R-matrix formulation of $Y(\widehat{gl}(1|1))$ and integrable systems of $N=2$ SCFT

Dmitry Kolyaskin, A.L. and Arkady Zhukov

- Integrable massive (gapped) field theory

$$\mathcal{A} = \mathcal{A}_{\mathsf{CFT}} + \lambda \int \Phi_{\Delta,\Delta}(z,\bar{z}) d^2 z, \qquad \Delta < 1.$$

- We are interested in finite volume spectrum $z = u + i\tau$, $u \sim u + 2\pi R$

$$H = L_0 + \bar{L}_0 + \lambda \int_0^{2\pi R} \Phi_{\Delta,\Delta} du, \qquad P = L_0 - \bar{L}_0$$

- Integrability implies the existence of local IM's $(\mathbf{I}_k, \bar{\mathbf{I}}_k)$

$$[\mathbf{I}_k, \mathbf{I}_l] = [\bar{\mathbf{I}}_k, \bar{\mathbf{I}}_l] = 0, \qquad \mathbf{I}_1 = L_0 + \frac{\lambda}{2} \int_0^{2\pi R} \Phi_{\Delta,\Delta} du \dots$$

- Expand at $\lambda \to 0$ (UV expansion): $\mathbf{I}_k = \mathbf{I}_k^{(0)} + \lambda \mathbf{I}_k^{(1)} + \dots$

$$[\mathbf{I}_k^{(0)}, \mathbf{I}_l^{(0)}] = 0, \qquad [\mathbf{I}_k^{(0)}, \mathbf{I}_l^{(1)}] + [\mathbf{I}_k^{(1)}, \mathbf{I}_l^{(0)}] = 0 \quad \text{etc}$$

- In particular

$$[\mathbf{I}_k^{(0)}, L_0] = 0, \quad \left[\mathbf{I}_k^{(0)}, \int \Phi du\right] + [\mathbf{I}_k^{(1)}, L_0] = 0$$

and hence

$$\left[\mathbf{I}_k^{(0)}, \int \Phi du\right]\Bigg|_{\mathcal{H}_\delta} = 0 \quad \text{where} \quad \mathcal{H}_\delta = \{|\chi\rangle \in \mathcal{H} : L_0|\chi\rangle = \delta|\chi\rangle\} \qquad (*)$$

- We can take (*) as a definition of $\mathbf{I}_k^{(0)}$

$$\mathbf{I}_k^{(0)} = \int G_{k+1}(z)dz : \quad \oint_{\mathcal{C}_z} \Phi(\xi)G_{k+1}(z)d\xi = \partial R_k \implies (*)$$

- For example for $\Phi = \Phi_{1,3}$ one has qKdV system

$$G_2 = T, \quad G_4 = T^2, \quad G_6 = T^3 + \frac{c+2}{12}T'^2 \quad \text{etc}$$

- The diagonalization problem for qKdV system has been considered by Bazhanov, Lukyanov and Zamolodchikov (1994-...)

$$\mathcal{T}(\lambda) = \Lambda(q\lambda) + \Lambda^{-1}(q^{-1}\lambda)$$
$$\log\Lambda(q\lambda) \sim m\lambda^{1+\xi} + \sum_k C_k \mathbf{I}_{2k-1}\lambda^{(1-2k)(1+\xi)}$$

- In 2003 Dorey and Tateo and also BLZ discovered the so called ODE/IM correspondence. Within this approach the spectrum of $\mathbf{I}_{2k-1}$ is provided by Gaudin like algebraic equations.

- In this talk we will use another approach based on affine Yangian symmetry.

In order to demonstrate the use of this approach, we consider affine $\mathfrak{gl}(n)$ Toda theory

$$S = \int \left(\frac{1}{8\pi}\Big(\partial_\mu \boldsymbol{\varphi} \cdot \partial_\mu \boldsymbol{\varphi}\Big) + \Lambda \sum_{k=1}^{n-1} e^{b(\varphi_{k+1}-\varphi_k)} + \Lambda e^{b(\varphi_1 - \varphi_n)} \right) d^2x.$$

With the last term dropped, we have CFT whose W-algebra can be described by the Lax representation

$$\Big(Q\partial - \partial\varphi_n\Big)\Big(Q\partial - \partial\varphi_{n-1}\Big)\dots\Big(Q\partial - \partial\varphi_1\Big) = (Q\partial)^n + \sum_{k=1}^{n} W^{(k)}(z)(Q\partial)^{n-k},$$

where $Q = b + b^{-1}$.

For $n = 2$, one has

$$W^{(1)} = \partial\varphi_1 + \partial\varphi_2, \quad W^{(2)} = \left(\frac{W^{(1)}}{2}\right)^2 - Q\frac{\partial W^{(1)}}{2} + T$$

where $[T, W^{(1)}] = 0$ and T generates Virasoro with $c = 1 + 6Q^2$.

In fact, one can drop any other exponent, leading to isomorphic $W-$algebra. For example, dropping the term $e^{b(\varphi_2-\varphi_1)}$, one has different formula

$$\Big(Q\partial-\partial\varphi_1\Big)\Big(Q\partial-\partial\varphi_n\Big)\dots\Big(Q\partial-\Big(Q\partial-\partial\varphi_2\Big)=(Q\partial)^n+\sum_{k=1}^{n}\tilde{W}^{(k)}(z)(Q\partial)^{n-k}.$$

By symmetry arguments, it is clear that local Integrals of Motion $\boldsymbol{I}_s$ should belong to the intersection of these two $W-$algebras. In particular, one can check that

$$\mathbf{I}_1=-\frac{1}{2\pi}\int\left[\sum_{i<j}^{n}(\boldsymbol{h}_i\cdot\partial\boldsymbol{\varphi})(\boldsymbol{h}_j\cdot\partial\boldsymbol{\varphi})\right]dx,$$

$$\mathbf{I}_2=\frac{1}{2\pi}\int\left[\sum_{i<j<k}^{n}(\boldsymbol{h}_i\cdot\partial\boldsymbol{\varphi})(\boldsymbol{h}_j\cdot\partial\boldsymbol{\varphi})(\boldsymbol{h}_k\cdot\partial\boldsymbol{\varphi})+Q\sum_{i<j}(\boldsymbol{h}_i\cdot\partial\boldsymbol{\varphi})(\boldsymbol{h}_j\cdot\partial^2\boldsymbol{\varphi})\right]dx,$$

. . . .

where $\boldsymbol{h}_i=\boldsymbol{e}_i-\frac{1}{n}\sum_{k=1}\boldsymbol{e}_k$ indeed satisfy this requirement.

This point of view that IM's should belong to intersection of two W-algebras automatically implies that the intertwining operator T_1

$$T_1 \tilde{W}^{(k)}(z) = W^{(k)}(z) T_1,$$

will be itself an Integral of Motion.

Actually it is natural to define more operators, which will map between different $W-$algebras corresponding to different permutations of factors in Lax operator. The Maulik-Okounkov $R-$matrix corresponds to elementary transposition

$$\mathcal{R}_{i,j}\Big(Q\partial - \partial\varphi_i\Big)\Big(Q\partial - \partial\varphi_j\Big) = \Big(Q\partial - \partial\varphi_j\Big)\Big(Q\partial - \partial\varphi_i\Big)\mathcal{R}_{i,j},$$

while the operator T_1 corresponds to the long cycle permutation

$$T_1 = \mathcal{R}_{1,2}\mathcal{R}_{1,3}\ldots\mathcal{R}_{1,n-1}\mathcal{R}_{1,n}.$$

The operator $\mathcal{R}_{i,j}$ acts in the tensor product of two Fock representations of Heisenberg algebra $(\widehat{\mathfrak{gl}}(1))$ with the highest weights u_i and u_j

$$\mathcal{F}_{u_i} \otimes \mathcal{F}_{u_j} \xrightarrow{\mathcal{R}_{i,j}} \mathcal{F}_{u_i} \otimes \mathcal{F}_{u_j}$$

and its matrix depends on difference $u_i - u_j$. Then it follows immediately from the definition that $\mathcal{R}_{i,j}(u_i - u_j)$ satisfies the Yang-Baxter equation

$$\mathcal{R}_{1,2}(u_1 - u_2)\mathcal{R}_{1,3}(u_1 - u_3)\mathcal{R}_{2,3}(u_2 - u_3) = \\ = \mathcal{R}_{2,3}(u_2 - u_3)\mathcal{R}_{1,3}(u_1 - u_3)\mathcal{R}_{1,2}(u_1 - u_2),$$

and hence the whole machinery of quantum inverse scattering method can be applied.

In particular, one can construct a family of twisted commuting transfer-matrices on $n-$sites

$$\mathbf{T}(u) = \mathrm{Tr}\Big(q^{L_0^{(0)}}\mathcal{R}_{0,1}(u-u_1)\mathcal{R}_{0,2}(u-u_2)\ldots\mathcal{R}_{0,n-1}(u-u_{n-1})\mathcal{R}_{0,n}(u-u_n)\Big)\Big|_{\mathcal{F}_u}.$$

The spectrum of ILW_n integrable system is governed by finite type Bethe ansatz equations which have been conjectured by Nekrasov and Okounkov and independently by A.L.

$$q\prod_{j\neq i}\frac{(x_i-x_j-\epsilon_1)(x_i-x_j-\epsilon_2)(x_i-x_j-\epsilon_3)}{(x_i-x_j+\epsilon_1)(x_i-x_j+\epsilon_2)(x_i-x_j+\epsilon_3)}\prod_{k=1}^{n}\frac{x_i-u_k+\frac{\epsilon_3}{2}}{x_i-u_k-\frac{\epsilon_3}{2}}=1$$

such that the eigenvalues of $\mathbf{I}_s(q)$ are symmetric polynomials in Bethe roots

$$\mathbf{I}_1(q)\sim-\frac{1}{2}\sum_{k=1}^{n}u_k^2+N,\qquad \mathbf{I}_2(q)\sim\frac{1}{3}\sum_{k=1}^{n}u_k^3-2i\sum_{j=1}^{N}x_j,\qquad \dots$$

Here we use Nekrasov notations

$$b=\sqrt{\frac{\epsilon_2}{\epsilon_1}},\qquad b^{-1}=\sqrt{\frac{\epsilon_1}{\epsilon_2}}\quad\text{and}\quad \epsilon_3\overset{\text{def}}{=}-\epsilon_1-\epsilon_2.$$

These Bethe ansatz equations were proven by

- Feigin et. al., by using the method of shuffle algebras
- Aganagic and Okounkov by geometric methods
- A.L. and Ilya Vilkoviskiy by "direct" algebraic methods

A.L. + I.V. proof uses current realization of $Y(\widehat{\mathfrak{gl}}(1))$

$$\mathcal{R}_{ij}(u-v)\mathcal{L}_i(u)\mathcal{L}_j(v) = \mathcal{L}_j(v)\mathcal{L}_i(u)\mathcal{R}_{ij}(u-v).$$

This algebra becomes an infinite set of quadratic relations between

$$\mathcal{L}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(u) \overset{\text{def}}{=} \langle u|a_{\boldsymbol{\lambda}}\mathcal{L}(u)a_{-\boldsymbol{\mu}}|u\rangle \quad \text{where} \quad a_{-\boldsymbol{\mu}}|u\rangle = a_{-\mu_1}a_{-\mu_2}\dots|u\rangle.$$

These results can be generalized in various direction:

- A.L. I.V. generalization to BCD CFT's (corresponds to "spin" chains with boundary)

- E. Chistyakova, A.L. and P. Orlov generalization $N = 1$ Virasoro: $\mathsf{Y}(\widehat{\mathfrak{gl}(1)}) \to \mathsf{Y}(\widehat{\mathfrak{gl}(2)})$ (admits straightforward generalization to $\mathsf{Y}(\widehat{\mathfrak{gl}(n)})$: paraVirasoro)

Today we are interested in $N = 2$ SUSY version. We will see, that it corresponds to $\mathsf{Y}(\widehat{\mathfrak{gl}(1|1)})$. We define the $R-$matrix for $\mathsf{Y}(\widehat{\mathfrak{gl}(1|1)})$ and derive current realization.

The $\mathcal{N}=2$ W_{n+1}-algebras correspond to $SU(n+1)/U(n)$ Kazama-Suzuki models. These algebras contain one $\mathcal{N}=2$ multiplet of holomorphic currents for each integers spin $s=1,\ldots,n$. In particular, for $s=1$ one has (J,G^{+},G^{-},T) of spins $(1,\frac{3}{2},\frac{3}{2},2)$, which form $\mathcal{N}=2$ Virasoro

$$
\begin{aligned}
T(z)T(w) &= \frac{c}{2(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}+\ldots,\\
T(z)J(w) &= \frac{J(w)}{(z-w)^2}+\frac{\partial J(w)}{z-w}+\ldots,\\
T(z)G^{\pm}(w) &= \frac{3G^{\pm}(w)}{2(z-w)^2}+\frac{\partial G^{\pm}(w)}{z-w}+\ldots,\\
J(z)J(w) &= \frac{c}{3(z-w)^2}+\ldots,\\
J(z)G^{\pm}(w) &= \pm\frac{G^{\pm}(w)}{z-w}+\ldots,\\
G^{+}(z)G^{-}(w) &= \frac{c}{3(z-w)^3}+\frac{J(w)}{(z-w)^2}+\frac{T(w)+\frac{1}{2}\partial J(w)}{z-w}+\ldots
\end{aligned}
$$

We will use free-field representation of $\mathcal{N}=2$ W_{n+1}-algebras suggested by Ito. We work in $\mathcal{N}=1$ formalism and introduce $2n+1$ real holomorphic superfields $\vec{\mathbf{\Phi}}(z,\theta)=(\mathbf{\Phi}_0(z,\theta),\mathbf{\Phi}_1(z,\theta),\ldots,\mathbf{\Phi}_{2n}(z,\theta))$

$$\mathbf{\Phi}_k(z,\theta)\overset{\text{def}}{=}\Phi_k(z)+\theta\chi_k(z),$$

$$\Phi_i(z)\Phi_j(w)=-\frac{\delta_{ij}}{2}\log(z-w)+\ldots,\qquad \chi_i(z)\chi_j(w)=\frac{\delta_{ij}}{2}\frac{1}{z-w}+\ldots$$

Then $\mathcal{N}=2$ W_{n+1}-algebra is defined by

$$\Big(b^{-1}D-iD\mathbf{\Phi}_0(z,\theta)\Big)\Big(b^{-1}D-D\mathbf{\Phi}_1(z,\theta)\Big)\ldots\Big(b^{-1}D-iD\mathbf{\Phi}_{2n}(z,\theta)\Big)$$

We denote the corresponding algebra as $W_{\mathfrak{gl}(n+1|n)}$.

Similarly $W_{\mathfrak{gl}(n|n)}$ corresponds to Lax operator with the first factor dropped.

It is convenient to use the notion of screening fields. The current $A(z)$ is said to "commute" with the screening field $\mathcal{S} = \int \mathcal{V}(\xi) d\xi$ if

$$\oint_{\mathcal{C}_z} \mathcal{V}(\xi) A(z) d\xi = 0.$$

.

For $W_{\mathfrak{gl}(n|n)}$ algebra use the complex fields $(k = 1, \ldots, n)$

$$\boldsymbol{X}_k(z,\theta) \stackrel{\text{def}}{=} \boldsymbol{\Phi}_{2k-1}(z,\theta) + i\boldsymbol{\Phi}_{2k}(z,\theta) = X_k(z) + \theta\psi_k(z),$$
$$\boldsymbol{X}_k^*(z,\theta) \stackrel{\text{def}}{=} \boldsymbol{\Phi}_{2k-1}(z,\theta) - i\boldsymbol{\Phi}_{2k}(z,\theta) = X_k^*(z) + \theta\psi_k^*(z),$$

where $(X_k(z), X_k^*(z), \psi(z), \psi^*(z))$ have the following OPE's

$$X_i(z)X_j^*(w) = -\delta_{ij}\log(z-w) + \ldots, \qquad \psi_i(z)\psi_j^*(w) = \frac{\delta_{ij}}{z-w} + \ldots$$

In these terms the screening operators for $W_{\mathfrak{gl}(n|n)}$ algebra

$$(S_1, S_{1,2}, S_2, S_{2,3}, \ldots, S_{n-1,n}, S_n)$$

have the form

$$S_k = b^{-1} \oint e^{b\boldsymbol{X}_k^*(z,\theta)} dz d\theta = \oint \psi_k^*(z) e^{bX_k^*(z)} dz,$$

and

$$S_{k,k+1} = 2b^{-1} \oint e^{\frac{b}{2}\left(\boldsymbol{X}_k(z,\theta) - \boldsymbol{X}_{k+1}(z,\theta) - \boldsymbol{X}_k^*(z,\theta) - \boldsymbol{X}_{k+1}^*(z,\theta)\right)} dz d\theta$$
$$= \oint e^{\frac{b}{2}\left(X_k(z) - X_{k+1}(z) - X_k^*(z) - X_{k+1}^*(z)\right)} \left(\psi_k(z) - \psi_{k+1}(z) - \psi_k^*(z) - \psi_{k+1}^*(z)\right) dz$$

The commutation relations of $\widehat{\mathfrak{gl}}(1|1)$ algebra can be written in terms of OPE of two bosonic currents $J^E(z)$, $J^N(z)$ and two fermionic currents $J^{\pm}(z)$:

$$J^N(z)J^E(w) = \frac{\kappa}{(z-w)^2} + \dots$$
$$J^+(z)J^-(w) = \frac{\kappa}{(z-w)^2} + \frac{J^E(w)}{(z-w)^2} + \dots$$
$$J^{\pm}(z)J^N(w) = \mp\frac{J^{\pm}(w)}{z-w} + \dots$$

The parameter κ called the level can be set $\kappa = 1$ by the simple rescaling $J^E(z) \to \kappa J^E(z)$ and $J^+(z) \to \kappa J^+(z)$.

This algebra can be realized (non-abelian bosonization) $bc\beta\gamma$-system is the system of complex fermions $\alpha(z)$, $\alpha^*(z)$ and bosons $\mathsf{a}(z)$, $\mathsf{a}^*(z)$ with

$$\alpha(z)\alpha^*(w) = \frac{1}{z-w} + \dots, \qquad \mathsf{a}(z)\mathsf{a}^*(w) = \frac{1}{z-w} + \dots$$

One can realize the currents $J^E(z)$, $J^N(z)$ and $J^{\pm}(z)$ as bilinears

$$\boldsymbol{E} \stackrel{\text{def}}{=} \begin{pmatrix} \frac{J^E}{2} + J^N & J^+ \\ J^- & \frac{J^E}{2} - J^N \end{pmatrix} = \begin{pmatrix} \alpha^*\alpha & \alpha^*\mathsf{a} \\ \mathsf{a}^*\alpha & \mathsf{a}^*\mathsf{a} \end{pmatrix}$$

For our purposes it will be convenient to further realize $(\alpha, \alpha^*, \mathsf{a}, \mathsf{a}^*)$ by the complex superfield $\boldsymbol{X}(z,\theta) = X(z) + \theta\psi(z)$, $\boldsymbol{X}^*(z,\theta) = X^*(z) + \theta\psi^*(z)$, as the Wick ordered fields

$$\begin{aligned} \alpha &= \psi e^{-\frac{X}{b}}, \qquad & \mathsf{a} &= e^{-\frac{X}{b}}, \\ \alpha^* &= \psi^* e^{\frac{X}{b}}, \qquad & \mathsf{a}^* &= \left(\psi\psi^* - b\partial X^*\right) e^{\frac{X}{b}}. \end{aligned} \tag{*}$$

Here b is an arbitrary parameter, which can be re-scaled out.

It is important to note that the currents (*) can be equivalently defined by the following Wakimoto type screening operator

$$\mathcal{S} = b^{-1} \oint e^{b\boldsymbol{X}^*(z,\theta)} dz d\theta = \oint \psi^*(z) e^{bX^*(z)} dz.$$

In order to define the R-matrix for $\widehat{\mathfrak{gl}}(1|1)$, we take two copies of $\widehat{\mathfrak{gl}}(1|1)$ algebra defined by two screening fields

$$\mathcal{S}_1 = \int \psi_1^*(z) e^{bX_1^*(z)} dz \quad \text{and} \quad \mathcal{S}_2 = \int \psi_2^*(z) e^{bX_2^*(z)} dz$$

and define the additional screening field

$$\tilde{\mathcal{S}}_{1,2} \stackrel{\text{def}}{=} \int (\psi_1(z) - \psi_2(z)) e^{\frac{b}{2}(X_1(z) - X_2(z) - X_1^*(z) - X_2^*(z))} dz,$$

which commutes with the diagonal $\widehat{\mathfrak{gl}}(1|1)$ current $\boldsymbol{E}_1 + \boldsymbol{E}_2$.

The commutant of this additional screening is actually larger and can be associated with the $W(\mathfrak{gl}(2|2))$ algebra introduced before. We note that the screening operator $\tilde{\mathcal{S}}_{1,2}$ differs from $S_{1,2}$ for $W(\mathfrak{gl}(2|2))$ by the replacement in the prefactor

$$\psi_1(z) - \psi_2(z) \to \psi_1(z) - \psi_2(z) - \psi_1^*(z) - \psi_2^*(z).$$

However two algebras defined by $(\mathcal{S}_1, \mathcal{S}_{1,2}, \mathcal{S}_2)$ and by $(\mathcal{S}_1, \tilde{\mathcal{S}}_{1,2}, \mathcal{S}_2)$ are actually isomorphic and are related by marginal deformations. It can be seen as follows.

Consider the Lax operator for $W(\mathfrak{gl}(1|1))$

$$\mathcal{L} = \left(b^{-1}D - D\mathbf{\Phi}_1\right)\left(b^{-1}D - iD\mathbf{\Phi}_2\right) =$$
$$= \left(b^{-1}D\right)^2 - D\boldsymbol{X}^* b^{-1}D + \frac{1}{2}D\boldsymbol{X}^* D\boldsymbol{X} + \frac{1}{2b}D^2\left(\boldsymbol{X}^* - \boldsymbol{X}\right),$$

where $\boldsymbol{X} = \mathbf{\Phi}_1 + i\mathbf{\Phi}_2$ and perform the formal scale transformation

$$\psi \to \lambda^{-\frac{1}{2}}\psi, \quad \psi^* \to \lambda^{\frac{1}{2}}\psi^*, \quad \theta \to \lambda^{\frac{1}{2}}\theta, \quad \frac{\partial}{\partial\theta} \to \lambda^{-\frac{1}{2}}\frac{\partial}{\partial\theta},$$

which does not change commutation relations. Then one has

$$\mathcal{L} = \mathcal{L}^{(0)} + \lambda\mathcal{L}^{(1)},$$

where (we use $D^2 = -\partial$)

$$\mathcal{L}^{(0)} = -\frac{\partial}{b^2} + \frac{1}{2}\left(b^{-1}\partial X - \psi\psi^* - b^{-1}\partial X^*\right) + \frac{1}{2b}\left(\partial\psi - b\psi\partial X^*\right)\theta - b^{-1}\psi^*\frac{\partial}{\partial\theta} + b^{-1}\partial X$$

and

$$\mathcal{L}^{(1)} = \left(b^{-1}\psi^*\partial + \frac{1}{2}\psi^*\partial X - \frac{1}{2b}\partial\psi^*\right)\theta.$$

It is clear that if $\mathcal{L}_1 \cdot \mathcal{L}_2$ "commutes" with the screening charges $(\mathcal{S}_1, \mathcal{S}_{1,2}, \mathcal{S}_2)$ then $\mathcal{L}_1^{(0)} \cdot \mathcal{L}_2^{(0)}$ "commutes" with $(\mathcal{S}_1, \tilde{\mathcal{S}}_{1,2}, \mathcal{S}_2)$.

Having defined the Lax operator $\mathcal{L}^{(0)}$, we define the $R-$matrix as an intertwining operator

$$\mathcal{R}_{ij}\mathcal{L}_i^{(0)}\mathcal{L}_j^{(0)} = \mathcal{L}_j^{(0)}\mathcal{L}_i^{(0)}\mathcal{R}_{ij}, \qquad (*)$$

where $\mathcal{L}_j^{(0)} \stackrel{\text{def}}{=} \mathcal{L}^{(0)}(X_i, X_i^*, \psi_i, \psi_i^*)$. We note that (*) automatically implies the Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

The $W_{\mathfrak{gl}(2|2)}$ algebra defined by $\mathcal{L}_1^{(0)}\mathcal{L}_2^{(0)}$ is obtained from the one defined by $\mathcal{L}_1\mathcal{L}_2$ in the limit $\lambda \to 0$. It consists of eight currents

$$\Psi_{12}^* \stackrel{\text{def}}{=} \psi_1^* + \psi_2^*,\ U_{12} \stackrel{\text{def}}{=} \partial X_1^* + \partial X_2^*,\ J_{12} \stackrel{\text{def}}{=} \psi_1\psi_1^* + \psi_2\psi_2^* - \frac{1}{b}\left(\partial X_1 + \partial X_2\right),$$

$$G_{12} \stackrel{\text{def}}{=} i\left(\psi_1\partial X_1^* + \psi_2\partial X_2^* - \frac{1}{b}\left(\partial\psi_1 + \partial\psi_2\right)\right),$$

$$G_{12}^* \stackrel{\text{def}}{=} i\left(\psi_1^*(\partial X_1 - \partial X_2^*) + \psi_2^*(\partial X_2 + \partial X_1^*) - \frac{1}{b}\left(\partial\psi_1^* - \partial\psi_2^*\right)\right),$$

$$T_{12} \stackrel{\text{def}}{=} -\partial X_1\partial X_1^* - \partial X_2\partial X_2^* + \frac{1}{b}(\partial^2 X_1^* - \partial^2 X_2^*) + \partial\psi_1\psi_1^* + \partial\psi_2\psi_2^* - \frac{1}{2}\partial J_{12},$$

and $\Theta_{12} = \ldots$ of spin 2 and $H_{12} = \ldots$ of spin $\frac{5}{2}$.

Working in a particular representation, $RLL = LLR$ relations can be used to define the matrix of R_{12}. We consider the tensor product of two NS representations: $\mathcal{F}_{\boldsymbol{u}} \otimes \mathcal{F}_{\boldsymbol{v}}$ of $\widehat{\mathfrak{gl}}(1|1)$ with the vacuum state

$$|\varnothing\rangle \overset{\text{def}}{=} |\boldsymbol{u}\rangle \otimes |\boldsymbol{v}\rangle.$$

We fix the normalization of $\mathcal{R}_{12}$ by $\mathcal{R}_{12}|\varnothing\rangle = |\varnothing\rangle$. Then using $RLL = LLR$ one can find the action of $\mathcal{R}_{12}$ on any state in $\mathcal{F}_{\boldsymbol{u}} \otimes \mathcal{F}_{\boldsymbol{v}}$. In particular, using two equations

$$\mathcal{R}_{12}\left(\Psi_{12}^{*}\right)_{-\frac{1}{2}}|\varnothing\rangle = \left(\Psi_{21}^{*}\right)_{-\frac{1}{2}}|\varnothing\rangle, \qquad \mathcal{R}_{12}\left(G_{12}^{*}\right)_{-\frac{1}{2}}|\varnothing\rangle = \left(G_{21}^{*}\right)_{-\frac{1}{2}}|\varnothing\rangle,$$

we find that

$$\mathcal{R}_{12}\Big(\psi_1^{*}\Big)_{-\frac{1}{2}}|\varnothing\rangle = \frac{(u-v)-(u^{*}-v^{*})}{(u-v)-(u^{*}+v^{*})+b^{-1}}\Big(\psi_1^{*}\Big)_{-\frac{1}{2}}|\varnothing\rangle - \\ - \frac{2u^{*}-b^{-1}}{(u-v)-(u^{*}+v^{*})+b^{-1}}\Big(\psi_2^{*}\Big)_{-\frac{1}{2}}|\varnothing\rangle,$$

etc

In general, it is clear that the matrix elements of $\mathcal{R}_{12}$ are some rational functions of three parameters

$$u - v, \quad u^* \quad \text{and} \quad v^*.$$

Indeed, from the relation $[\mathcal{R}_{12}, U_{12}] = 0$ it follows that $\mathcal{R}_{12}$ does not depend on $u + v$.

Having defined the $R-$matrix, we study RLL algebra

$$\mathcal{R}_{12}(\boldsymbol{u},\boldsymbol{v})\mathcal{L}_1(\boldsymbol{u})\mathcal{L}_2(\boldsymbol{v}) = \mathcal{L}_2(\boldsymbol{v})\mathcal{L}_1(\boldsymbol{u})\mathcal{R}_{12}(\boldsymbol{u},\boldsymbol{v}), \qquad (*)$$

where as usual $\mathcal{L}_1(\boldsymbol{u})$ and $\mathcal{L}_2(\boldsymbol{v})$ are treated as matrices acting in $\widehat{gl}(1|1)$ modules $\mathcal{F}_{\boldsymbol{u}}$ and $\mathcal{F}_{\boldsymbol{v}}$ correspondingly, whose entries are operators acting in some unspecified space (called quantum space).

The relation (*) is equivalent to the infinite set of quadratic relations between the matrix elements

$$\langle l|\mathcal{L}(u)|k\rangle,$$

where $|k\rangle$ and $|l\rangle$ are arbitrary states from $\mathcal{F}_{\boldsymbol{u}}$.

We use the so called current realization of the same algebra. The isomorphism between the $R-$matrix and current realizations is well understood in the case of classical algebras.

We note that the R-matrix constructed above is rather unusual since

$$\mathcal{R}_{12}(u-v, u^*, v^*)$$

The well known example of non-difference R-matrix has been found by Shastry and depends on two spectral parameters individually rather than on their difference.

Let us define the following currents

$$h(\boldsymbol{u}) \stackrel{\text{def}}{=} \langle\varnothing|\mathcal{L}(\boldsymbol{u})|\varnothing\rangle,$$

$$e(\boldsymbol{u}) \stackrel{\text{def}}{=} h^{-1}(\boldsymbol{u})\langle\varnothing|\mathcal{L}(\boldsymbol{u})\psi_{-\frac{1}{2}}|\varnothing\rangle,\ f(\boldsymbol{u}) \stackrel{\text{def}}{=} \langle\varnothing|\psi_{\frac{1}{2}}\mathcal{L}(\boldsymbol{u})|\varnothing\rangle h^{-1}(\boldsymbol{u}),$$

$$e^*(\boldsymbol{u}) \stackrel{\text{def}}{=} \frac{1}{2u^* - b^{-1}} h^{-1}(\boldsymbol{u})\langle\varnothing|\mathcal{L}(\boldsymbol{u})\psi^*_{-\frac{1}{2}}|\varnothing\rangle,$$

$$f^*(\boldsymbol{u}) \stackrel{\text{def}}{=} \frac{1}{2u^* + b^{-1}}\langle\varnothing|\psi^*_{\frac{1}{2}}\mathcal{L}(\boldsymbol{u})|\varnothing\rangle h^{-1}(\boldsymbol{u}).$$

Taking vacuum average of $RLL = LLR$ relation, one finds

$$[h(\boldsymbol{u}), h(\boldsymbol{v})] = 0,$$

$$\Big((u+u^*)-(v-v^*)+b^{-1}\Big)h(\boldsymbol{u})e(\boldsymbol{v}) = \Big((u-u^*)-(v-v^*)\Big)\, e(\boldsymbol{v})h(\boldsymbol{u}) + \Big(2u^* + b^{-1}\Big)h(\tag{1}$$

$$\Big((u-u^*)-(v+v^*)+b^{-1}\Big)h(\boldsymbol{u})e^*(\boldsymbol{v}) = \Big((u+u^*)-(v+v^*)\Big)\, e^*(\boldsymbol{v})h(\boldsymbol{u}) - \Big(2u^* - b^{-1}\Big)$$

$$\Big((u-u^*)-(v+v^*)+b^{-1}\Big)f(\boldsymbol{v})h(\boldsymbol{u}) = \Big((u+u^*)-(v+v^*)\Big)\, h(\boldsymbol{u})f(\boldsymbol{v}) - \Big(2u^* - b^{-1}\Big)f$$

$$\Big((u+u^*)-(v-v^*)+b^{-1}\Big)f^*(\boldsymbol{v})h(\boldsymbol{u}) = \Big((u-u^*)-(v-v^*)\Big)\, h(\boldsymbol{u})f^*(\boldsymbol{v}) + \Big(2u^* + b^{-1}$$

The terms shown in blue are called local terms, since they depend only on $\boldsymbol{u}$, but not on $\boldsymbol{v}$.

Consider for example the relation (1), set $v - v^* = u - u^*$ and multiply it by $h^{-1}(\boldsymbol{u})$ from the left. Then the first term in the r.h.s. of (1) vanishes and one obtains

$$e(v + v^*, u - u^*) = e(u + u^*, u - u^*) \implies e = e(u - u^*).$$

Similarly one finds

$$f = f(u + u^*), \quad e^* = e^*(u + u^*), \quad \text{and} \quad f^* = f^*(u - u^*).$$

We will also need the following currents

$$\psi(\boldsymbol{u}) \stackrel{\text{def}}{=} \langle\varnothing\psi^*_{\frac{1}{2}}\mathcal{L}(\boldsymbol{u})\psi_{-\frac{1}{2}}|\varnothing\rangle h^{-1}(\boldsymbol{u}) - \langle\varnothing|\psi^*_{\frac{1}{2}}\mathcal{L}(\boldsymbol{u})|\varnothing\rangle h^{-1}(\boldsymbol{u})\langle\varnothing|\mathcal{L}(\boldsymbol{u})\psi_{-\frac{1}{2}}|\varnothing\rangle h^{-1}(\boldsymbol{u}),$$

and

$$\psi^*(\boldsymbol{u}) \stackrel{\text{def}}{=} \langle\varnothing|\psi_{\frac{1}{2}}\mathcal{L}(\boldsymbol{u})|\varnothing\rangle h^{-1}(\boldsymbol{u})\langle\varnothing|\mathcal{L}(\boldsymbol{u})\psi^*_{-\frac{1}{2}}|\varnothing\rangle h^{-1}(\boldsymbol{u}) - \langle\varnothing|\psi_{\frac{1}{2}}\mathcal{L}(\boldsymbol{u})\psi^*_{-\frac{1}{2}}|\varnothing\rangle h^{-1}(\boldsymbol{u}$$

which can be shown to satisfy

$$\psi(\boldsymbol{u}) = \psi(u - u^*) \quad \text{and} \quad \psi^*(\boldsymbol{u}) = \psi(u + u^*).$$

$$\left[\psi_i(u), \psi_j(v)\right] = 0, \quad \{e_i(u), e_i(v)\} = 0, \quad \{f_i(u), f_i(v)\} = 0,$$

$$\left\{e_i(u), f_j(v)\right\} = \delta_{ij}\frac{\psi_i(u) - \psi_i(v)}{u - v},$$

$$(u - v - h_1)(u - v + h_1)\left(e_1(u)e_2(v) + \text{locals}\right) =$$
$$= -(v - u - h_2)(v - u + h_2)\left(e_2(v)e_1(u) + \text{locals}\right)$$

$$(u - v - h_1)(u - v + h_1)\left(f_2(u)f_1(v) + \text{locals}\right) =$$
$$= -(v - u - h_2)(v - u + h_2)\left(f_1(v)f_2(u) + \text{locals}\right)$$

$$\psi_i(u)e_j(v) = \frac{(u - v - h_j)(u - v + h_j)}{(u - v - h_i)(u - v + h_i)}e_j(v)\psi_i(u) + \text{locals}$$

$$\psi_i(u)f_j(v) = \frac{(u - v - h_i)(u - v + h_i)}{(u - v - h_j)(u - v + h_j)}f_j(v)\psi_i(u) + \text{locals}$$

These relations should be understood in terms of modes at $u = \infty$

$$e_k(u) = \frac{e_k^{(0)}}{u} + \frac{e_k^{(1)}}{u^2} + \ldots, \quad f_k(u) = \frac{f_k^{(0)}}{u} + \frac{f_k^{(1)}}{u^2} + \ldots, \quad \ldots$$

Concluding remarks:

- Using current realization of $Y(\widehat{\mathfrak{gl}}(1|1))$ it is straightforward to derive Bethe ansatz equations. These Bethe ansatz equations correspond in particular to quantum version of $\mathcal{N}=2$ KdV hierarchy etc.

- The relations above should be supplemented by Serre relations to define the correct algebra. The derivation of these relations starting from RLL algebra is rather tedious.

- It is interesting to describe explicitly the embedding

$$Y(\widehat{\mathfrak{gl}}(1)) \oplus Y(\widehat{\mathfrak{gl}}(1)) \in Y(\widehat{\mathfrak{gl}}(1|1)).$$

  This should correspond to the definition of $Y(\widehat{\mathfrak{gl}}(1|1))$ in the so called glued basis (Gaberdiel et al).