# Constrained modified gravities: generalized unimodular gravity and beyond

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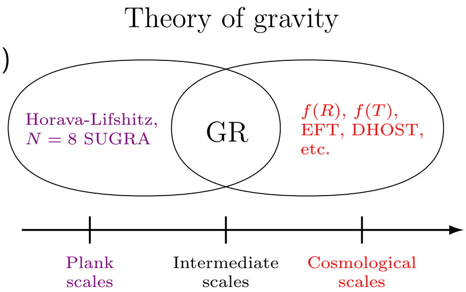
# Plan of the talk

1. Review of modified gravities
2. Constrained modified gravities and its' covariantization
3. Generalized unimodular gravity and it's cosmological applications
4. Models beyond GUMG
5. Some further ideas

# Modified gravities I

What problems do modified gravities try to solve?

- General Relativity perfectly describes physics at intermediate scales, however, there is no
    - consistent quantum theory of gravity (small scales)
    - natural model of late time acceleration (large scales)
- One of the ideas is to modify General Relativity in such a way that it becomes either UV-complete or naturally generates accelerating stage of present Universe
- What are these modifications?



# Modified gravities II

How to modify General Relativity? Just add additional d.o.f!

- Explicit addition of gravitationally interacting field
  - Scalar-tensor theories (k-essence, Horndeski, etc.)
  - Aether (vector) theories
  - etc.
- Implicit inclusion of d.o.f., like in $f(R)$ gravity
- Breaking of general covariance!

# Modified gravities III

Breaking general covarinace — motivations and methods

- Horava gravity. Breaking of coorditane-depenent temporal diffeomorphism invariance

$$t \mapsto t + \xi(t), \qquad x^i \mapsto x^i + \xi^i(t, x^i)$$

  plus imbalance of space and time scaling dimensions

$$t \ \mapsto \ b^{-z} t, \qquad x^i \mapsto b^{-1} x^i$$

  The new invariants are scalars, composed of 3-dimensional curvature and extrinsic curvature. Coupling constants dimensions are nonnegative. Resulting theory is power-counting renormalizeable.

- Massive gravity and self-gravitating media.

$$\mathcal{L}_m = m_0^2\, h_{00} h_{00} + 2m_1^2\, h_{0i} h_{0i} - m_2^2 h_{ij} h_{ij} + m_3^2\, h_{ii} h_{jj} - 2m_4^2\, h_{00} h_{ii}$$

  General case is $S_m = S_{\text{EH}} + \int d^4x \sqrt{-g}\, U(g^{\mu\nu})$.

- "Constrained" modified gravities (UMG, GUMG, etc.)

# Constrained modified gravities I

Algebraic constraints on metric coefficients — naive d.o.f. estimations.
Let us impose some algebraic constraint $U(g^{\mu\nu}) = 0$ on metric coefficients. Then,

- naively, constraint should decrease d.o.f. number
- however, it is not the case for gauge systems. $U(g^{\mu\nu}) = 0$ is second-class Hamiltonian constraint (i.e. it doesn't commute with the superhamiltonians—generators of diffeomorphisms). Stability condition of this constraint may lead to the chain of new constraints, that usually changes the balance of the first and second-class constraints and, consequently, d.o.f. number.
- from pure Lagrangean point of view, such constraint explicitly breaks general covariance, so we have less symmetries to gauge out part of d.o.f. Howevere, it's not always clear what part of gauge symmetry is broken.

Covariantization issue of constrained modified gravities: Stuckelberg trick and equivalence to non-covariant formulation

- Let us enforce the constraint by Lagrange multiplier procedure

$$S[g_{\mu\nu}, \Lambda] = S_{\mathrm{EH}}[g_{\mu\nu}] + \int d^4x \sqrt{-g}\,\Lambda\, U(g^{\mu\nu})$$

- After that, let's introduce four Stuckelberg fields $\phi^A(x)$ and make the change of coordinates $x^A \mapsto \phi^A(X)$, after which the action takes the form

$$S[g_{\mu\nu}, \Lambda, \phi^A] = S_{\mathrm{EH}}[g_{\mu\nu}] + \int d^4x \sqrt{-g}\,\Lambda\, U(C^{AB}), \qquad C^{AB} = g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B$$

Now, if one treats $\phi^A$ and $\Lambda$ as scalar fields, the last action is general covariant.

# Constrained modified gravities II II

- Gauge $\phi^A(x) = x^A$ brings us back to noncovariant formulation. However, it's not the proof of the equivalence yet. One should also show that Stugelberg field equaitons doesn't dynamics new dynamics. Indeed, having stress-energy tensor of the theory

$$T_{\mu\nu} = \Lambda\, U\, g_{\mu\nu} - 2\,\Lambda \frac{\partial U}{\partial C^{AB}} \partial_\mu \phi^A \partial_\nu \phi^B,$$

we see that Stuckelberg field equations are automatically satisfied due to Bianchi identity and constraint equation

$$\begin{cases} \nabla^\mu T_{\mu\nu} = 0, \\ U(C^{AB}) = 0, \end{cases} \quad \Rightarrow \quad 0 = \nabla^\mu \left( \Lambda \frac{\partial U}{\partial C^{AB}} \nabla_\mu \phi^B \right) \propto \frac{\delta S}{\delta \phi^A}$$

The only condition is $\det \partial_\mu \phi^A \neq 0$.

# CMG

Let us consider particular models, which are the most relevant from cosmological ground. Namely, we want to restrict ourselves to functions $U(C^{AB})$, such that

- It's $SO(3)$-**invariant** (under spatial Stuckelberg field $\phi^a$, $a = 1, 2, 3$ rotations)
- Stress-energy tensor, generated by Lagrange multiplier term $\int d^4x \sqrt{-g}\, \Lambda\, U(C^{AB})$ is a **perfect fluid stress-energy tensor**, i.e.

$$T_{\mu\nu} = p\, g_{\mu\nu} + (p + \rho) u_\mu u_\mu$$

Well-studied classification of self-gravitating media shows that there are two types of $U(C^{AB})$ functions, satisfying the above restrictions

$$U(C^{AB}) = V_1(X, Z), \qquad U(C^{AB}) = V_2(b, Z),$$

where

$$X \equiv C^{00}, \quad b \equiv \sqrt{\det C^{ab}}, \quad Z \equiv \sqrt{-\det C^{AB}}$$

Let us consider the first type of constraints $V_1(X, Z) = 0$ in detail.

- Returning to noncovariant setup by $\phi^A(x) = x^A$ gauge, we obtain the following constraint on metric coefficients $V_1(g^{00}, (-g)^{-1/2}) = 0$, or

$$(-g^{00})^{-1/2} = N(\gamma), \qquad \gamma = \det g_{ij}$$

  for some function $N(\gamma)$.
- General relativity, equipped by such type of the constraint is recently proposed[1] and subsequently studied[2] generalized unimodular gravity (GUMG), originally defined by the action

$$S_{\mathrm{GUMG}}[g_{\mu\nu}, \lambda] = S_{\mathrm{EH}}[g_{\mu\nu}] + \int d^4x\, \lambda \left[(-g^{00})^{-1/2} - N(\gamma)\right]$$

- The main motivations of original paper on GUMG theory was
  - An analogy with unimodular gravity (UMG), which is equivalent to GR with the cosmological constant as global d.o.f. with trivial dynamics.
  - "Perfect fluidity" and $SO(3)$-invariance again.

[1] Barvinsky, Kamenshchik, 2017;

[2] Barvinsky, N.K., Kurov, Nesterov, 2019; Barvinsky, N.K., 2019;

# GUMG I

GUMG theory exhibits some peculiar properties

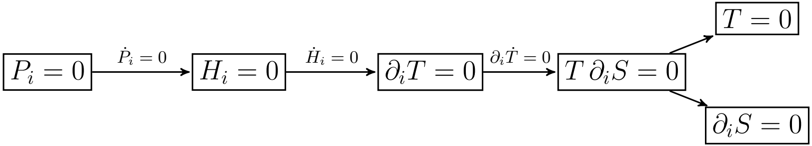
- After transition to ADM variables $\mathcal{N}, \mathcal{N}_i, \gamma_{ij}$

$$g_{\mu\nu} = \begin{pmatrix} -\mathcal{N}^2 + \mathcal{N}^i\mathcal{N}_i & \mathcal{N}_i \\ \mathcal{N}_j & \gamma_{ij} \end{pmatrix}, \qquad i, j = 1, 2, 3,$$

the constraint takes the form $\mathcal{N} = N(\gamma)$ and can be explicitly resolved. Legendre transform brings the action to the following form

$$S_H[\gamma_{ij}, \mathcal{N}^i, \pi^{ij}, P_i, u^i] = \int dt\, d^3x\, (\pi^{ij}\dot{\gamma}_{ij} + P_i\dot{\mathcal{N}}^i - N(\gamma)H_\perp - \mathcal{N}^i H_i - u^i P_i)$$

where $P_i$ is primary constraint, since $\mathcal{N}^i$ is nondynamical. It's conservation condition leads to the chain of secondary (tertiary, etc.) constraints, which bifurcates at the third step




Thus, phase space of the theory divides into two branches with different set of constraints

1. $(P_i, H_i, T)$ — 7 first class constraints, **2 d.o.f.**—graviton polarizations
2. $(P_i, H_i, \partial_i T, \partial_i S)$ — 4 first class and 4 second class constraints, **3 d.o.f.**—2 graviton polarozations plus something new.

- The role of third d.o.f. was examined on the Friedmannian background

$$ds^2 = -N^2(a)\,dt^2 + a^2(t)\,\delta_{ij}dx^i dx^j,$$

which satisfies Friedmann equation

$$H^2 = \frac{1}{3M_P^2}\frac{C}{N(a)a^3}$$

- The idea is to test, whether third d.o.f. can be responsible for primordial scalar CMB spectrum. For this purpose we consider arbitrary metric perturbations

$$\delta\gamma_{ij} = a^2\left(t_{ij} + 2\partial_{(i}F_{j)} - 2\psi\,\delta_{ij} + 2\partial_i\partial_j E\right), \qquad \delta\mathcal{N}^i = \partial^i B + V^i,$$
$$t^i{}_i = \partial^i t_{ij} = \partial_i F^i = \partial_i V^i = 0.$$

After resolution of the constraints only conformal mode $\psi$ survives and has the action

$$S_s = \frac{1}{2}\int d\eta\, d^3x\, z^2\left(\psi'^2 + c_s^2\,\psi\Delta\psi\right), \quad z^2 = \frac{3\Omega}{2w}a^2, \quad c_s^2 = \frac{w}{\Omega}(1+w), \quad \Omega = 1+w+\frac{1}{3}\frac{d\ln w}{d\ln a}$$

If one impose the conditions of 1) $\sim 60$ e-folding inflationary stage, 2) flatnass of the spectrum tilt $n_s - 1 \ll 1$, 3) absence of mass and gradient instabilities, the parameter of the model $N(\gamma)$ becomes fixed as

$$N(\gamma) = \frac{1}{\sqrt{\gamma}}\left[1 + \left(\frac{\gamma}{\gamma_*}\right)^{\frac{1}{2}} + B\left(\frac{\gamma}{\gamma_*}\right)^{\frac{3}{2}+\frac{4}{3}(n_s-1)} + \dots\right]$$

# Duality to k-essence I

Let us reinterpret these results in the convariant setup.

- First of all, let us assume that the first type of the constraint $V_1(X, Z) = 0$ can be resolved as $P(X) - Z = 0$, and the action becomes

$$S[g_{\mu\nu}, \Lambda, \phi^A] = S_{\text{EH}}[g_{\mu\nu}] + \int d^4x \sqrt{-g}\, \Lambda\, (P(X) - Z), \qquad X = \nabla^\mu \phi^0 \nabla_\mu \phi^0, \;\; Z = \frac{\det \partial_\mu \phi^A}{\sqrt{-g}}$$

- Let us project stress-energy tensor

$$T_{\mu\nu} = \Lambda\, (P g_{\mu\nu} + 2 P_X X v_\mu v_\nu), \quad v_\mu = -\frac{\partial_\mu \phi^0}{\sqrt{-X}}$$

onto the linear space, orthogonal to comoving vector $v_\mu$

$$(\delta^\nu_\sigma + v^\nu v_\sigma) \nabla^\mu T_{\mu\nu} = P(X)\, (\delta^\nu_\sigma + v^\nu v_\sigma) \partial_\nu \Lambda = 0$$

The only solution is $\Lambda = K(\phi^0)$.

# Duality to k-essence II

- Using the last result, one can see that the variation of the action with respect to $\phi^0$

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\phi^0} = \left(\frac{\partial}{\partial\phi^0} - \nabla_\mu \frac{\partial}{\partial(\nabla_\mu\phi^0)}\right) K(\phi^0)P(X)$$

and stress-energy tensor (after the substitution $\Lambda = K(\phi^0)$) can be obtained from the following k-essence action

$$S_k[g_{\mu\nu},\phi^0] = S_{\text{EH}}[g_{\mu\nu}] + \int d^4x\sqrt{-g}\,K(\phi^0)P(X)$$

it is the statement of the duality.

- **Disclaimer:** the correspondence is not complete, since $K(\phi^0)$ can depend on initial condition of original theory. Thus, k-essence action obtained should be used with only these initial conditions. Moreover, we don't know, how to reconstruct $K(\phi^0)$ on arbitrary background.

# Duality to k-essence on Friedmannian background I

Nevertheless, $K(\phi)$ can be easily reconstructed from $P(X)$ on Friedmannian background.

- To develop this reconstruction procedure, we consider a homogeneous background

$$ds^2 = -\mathcal{N}^2(t)\,dt^2 + a^2(t)\,\delta_{ij}dx^i dx^j, \quad \phi^0 = \phi(t), \quad \phi^a = x^a, \quad \Lambda = \Lambda(t).$$

so that the covariant action takes the form

$$S[a,\mathcal{N},\phi,\Lambda] = -3M_P^2 \int dt\,\mathcal{N} a^3 \frac{\dot{a}^2}{\mathcal{N}^2 a^2} + \int dt\,\mathcal{N}\,\Lambda\left(a^3 P(X) - \sigma\sqrt{-X}\right),$$

where $\sigma \equiv \mathrm{sign}(\dot{\phi}/\mathcal{N})$, $X = -\dot{\phi}^2/\mathcal{N}^2$ whereas background constraint reads

$$a^3 = \sigma\sqrt{-X}/P(X)$$

- Stuckelberg field $\phi^0 = \phi$ e.o.m. can be integrated as

$$-\Lambda\,(2\sigma a^3 P_X\sqrt{-X} + 1) = C \quad \Rightarrow \quad \Lambda = C\,w(X)$$

# Duality to k-essence on Friedmannian background II

- The idea of $K(\phi)$ reconstruction is to evaluate $X$ as a function of $\phi$. Indeed, dividing Friedmann equation by $\dot{\phi}^2/\mathcal{N}^2 = -X$

$$\frac{\dot{a}^2}{\mathcal{N}^2 a^2} = \frac{\Lambda}{3M_P^2}(2P_X X - P) \quad \Rightarrow \quad \frac{d\ln a^3}{d\phi} = M_P^{-1}\sqrt{\frac{3CP(X)}{-X}}$$

one gets a differential equation, which can be integrated as

$$\phi(a) - \phi_0 = M_P \int_{a_0}^{a} da'\, a'^2 \sqrt{\frac{3P(X(a'))}{C}} \equiv \Phi(a) - \Phi(a_0)$$

- Substitution $X \mapsto X(a) \mapsto X(a(\phi))$ gives the definition of $K(\phi)$

$$\Lambda = Cw(X(a(\phi))) \equiv K(\phi)$$

# Duality to k-essence on Friedmannian background III

- The resulting k-essence theory $\mathcal{L}_k = K(\phi)P(X)$ should be used with very specific initial conditions
$$\phi(t_0) = \Phi(a_0), \qquad \frac{\dot{\phi}}{\mathcal{N}}\bigg|_{t_0} = \sigma\sqrt{-X(a_0)}$$
because they should satisfy the constraint equation.
- It's easy to see that perturbation parameters of obtained k-essence theory coincide with those obtained in noncovariant GUMG theory. The main ingredient is the connection between GUMG theory parameter $N(\gamma)$ and covariant formulation parameter $P(X)$
$$P(X) \equiv \left(\frac{-X}{\Gamma(1/\sqrt{-X})}\right)^{1/2}, \quad \Gamma(N(\gamma)) \equiv \gamma.$$

Let us provide some examples of $K(\phi)$-reconstruction from known $P(X)$—the ingredients of dual k-essence Lagrangian $\mathcal{L}_k(\phi, X) = K(\phi)P(X)$

1. The first example is not interesting from the physical ground, but in this case $K(\phi)$ can be reconstructed explicitly

$$P(X) = \frac{-\alpha X}{\beta^2 - \sqrt{-X}} \quad \Rightarrow \quad K(\phi) = C\,\tanh^2\frac{\phi}{\tilde{\phi}}, \quad \tilde{\phi} \equiv \frac{2\beta M_P}{\sqrt{3C\alpha}}$$

   initial conditions should be chosen as

$$\phi\Big|_{t_0} = \tilde{\phi}\,\mathrm{arsinh}\sqrt{\alpha a_0^3}, \quad \frac{\dot{\phi}}{\mathcal{N}}\Big|_{t_0} = \frac{\beta^2}{1 + \alpha\, a_0^3}$$

2. The second example corresponds to the model of canonically normalized kinetic term theory

$$P(X) = -\beta X - \alpha, \qquad \sqrt{-X_{\pm}(a)} = \frac{1}{2\beta\, a^3}\left(\sqrt{1 + 4\alpha\beta\, a^6} \pm 1\right)$$

Indeed, after the following change of coordinates

$$\varphi = \int_0^{\phi} d\phi' \sqrt{2\beta K(\phi')}$$

the Lagrangian $\mathcal{L}_k(\phi, X) = K(\phi)P(X)$ takes the form

$$\mathcal{L}_\varphi = \frac{1}{2}\frac{\dot{\varphi}}{\mathcal{N}^2} - V(\varphi), \qquad V(\varphi) = -\alpha K(\phi(\varphi)).$$

Unfortunately, $K(\phi)$ and corresponding $V(\varphi)$ cannot be found analytically, but asymptotics can be found

Constraint solution

$$\sqrt{-X_{\pm}(a)} = \frac{1}{2\beta\, a^3}\left(\sqrt{1+4\alpha\beta\, a^6} \pm 1\right)$$

- $+$ **sign**

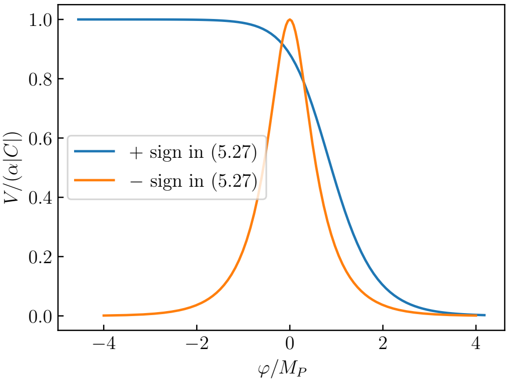
$$V(\varphi) = -C\alpha + O\left(e^{-\sqrt{6}|\varphi|/M_P}\right), \quad \frac{\varphi}{M_P} \to -\infty$$

$$V(\varphi) = -C\alpha\, O\left(e^{-\sqrt{3}\varphi/M_P}\right), \quad \frac{\varphi}{M_P} \to +\infty.$$

- $-$ **sign**

$$V(\varphi) = -C\alpha\left(1 - 3(\varphi/M_P)^2 + O((\varphi/M_P)^4)\right), \quad |\varphi/M_P| \ll 1$$

$$V(\varphi) = C\alpha\, O(e^{-\sqrt{3}|\varphi/M_P|}), \qquad |\varphi/M_P| \gg 1$$



3. The last example corresponds to particular GUMG model, which leads to primordial spectrum, consistent with observations. We should take $N(\gamma)$ found in concovariant formalism and convert it into $P(X)$ using the formula

$$P(X) \equiv \left(\frac{-X}{\Gamma(1/\sqrt{-X})}\right)^{1/2}, \quad \Gamma(N(\gamma)) \equiv \gamma.$$

The result is

$$P(X) = 1 - \left(\frac{-X}{\gamma_*}\right)^{\frac{1}{2}} - B\left(\frac{-X}{\gamma_*}\right)^{\frac{3}{2}+4\frac{n_s-1}{3}} + \ldots = 1 - \delta - B\,\delta^{3+8\frac{n_s-1}{3}}, \quad \delta \equiv \left(\frac{-X}{\gamma_*}\right)^{\frac{1}{2}}$$

$$K(\phi) = -3M_P^2H_0^2\left[1 - \varepsilon + \frac{3}{4}\varepsilon^2 - \frac{1}{2}\varepsilon^3 - \left(3 + 8\,\frac{n_s-1}{3}\right)B\,\varepsilon^{3+8\frac{n_s-1}{3}} + \ldots\right], \quad \varepsilon \equiv \frac{3H_0\phi}{\sqrt{\gamma_*}}$$

# Duality to k-essence: examples V

For small spectrum tilt $n_s - 1 \ll 0$ it looks like radiative corrections terms. Indeed

$$\delta^{3+8\frac{n_s-1}{3}} \simeq \left(\frac{-X}{\gamma_*}\right)^{\frac{3}{2}} + 4\,\frac{n_s-1}{3}\left(\frac{-X}{\gamma_*}\right)^{\frac{3}{2}} \ln\frac{-X}{\gamma_*},$$

$$\varepsilon^{3+8\frac{n_s-1}{3}} \simeq \left(\frac{3H_0\phi}{\sqrt{\gamma_*}}\right)^3 + 8\frac{n_s-1}{3}\left(\frac{3H_0\phi}{\sqrt{\gamma_*}}\right)^3 \ln\frac{3H_0\phi}{\sqrt{\gamma_*}}.$$

# Beyond GUMG: spatial media I

- Let us remember that until now we have considered only the first of two types of the constraints, that are $SO(3)$-invariant and provide a perfect fluid stress energy tensor

$$U(C^{AB}) = V_1(X, Z), \qquad U(C^{AB}) = V_2(b, Z),$$

where

$$C^{AB} \equiv g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B, \quad X \equiv C^{00}, \quad b \equiv \sqrt{\det C^{ab}}, \quad Z \equiv \sqrt{-\det C^{AB}}$$

- The second class is also interesting and also admits decoupling of the part of Stuckelberg fields. Resolving the constraint $V_2(b, Z) = 0$ as $Q(b) - Z = 0$ we obtain the action

$$S[g_{\mu\nu}, \Lambda, \phi^A] = S_{\rm EH}[g_{\mu\nu}] + \int d^4x\sqrt{-g}\,\Lambda\,(Q(b) - Z)$$

whose stress-energy tensor is

$$T_{\mu\nu} = \Lambda\left[(Q - b\,Q_b)\,g_{\mu\nu} - b\,Q_b\,u_\mu u_\nu\right], \qquad u^\mu = \frac{1}{3!\sqrt{-g}\,b}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abc}\partial_\nu\phi^a\partial_\rho\phi^b\partial_\sigma\phi^c,$$

# Beyond GUMG: spatial media II

- Projecting Bianchi identities on the comoving vector, one obtains

$$u^\nu \nabla^\mu T_{\mu\nu} = Q(b)\, u^\mu \partial_\mu \Lambda = 0.$$

that implies $\Lambda = L(\phi^a)$

- Similar to covariantized GUMG case, e.o.m. of the theory can be obtained from the action

$$S[g_{\mu\nu}, \phi^a] = S_{\text{EH}}[g_{\mu\nu}] + \int d^4x \sqrt{-g}\, L(\phi^a)\, Q(b)$$

- On the Friedmann background $L = \text{const}$ so that the theory is equivalent to self-gravitating media model. It is shown in literature that such theories can also be responsible for primordial CMB spectrum[3].

[3] Endlich, Nicolis, Wang, 2012;

# Conclusion and further ideas I

Conclusion

- We have discussed modified gravity theories that differ from GR by algebraic constraint on metric coefficients. We call it "constraint" modified gravity here.
- Next, we covariantize such theories by means of Stuckelberg fields and concentrate on the constraints that are $SO(3)$-invariant and provide perfect fluid stress-energy tensor.
- We found that there two type of such constraints. The first is nothing that recently proposed GUMG theory, while the second seems like something new.
- Further, we briefly review the properties of GUMG theory in noncovariant setup and notice that it's cosmological perturbation theory is very similar to k-essence perturbation theory.
- We found that covariantized GUMG theory is really equivalent to some k-essence theory. It's true on arbitrary background.

# Conclusion and further ideas II

- We develop a reconstruction procedure of k-essence action dual to GUMG theory on Frieadmannian background and reinterpret cosmological applications of GUMG theory in the covariant setup.
- We discuss the second type of the constraint on the metric defined above, and found that in this case decoupling of part of Stuckelberg fields also takes place.

Todo

- Matter content?
- Graceful exit?
- Models beyond perfect fluids?

Thank you for your attention!