\vec{k}}\right)\right)$$
$$+\frac{e^{2i\Omega_{\vec{k}}t}}{4\Omega_{\vec{k}}}\left(\dot{a}^\dagger_{\vec{k}}\,\dot{a}^\dagger_{-\vec{k}} + i\Omega_{\vec{k}}\left(\dot{a}^\dagger_{\vec{k}}\,a^\dagger_{-\vec{k}} + a^\dagger_{\vec{k}}\,\dot{a}^\dagger_{-\vec{k}}\right)\right)$$

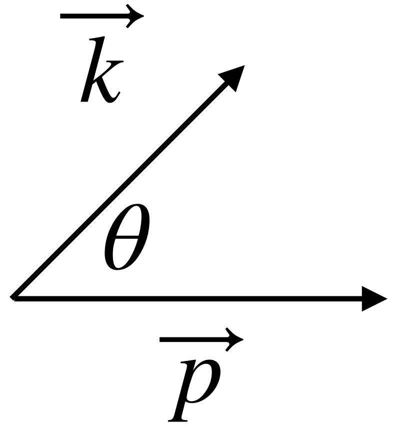
**we define** $\left\langle n_{\vec{k}}(t)\right\rangle = \langle 0 \big|\, a_{\vec{k}}(t) a_{\vec{k}}(t)^\dagger \,\big| 0\rangle$

$$\langle 0 \big|\, R(t) \,\big| 0\rangle = -\frac{1}{2}\epsilon_{\vec{k}}\left\langle n_{\vec{k}}(t)\right\rangle$$

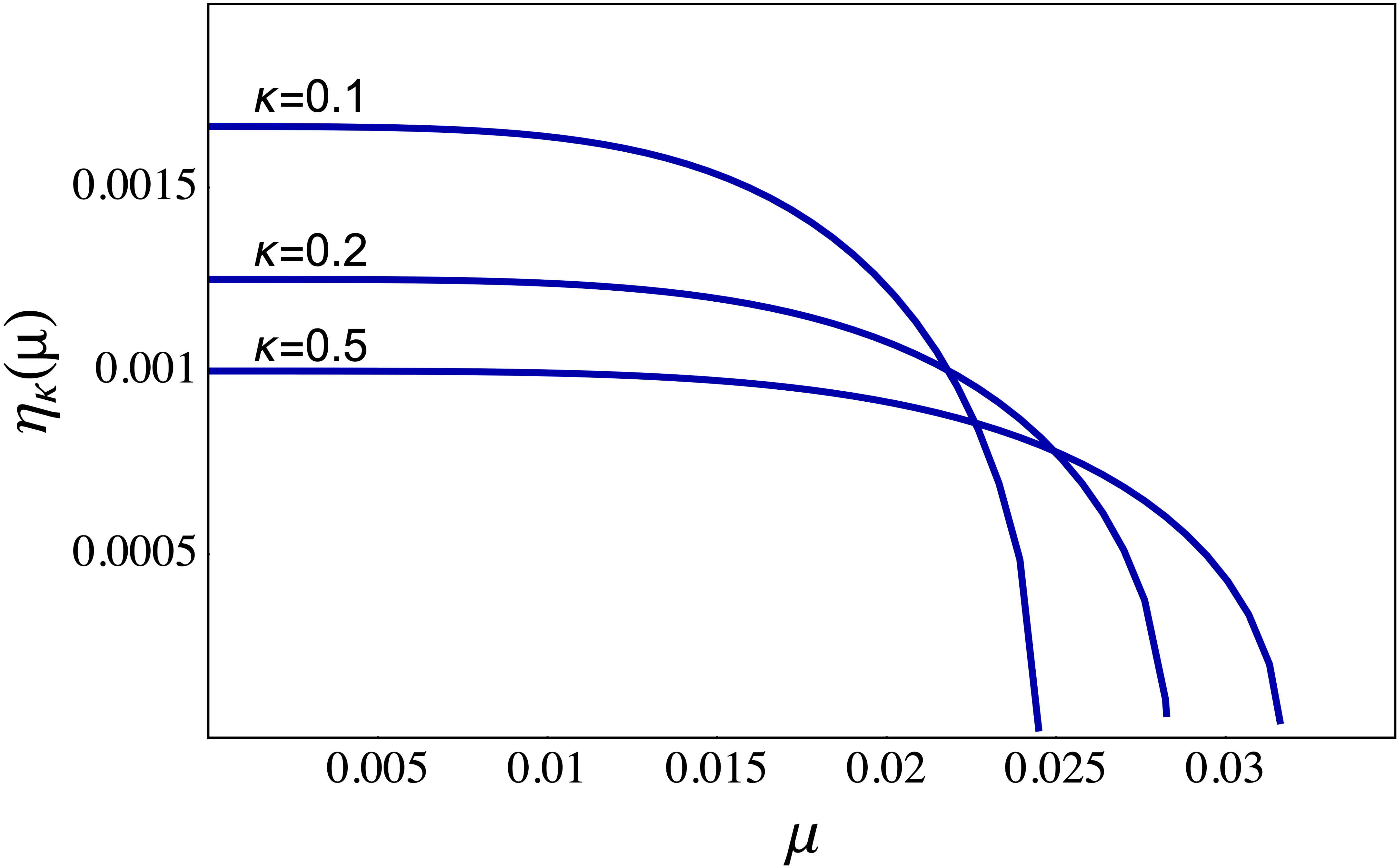
$$\left\langle E_\chi(t)\right\rangle = \frac{\omega_{\vec{p}}}{2}\int\frac{d^3k}{(2\pi)^3}\left\langle n_{\vec{k}}(t)\right\rangle$$

## Case (2)

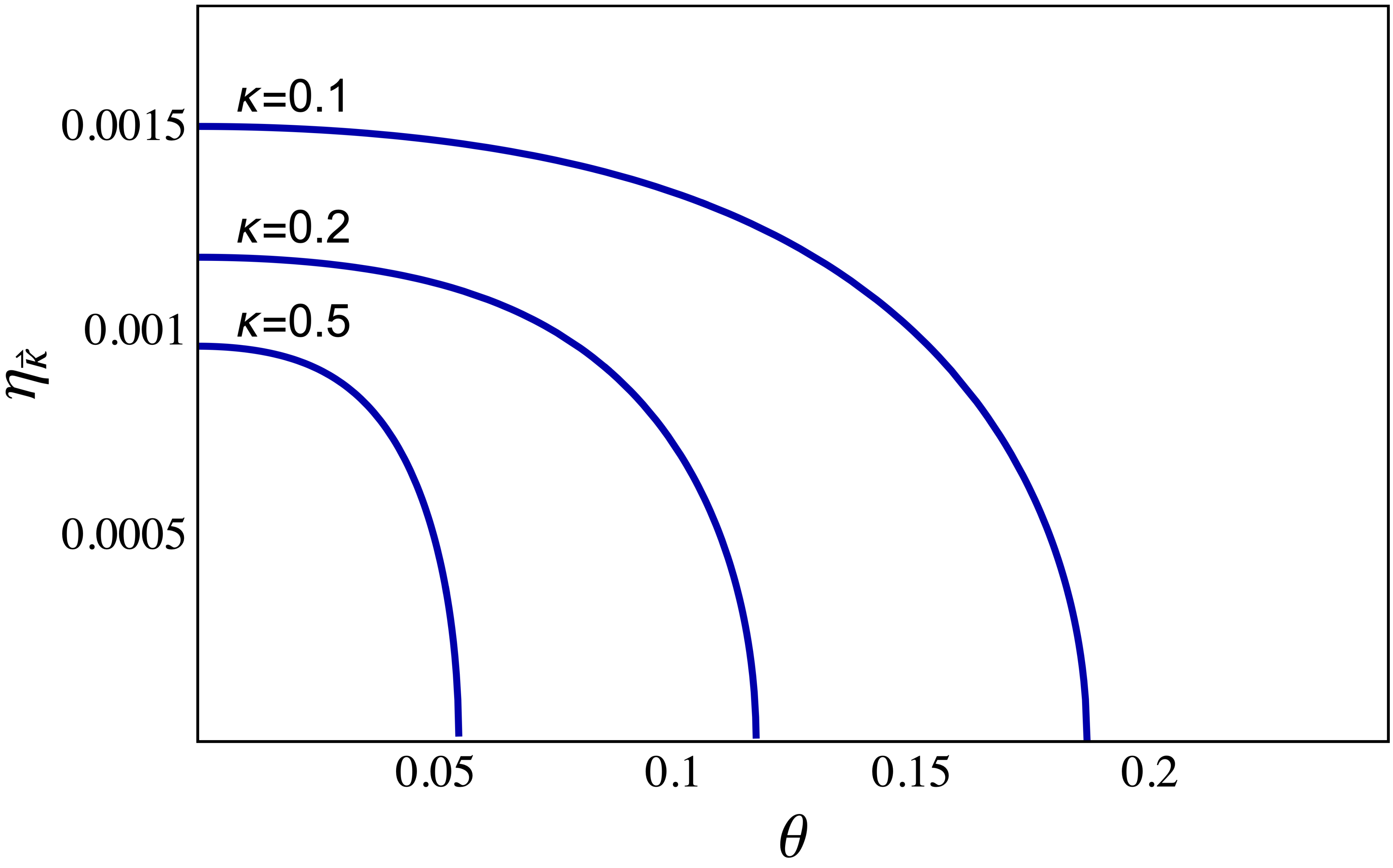
$v = 1$




## For $\theta = 0$ and $\alpha = 10^{-3}$



**For** $\mu = \sqrt{\alpha}/2$ **and** $\alpha = 10^{-3}$



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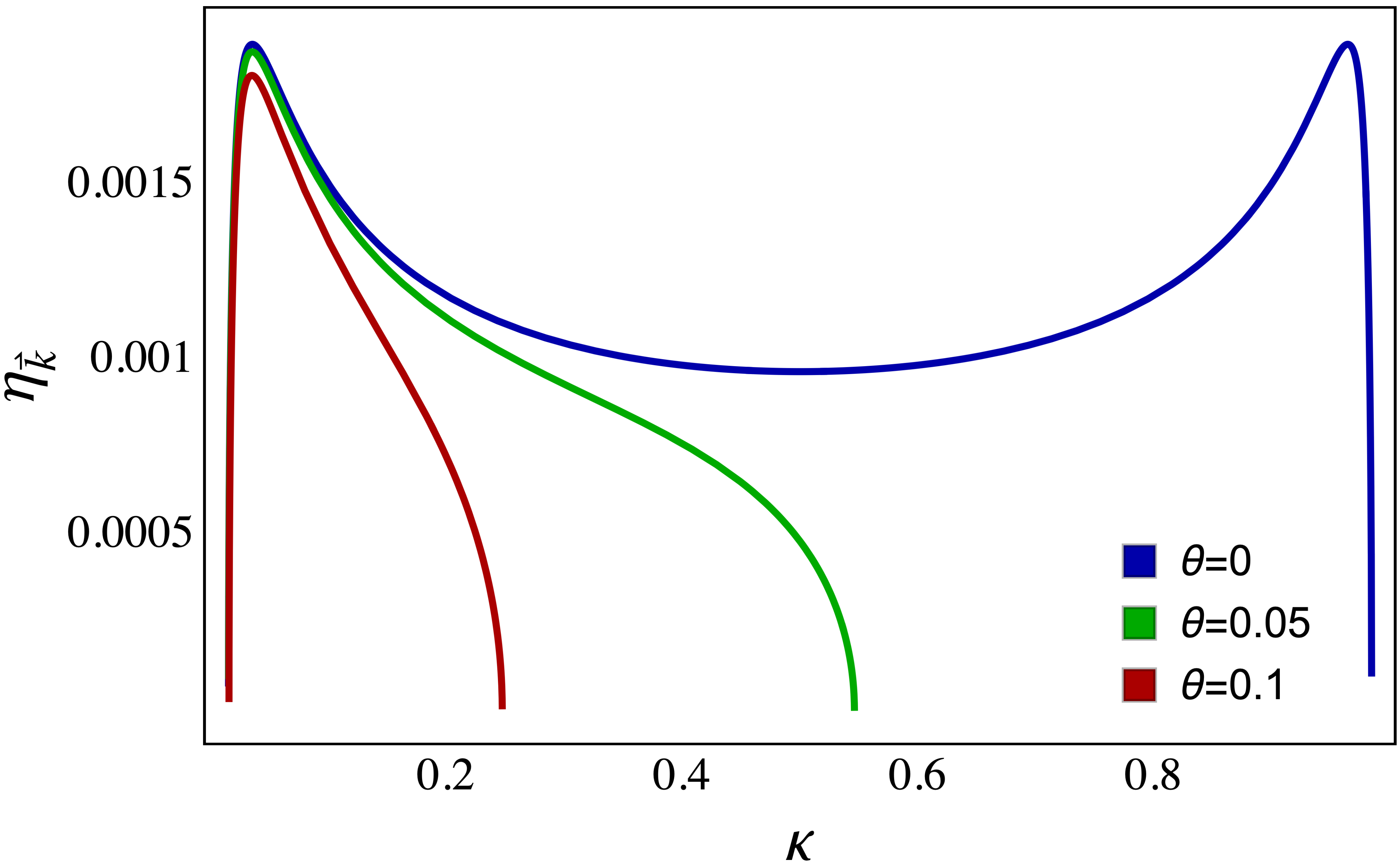


**For** $\mu \ll 1$ **and** $\theta \ll 1$

$$\epsilon_{\vec{k}} = \frac{p(m_\chi^2 + k^2\theta^2)}{2k(p-k)}$$

**limits of the instability for** $\theta = 0$

$$\kappa_- < \kappa < \kappa_+ \qquad \kappa_\pm = \frac{1 \pm \sqrt{1 - \mu^4/\alpha^2}}{2}$$

$$\delta\kappa = \sqrt{1 - \mu^4/\alpha^2}$$ **Bandwidth**

**Instability condition:** $\mu < \sqrt{\alpha}$ $\Rightarrow$ $\langle \rho \rangle > \dfrac{p^2 m_\chi^4}{2g^2}$

**in case (2) the quantity** $\langle \rho \rangle / p^2$ **is Lorentz invariant**

## The RWA was used to find the instability in a frame where $p \gg m_\chi$

## What about a frame where $p \ll m_\chi$ ?

$$\left(\partial_t^2 + \Omega_{\vec{k}}^2\right) A_{\vec{k}} = -\,\omega_{\vec{p}}^2\, \alpha \sqrt{\frac{\Omega_{\vec{k}}}{\Omega_{\vec{k}-\vec{p}}}}\, A_{\vec{k}-\vec{p}}\, e^{-i\omega_{\vec{p}}\, t}$$

$$\Rightarrow \quad \left(\partial_t^2 + m_\chi^2\right) A_{\vec{k}} = -\,\omega_{\vec{p}}^2\, \alpha A_{\vec{k}-\vec{p}}\, e^{-ip\, t}$$

**again, the instability occurs for** $\mu < \sqrt{\alpha}$

$t \gg 0$

**when the modes $\vec{k}$ and $\vec{p} - \vec{k}$ are occupied**

**Case (1)**

**the others $\epsilon^{(i)}_{\vec{k}}$ are still big,**

**so there are no other processes involved**

**Case (2)**

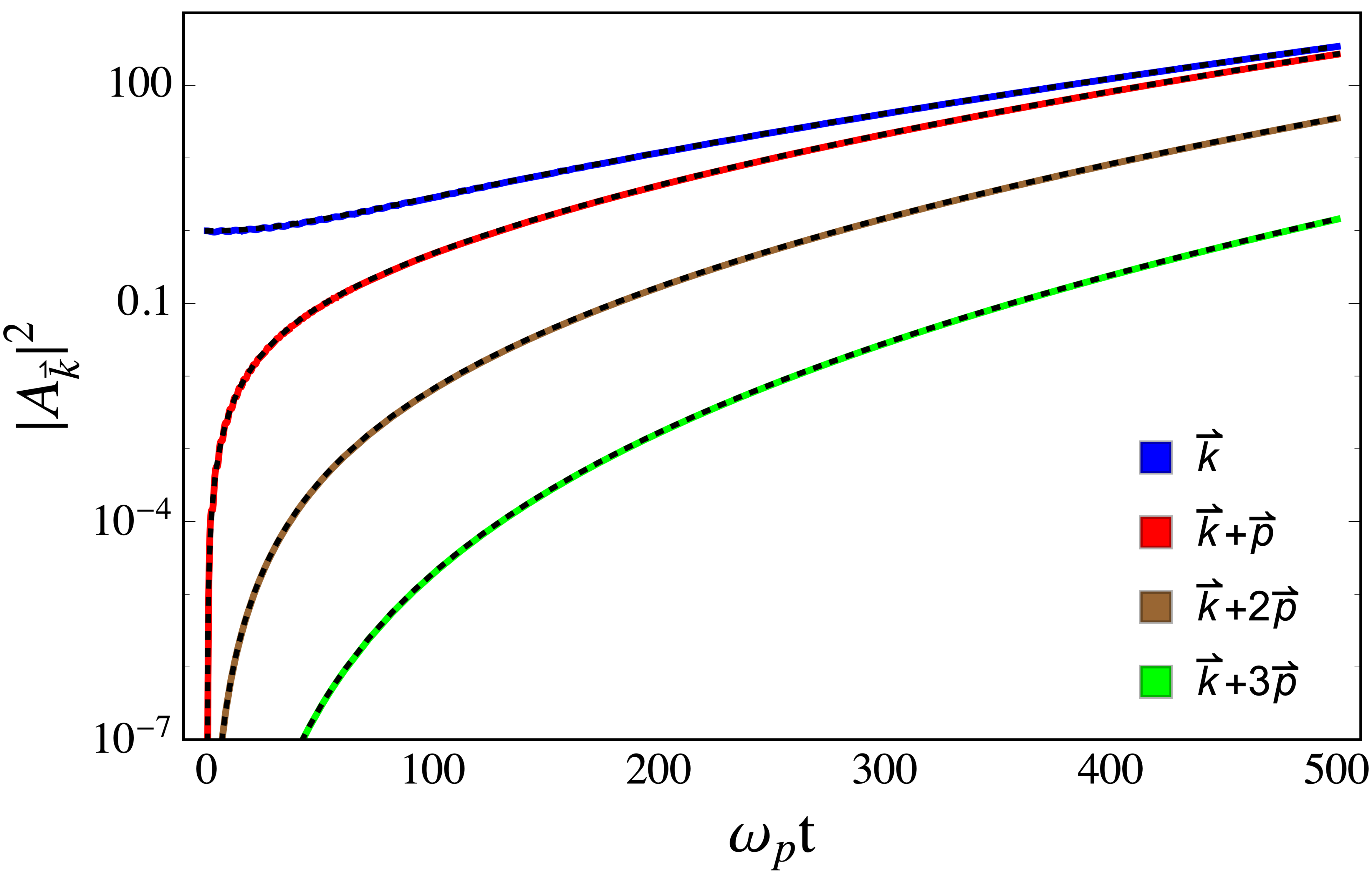
$\epsilon^{(3)}_{\vec{k}}$ **is as small as** $\epsilon^{(2)}_{\vec{k}}$

$$\phi_{\vec{p}} + \chi_{\vec{k}+n\vec{p}} \to \chi_{\vec{k}+(n+1)\vec{p}}$$

$$\phi_{\vec{p}} + \chi_{n\vec{p}-\vec{k}} \to \chi_{(n+1)\vec{p}-\vec{k}}$$

**they gradually become important**

$\overrightarrow{k} = \overrightarrow{p}/2$



# Spontaneous decay rate

$$S_{fi} = \int_0^T dt \int d^3x \, \langle f | \, \mathscr{H}_I \, | i \rangle$$

$$= \frac{1}{V} \sigma_{\vec{k}} \, (2\pi)^3 \, \delta^3(\vec{k} + \vec{q} - \vec{p}) \, e^{i\epsilon_{\vec{k}} T/2} \, \frac{\sin(\epsilon_{\vec{k}} T/2)}{\epsilon_{\vec{k}}/2}$$

$$\Gamma_{\phi \to 2\chi} = V \int \frac{d^3k}{(2\pi)^3} \sigma^2_{\vec{k}} \, \frac{\sin(\epsilon_{\vec{k}} T)}{\epsilon_{\vec{k}}}$$

$$T = \frac{1}{2s_{\vec{k}}}$$

$$\sin(\epsilon_{\vec{k}}/2s_{\vec{k}}) \approx \epsilon_{\vec{k}}/2s_{\vec{k}}$$

$$\Gamma_{\phi \to 2\chi} = V \int \frac{d^3k}{(2\pi)^3} \frac{\sigma^2_{\vec{k}}}{\sqrt{4\sigma^2_{\vec{k}} - \epsilon^2_{\vec{k}}}}$$

**Case (1)**

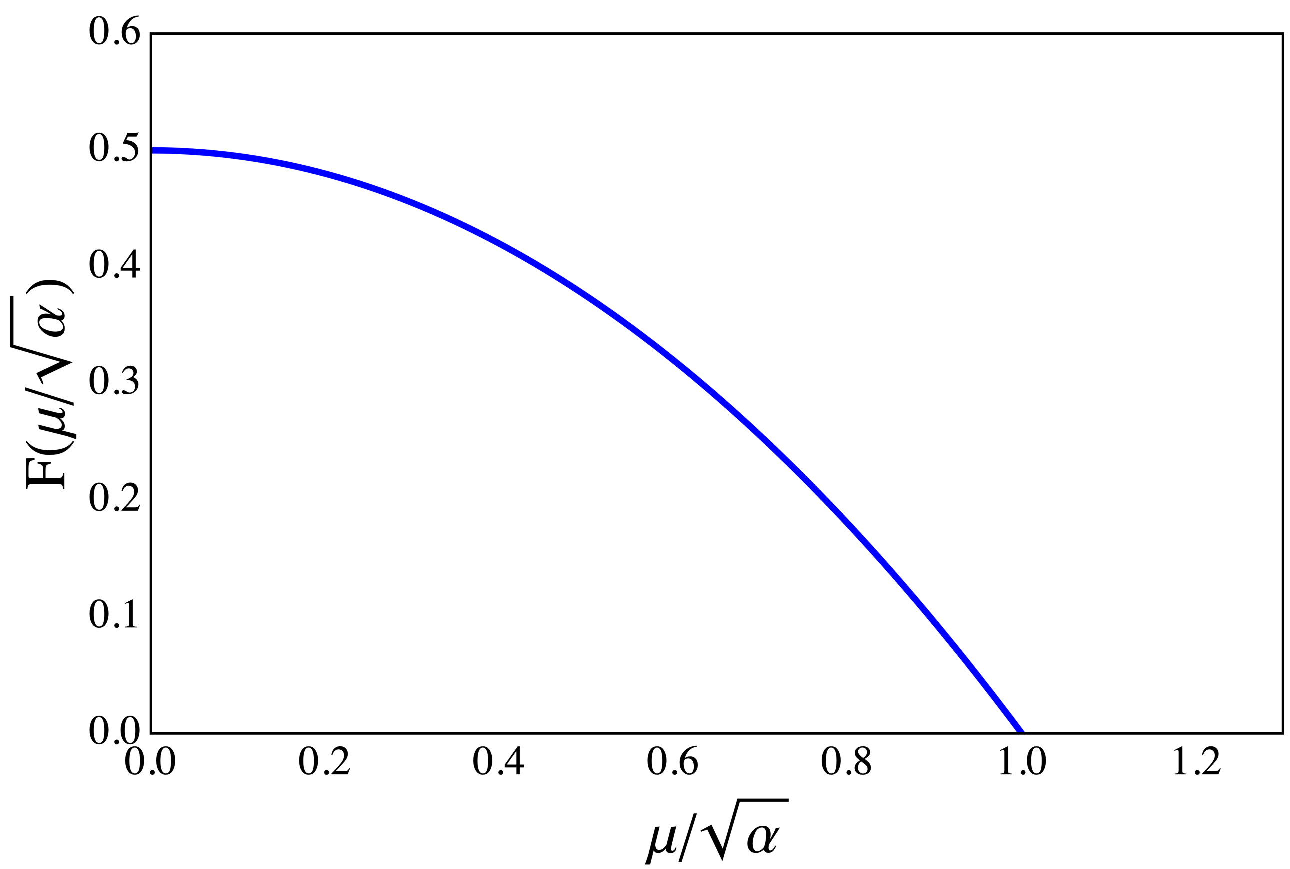
$$\Gamma^{(1)}_{\phi\to 2\chi} = \frac{g^2 N}{8\pi m_\phi}$$

**Case (2)**

$$\Gamma^{(2)}_{\phi\to 2\chi} = \frac{g^2 N}{8\pi p} F(\mu/\sqrt{\alpha})$$

$$F(x) = \frac{1}{\pi}\int_{\kappa_-(x)}^{\kappa_+(x)} d\kappa \left(\frac{\pi}{2} - \arcsin\left(\frac{x^2}{2\sqrt{\kappa(1-\kappa)}}\right)\right)$$

$$\frac{\Gamma^{(2)}_{\phi\to 2\chi}}{\Gamma^{(1)}_{\phi\to 2\chi}} = F(\mu/\sqrt{\alpha})$$



# Thanks for your attention