ITMP seminar, Zoom, May 2025

Amplitudes for Hawking Radiation

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With

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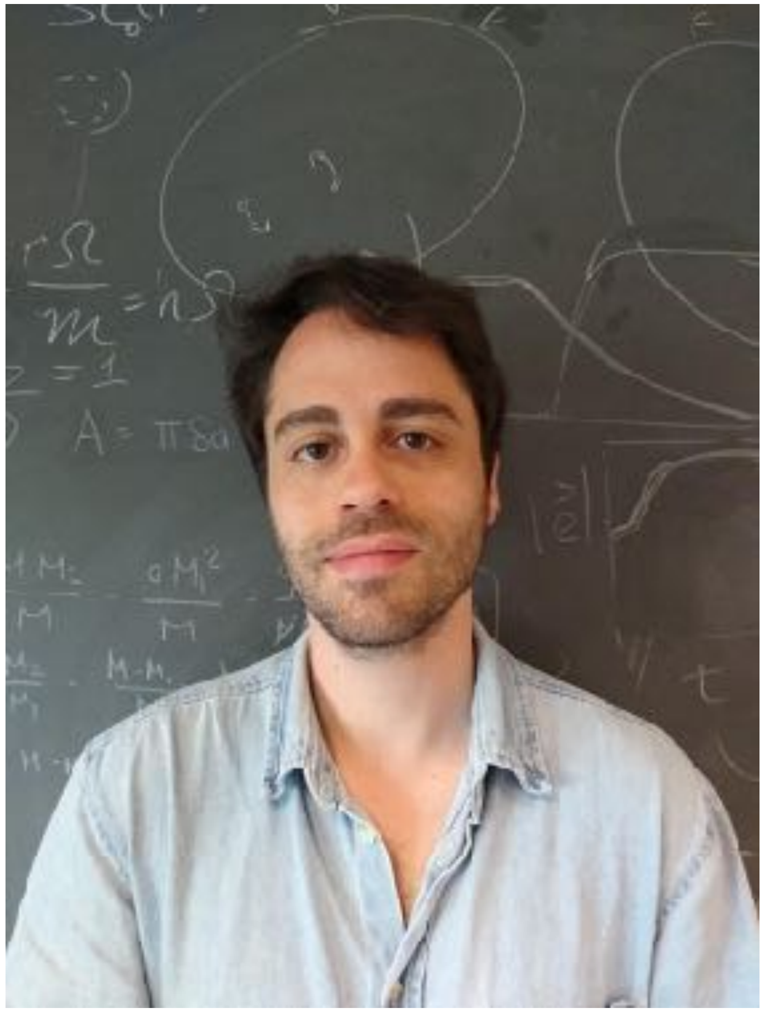
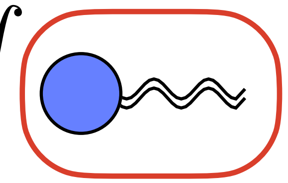


Matteo Sergola
IPhT

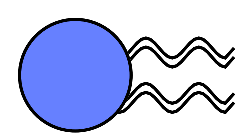
Motivation

Motivation

Classical gravitational waveform: from (generalised) amplitudes:

$$\text{waveform}(t) = \int$$  $$+ \cdots$$

Single-graviton production
amplitude (in-in)

Amplitudes: quantum. Pair production? 

Motivation

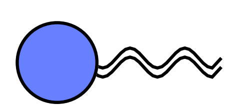
Gravitational wave ≠ graviton?

classical

Graviton number state

$$\langle n | \mathbb{h}_{\mu\nu} | n \rangle \sim \langle n | a(k) | n \rangle = 0$$

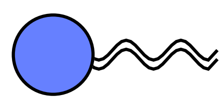
Resolution: wave is a coherent state of gravitons

$$S\,|\psi\rangle \supset \exp\left[\int_k$$

$$a^\dagger(k)\right]|0\rangle$$

Single-graviton production amplitude

Motivation

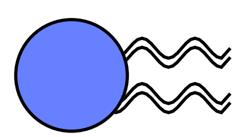
Classically:

$S\,|\psi\rangle \supset \exp\Big[\int_k$  $a^\dagger(k)\Big]\,|0\rangle$

Coherent state
Classical waveform

Uncorrelated single graviton production

Semiclassically:

$S\,|\psi\rangle \supset \exp\Big[\int_{k_1,k_2}$  $a^\dagger(k_1)a^\dagger(k_2)\Big]\,|0\rangle$

Squeezed state

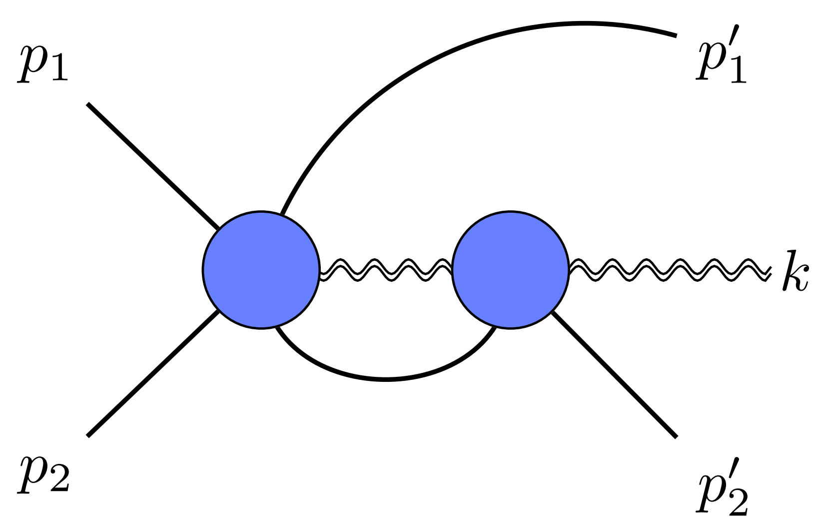
Uncorrelated production of pairs

Hawking radiation: pair production from BH

Motivation

So study how to compute Hawking radiation from amplitudes

Eg usual radiation backreaction

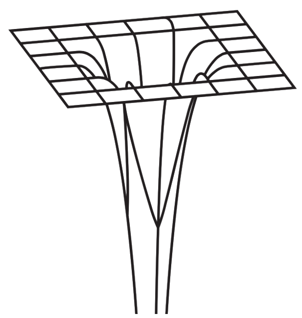


Compute reaction to Hawking pairs?

This talk

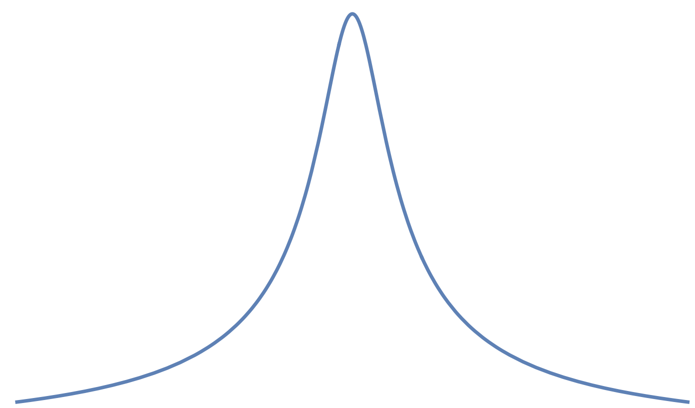
1. Waveforms and (generalised) amplitudes
2. Amplitudes and Bogoliubov
3. Hawking's scattering process, spectrum
4. Outlook

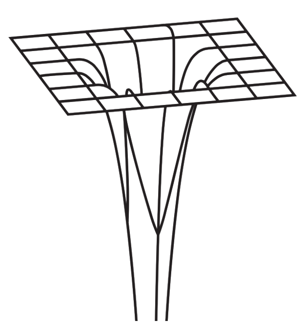
Waveforms & Amplitudes



b

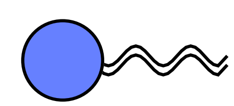
$$\text{waveform}(t) =$$





$$= \int \boxed{\langle p_1' p_2' | S^\dagger a(k) S | p_1 p_2 \rangle} + \cdots$$

Generalised amplitude



Classical point particle approximation (corrected at high order)

Kosower, Maybee & DOC

Cristofoli, Gonzo, Kosower & DOC

Waveforms & Amplitudes

Amplitudes: from time evolution (interaction picture)

$$|\psi\rangle \xrightarrow{\text{time}\to\infty} \boxed{U(\infty,-\infty)|\psi\rangle}$$

Initial state
Far past

Final state
Far future

Waveforms & Amplitudes

Amplitudes: from time evolution (interaction picture)

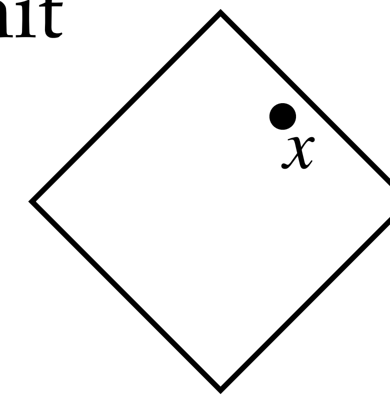
$$|\psi\rangle \xrightarrow{\text{time}\to\infty} S|\psi\rangle \qquad U(-\infty,\infty) = S = 1 + iT$$

Matrix elements of T are the amplitudes

$$\langle q_1 \cdots q_m|T|p_1 \cdots p_n\rangle = \mathcal{A}(p_1 \ldots p_n \to q_1 \ldots q_m)\,\delta^4(\text{total momentum})$$

Waveforms & Amplitudes

Measure *expectation* of curvature component in classical limit



Riemann curvature
operator

$$\text{waveform} \equiv \langle \psi | S^\dagger \mathbb{R}_{....}(x) S | \psi \rangle$$

Final state
Amplitudes!

Dominant term at large distances
(Ψ_4)

$$|\psi\rangle \sim \int \bigwedge \bigwedge e^{ip_1 \cdot b} |p_1\, p_2\rangle$$

Classical initial conditions

Waveforms & Amplitudes

Measure *expectation* of curvature component in classical limit

$$\mathbb{R}_{....}(x) = \partial.\partial.\mathbb{h}_{..}(x)$$

Leading in distance

Graviton polarisation

$$\text{waveform} = \int \mathrm{d}\Phi(k)\left[kk\,\varepsilon\varepsilon\,e^{-ik\cdot x}\langle\psi|S^\dagger a(k) S|\psi\rangle + \text{c.c.}\right]$$

On-shell
phase space

Waveforms & Amplitudes

$$\text{waveform} = \int \mathrm{d}\Phi(k)\,\left[kk\,\varepsilon\varepsilon\,e^{-ik\cdot x}\langle\psi|S^\dagger a(k)S|\psi\rangle + \text{c.c.}\right]$$

Key object to understand

$$\langle\psi|S^\dagger a(k)S|\psi\rangle = \int_{p_i,p_i'} \psi^*(p_1',p_2')\psi(p_1,p_2)\,\boxed{\langle p_1',p_2'|S^\dagger a(k)S|p_1,p_2\rangle}$$

Generalised amplitude

Caron-Huot, Giroux, Hannesdóttir, Mizera

Elkhidir, Sergola, Vazquez-Holm, DOC

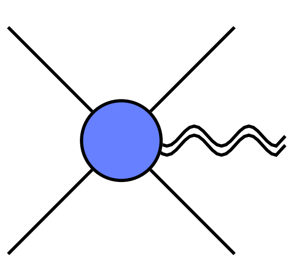
Waveforms & Amplitudes

Generalised amplitude closely related to amplitudes

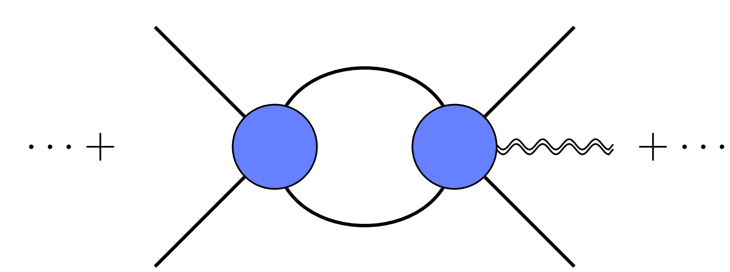
$$\mathcal{E}\,\delta^4(\text{momentum}) \equiv -i\langle p_1' p_2' | S^\dagger a(k) S | p_1 p_2 \rangle$$

$$= \langle p_1' p_2' k | T | p_1 p_2 \rangle - i \langle p_1' p_2' | T^\dagger a(k) T | p_1 p_2 \rangle$$

5 point amplitude



Cut (subtraction) term

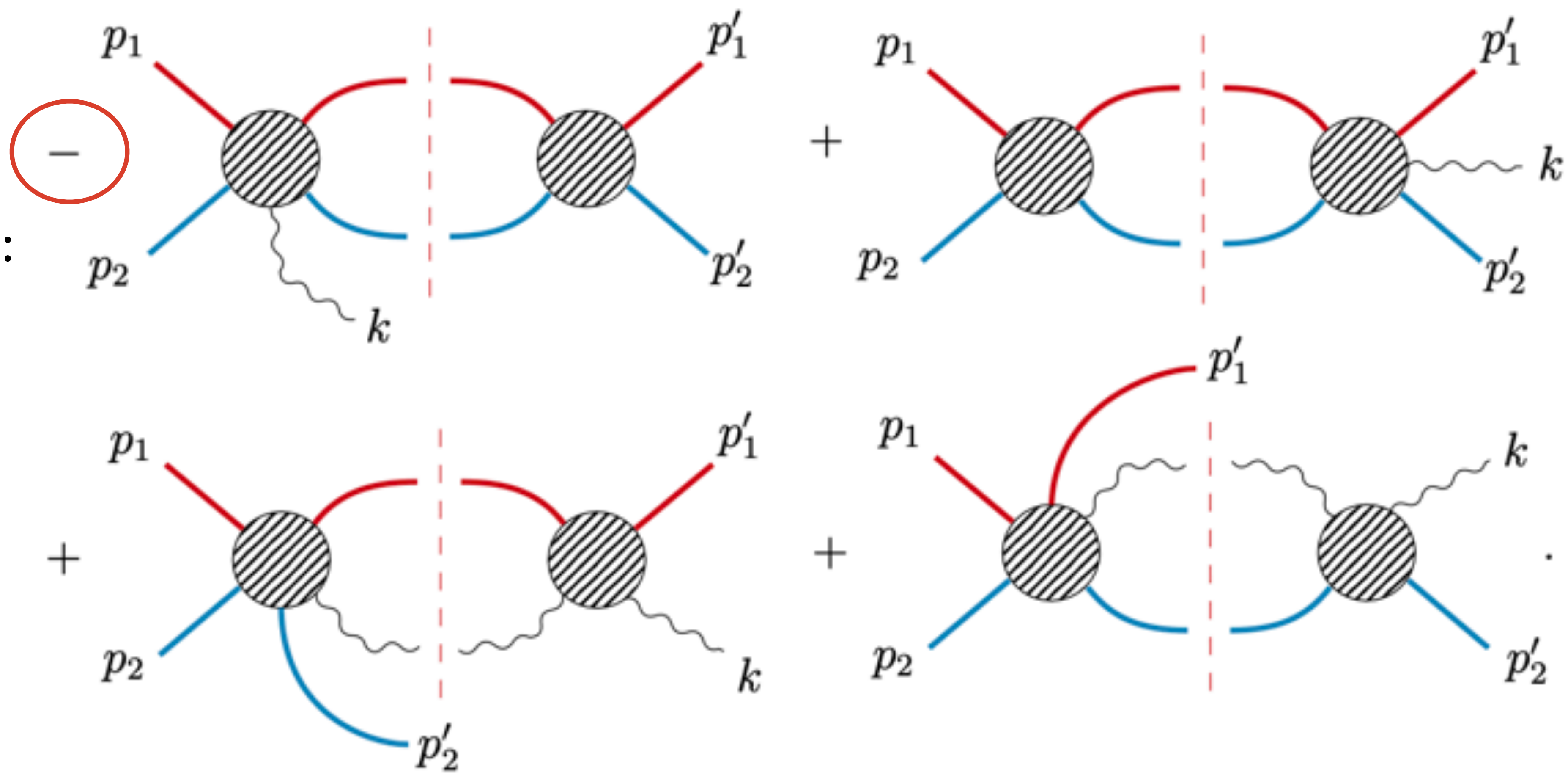


Waveforms & Amplitudes

At NLO only difference:

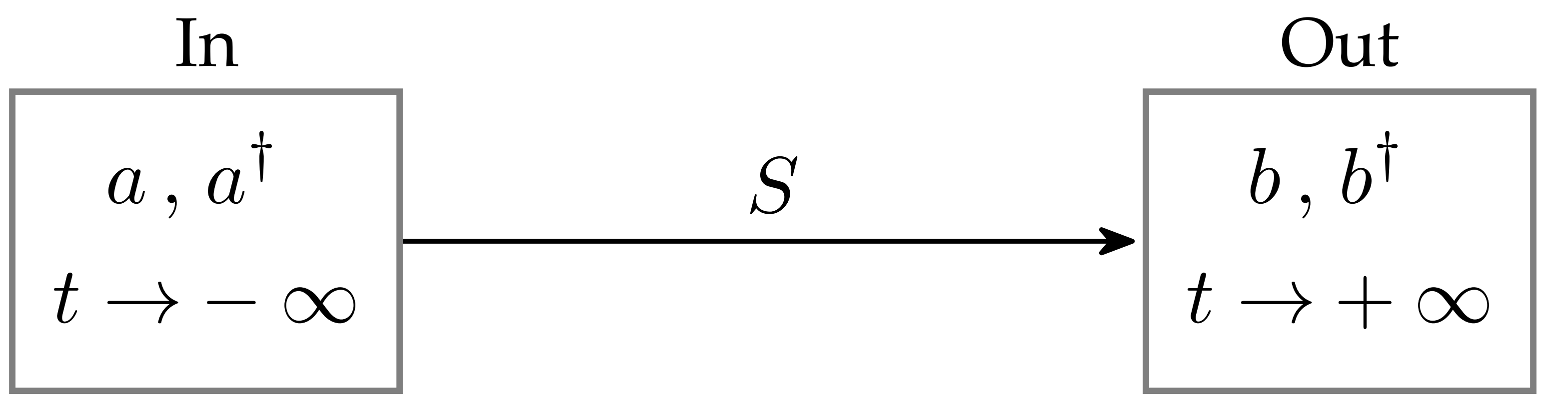
$$2\,\mathrm{Im}'\,\mathcal{E}(p_1 p_2 \to p_1' p_2' k_\eta)\hat{\delta}^D(p_1 + p_2 - p_1' - p_2' - k) =$$

Amplitude:
plus sign



Waveforms & Amplitudes

Past/future ladder operators



$$b(p) = S^\dagger a(p) S$$

Waveforms & Amplitudes

Past / future ladder operators

$$\boxed{\begin{matrix} a\,,\,a^\dagger \\ t \to -\infty \end{matrix}} \xrightarrow{\quad S \quad} \boxed{\begin{matrix} b\,,\,b^\dagger \\ t \to +\infty \end{matrix}}$$

$$i\mathcal{A}\delta^4(\mathrm{momentum}) = \langle 0| b(p_1')b(p_2')b(k)a^\dagger(p_1)a^\dagger(p_2)|0\rangle$$

$$i\mathcal{E}\delta^4(\mathrm{momentum}) = \langle 0| a(p_1')a(p_2')b(k)a^\dagger(p_1)a^\dagger(p_2)|0\rangle$$

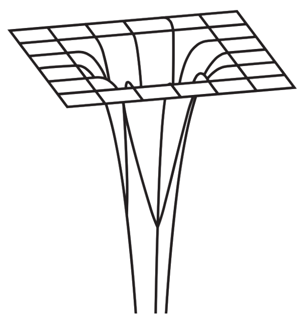
Physics: different $i\epsilon$ prescription. In-out vs in-in, Schwinger-Keldysh

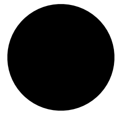
Expectation admits classical limit, amplitudes doesn't

Caron-Huot, Giroux, Hannesdóttir, Mizera

Waveforms & Amplitudes

Scattering on background:





$$\text{waveform}(t) = \int \boxed{\langle \Omega | a(p_1') S^\dagger a(k) S a^\dagger(p_1) | \Omega \rangle}$$

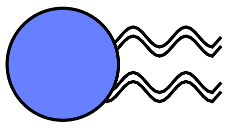
Generalised 3-point amplitude
on background

Appropriate at LO in m/M

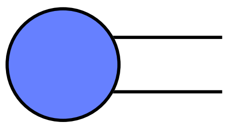
Amplitudes & Bogoliubov

Amplitudes & Bogoliubov

Pair production: any species



Graviton pair production



Scalar pair production: simpler

Amplitudes & Bogoliubov

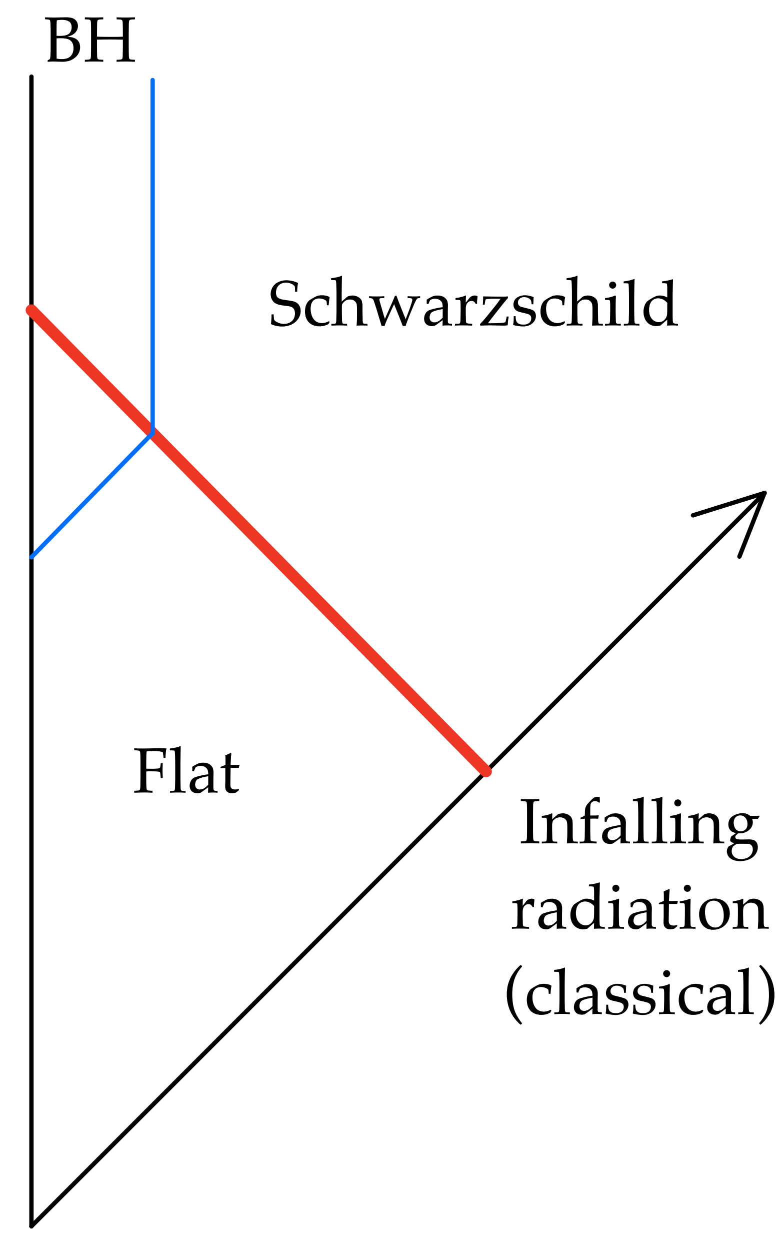
Produced from dynamical background metric “Vaidya”

Thin shell of radiation
falling in to origin

$$g^{\mu\nu} = \eta^{\mu\nu} + \frac{2GM\Theta(t+r)}{r}k^\mu k^\nu$$

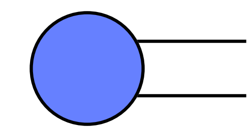
Kerr-Schild vector

$$k_\mu \mathrm{d}x^\mu = \mathrm{d}t + \mathrm{d}r$$



Amplitudes & Bogoliubov

$$\text{action} = \frac{1}{2}\int \mathrm{d}^4x\,\partial_\mu\phi\partial_\nu\phi\,(\eta^{\mu\nu} + h^{\mu\nu})$$



Bogoliubov very useful:

$$\underset{\text{In}}{\boxed{\begin{matrix} a\,,\,a^\dagger \\ t\to-\infty \end{matrix}}} \xrightarrow{\quad S \quad} \underset{\text{Out}}{\boxed{\begin{matrix} b\,,\,b^\dagger \\ t\to+\infty \end{matrix}}}$$

$$b(p) = S^\dagger a(p) S$$

Amplitudes & Bogoliubov

Exact field eom $$\partial^2\phi + \partial_\mu h^{\mu\nu}(x)\partial_\nu\phi = 0$$

Past solutions $P(x,p)$

$$P(x,p) \xrightarrow[t\to-\infty]{} e^{-ix\cdot p}$$

Future solutions $F(x,p)$

$$F(x,p) \xrightarrow[t\to\infty]{} e^{-ix\cdot p}$$

$$\phi(x) = \int \mathrm{d}\Phi(p)\left[P(x,p)a(p) + \text{h.c.}\right]$$

$$\phi(x) = \int \mathrm{d}\Phi(p)\left[F(x,p)b(p) + \text{h.c.}\right]$$

Amplitudes & Bogoliubov

$$b(k) = S^\dagger a(k) S = \int \mathrm{d}\Phi(p) \big(A(k,p) a(p) + B(k,p) a^\dagger(p) \big)$$

$$P \sim \Sigma \left(AF + \bar{B}\bar{F} \right)$$

F is a basis

$$\phi(x) = \int \mathrm{d}\Phi(p) \big[P(x,p) a(p) + \text{h.c.} \big] \;=\; \phi(x) = \int \mathrm{d}\Phi(p) \big[F(x,p) b(p) + \text{h.c.} \big]$$

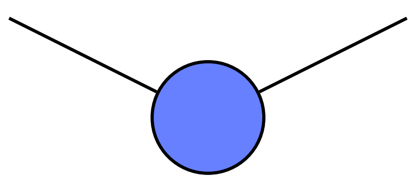
Amplitudes & Bogoliubov

$$b(k) = S^\dagger a(k) S = \int \mathrm{d}\Phi(p) \big(A(k,p) a(p) + B(k,p) a^\dagger(p) \big)$$

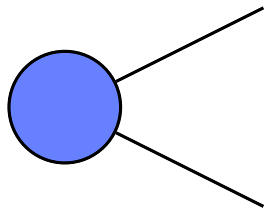
Bogoliubov coefficients = (generalised) amplitudes

Vaidya vacuum

$$A(k,p) = \langle \Omega | S^\dagger a(k) S a^\dagger(p) | \Omega \rangle$$



$$B(k,p) = \langle \Omega | a(p) S^\dagger a(k) S | \Omega \rangle$$



Amplitudes & Bogoliubov

Count number of particles in future

$$\boxed{\begin{matrix} a\,,\,a^\dagger \\ t \to -\infty \end{matrix}} \xrightarrow{\quad S \quad} \boxed{\begin{matrix} b\,,\,b^\dagger \\ t \to +\infty \end{matrix}}$$

$$\begin{aligned} \langle \Omega | b^\dagger(p) b(p) | \Omega \rangle &= \langle \Omega | S^\dagger a^\dagger(p) a(p) S | \Omega \rangle \\ &= \int \mathrm{d}\Phi(k)\, \bar{B}(p,k) B(p,k) \end{aligned}$$

Just need Bogoliubov ***B***

Amplitudes & Bogoliubov

"Crossing": follows since F and P are both bases $P \sim \Sigma\left(AF + \bar{B}\bar{F}\right)$

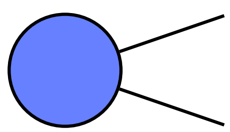
$$A(k,p) = \int \mathrm{d}^3x \left[F^*(x,k)\, i\partial_t e^{-ip\cdot x} - e^{-ip\cdot x}\, i\partial_t F^*(x,k)\right]$$

$$B(k,p) = \int \mathrm{d}^3x \left[F^*(x,k)\, i\partial_t e^{+ip\cdot x} - e^{+ip\cdot x}\, i\partial_t F^*(x,k)\right]$$

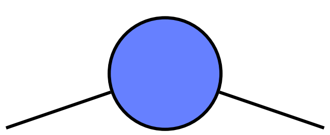
Analytically continue $p \to -p$

Amplitudes & Bogoliubov

"Crossing"



Scalar pair production



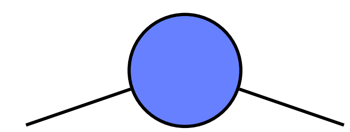
Scalar scattering: more familiar

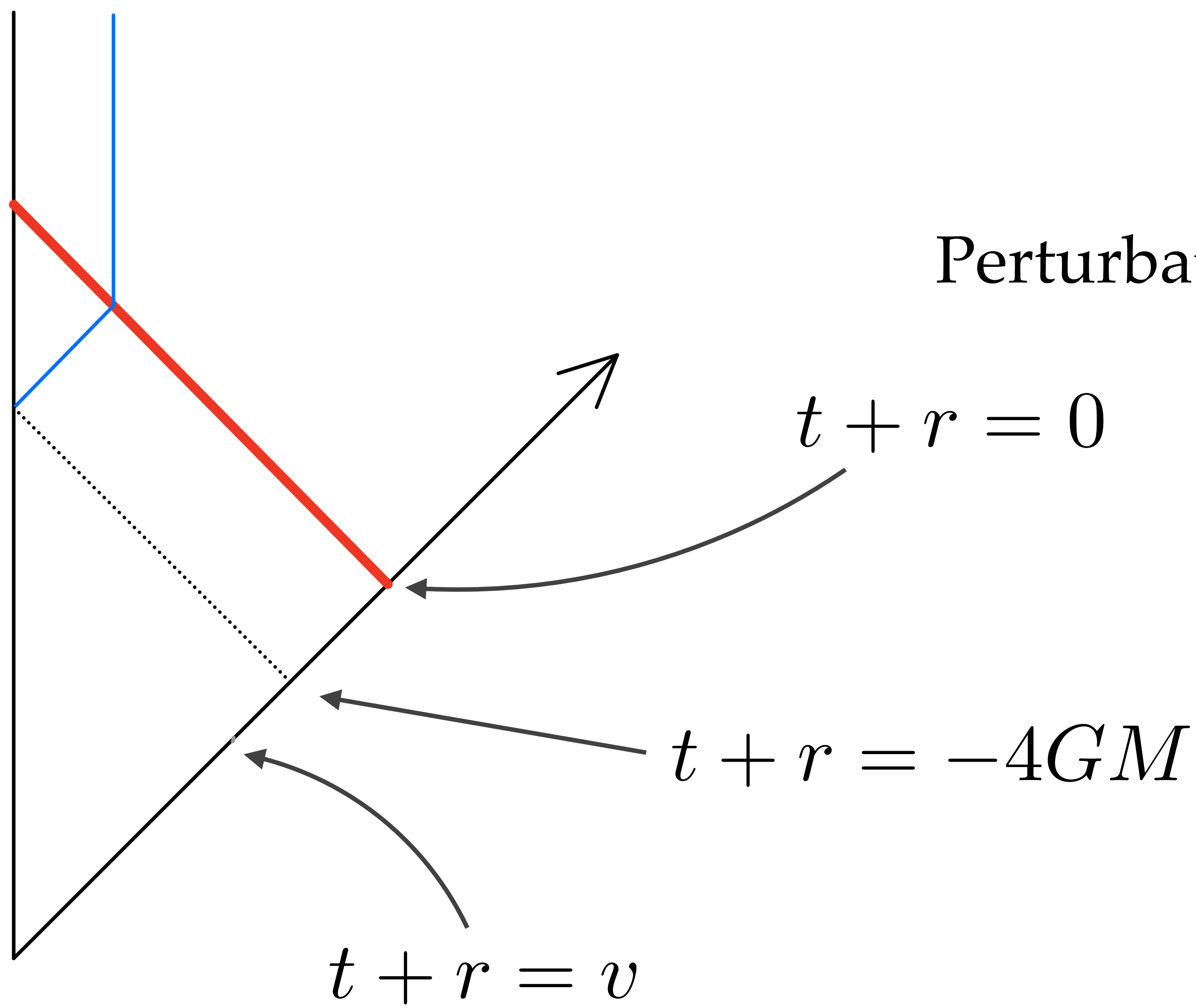
"Hawking" scattering
Quantum process

Hawking's Scattering Process

Hawking Scattering

Scatter spherical particle through background





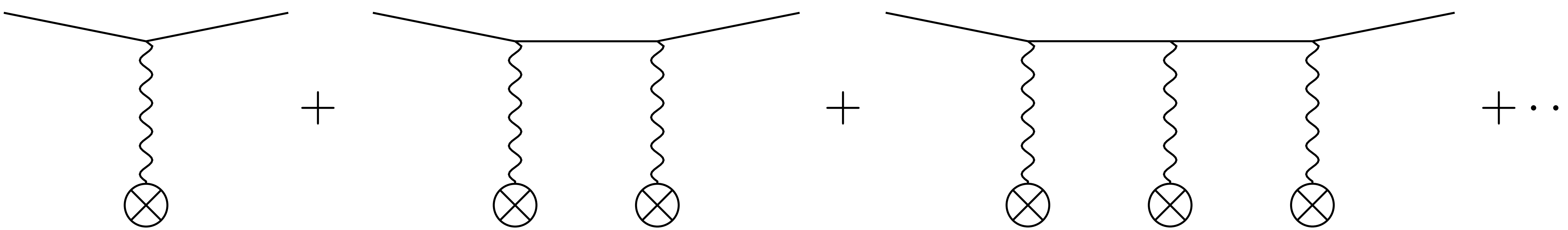
Perturbative parameter

$$\frac{v}{4GM}$$

Hawking Scattering

Lippmann-Schwinger approach

$$\partial^2\phi + \partial_\mu h^{\mu\nu}(x)\partial_\nu\phi = 0$$



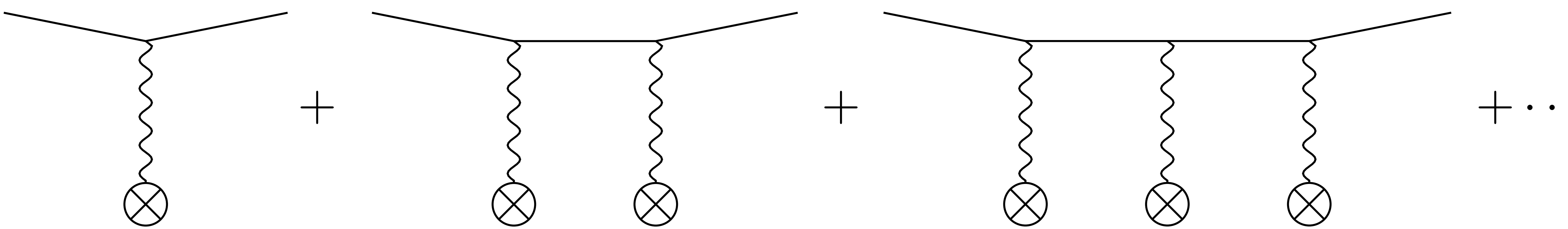
Hawking Scattering

Exact quantum scattering: hard

Analogous to $\hbar$

Simplify: high energy "geometric optics" $\sim e^{iS(x)/\eta}$

$$\mathcal{A}_0^{(-1)} \sim \frac{1}{\eta} \qquad \mathcal{A}_1^{(-2)} \sim \frac{1}{\eta^2} \qquad \mathcal{A}_2^{(-3)} \sim \frac{1}{\eta^3}$$



Hawking Scattering

Scatter massless spherical particle through background

$$|\psi\rangle = \int \mathrm{d}\Phi(p)\,\boxed{\varphi(p)}\,|p\rangle = \int \mathrm{d}\Phi(p)\,\boxed{\int \mathrm{d}v\,\varphi(v)e^{iEv}}\,|p\rangle$$

$$= \int \mathrm{d}\Phi(p) \int \mathrm{d}v\,\varphi(v)e^{ip\cdot b(v)}\,|p\rangle$$

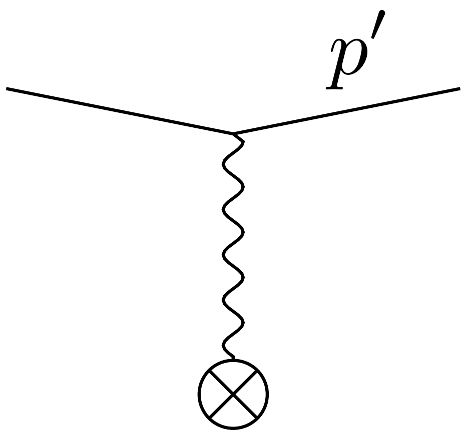
Spherical symmetry

$E = |\mathbf{p}|$

Time-like "impact parameter"

Hawking Scattering

Tree



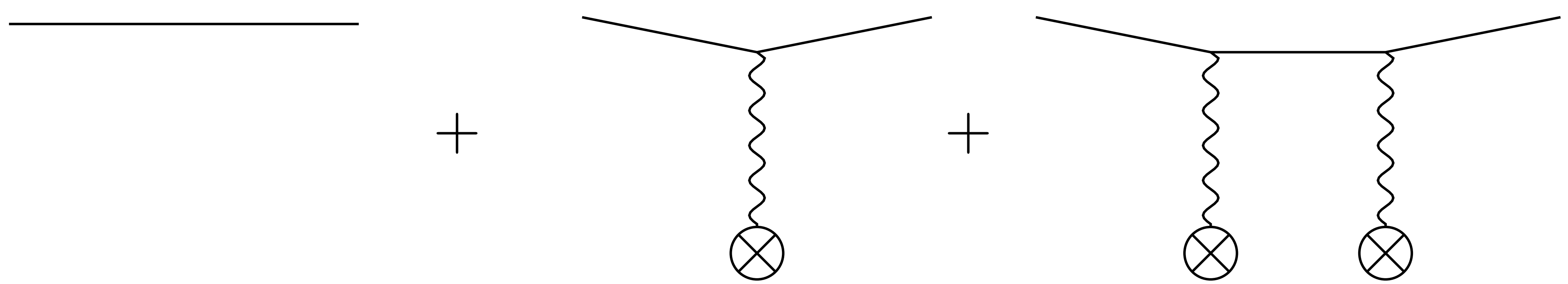
$$S_{\rm int} = \frac{1}{2}\int \mathrm{d}^4x\, h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

Compute using GW methods:

$$\text{diagram} = \int \mathrm{d}v\, \varphi(v) e^{ip'\cdot b(v)}\big(-4GME'\log(-v/\mu)\big)$$

Hawking Scattering

Iterates & exponentiates



$$\mathcal{A} = \mathcal{N} \int \mathrm{d}v \, \phi(v) \, e^{i(E'-E_0)v} \exp\left(-4iGME' \log(-v/\mu)\right)$$

Spherical energy eigenstate $\quad |\psi\rangle = \dfrac{1}{4\pi} \displaystyle\int \mathrm{d}\Omega \, |E_0, E_0 \mathbf{n}\rangle$

Hawking Scattering

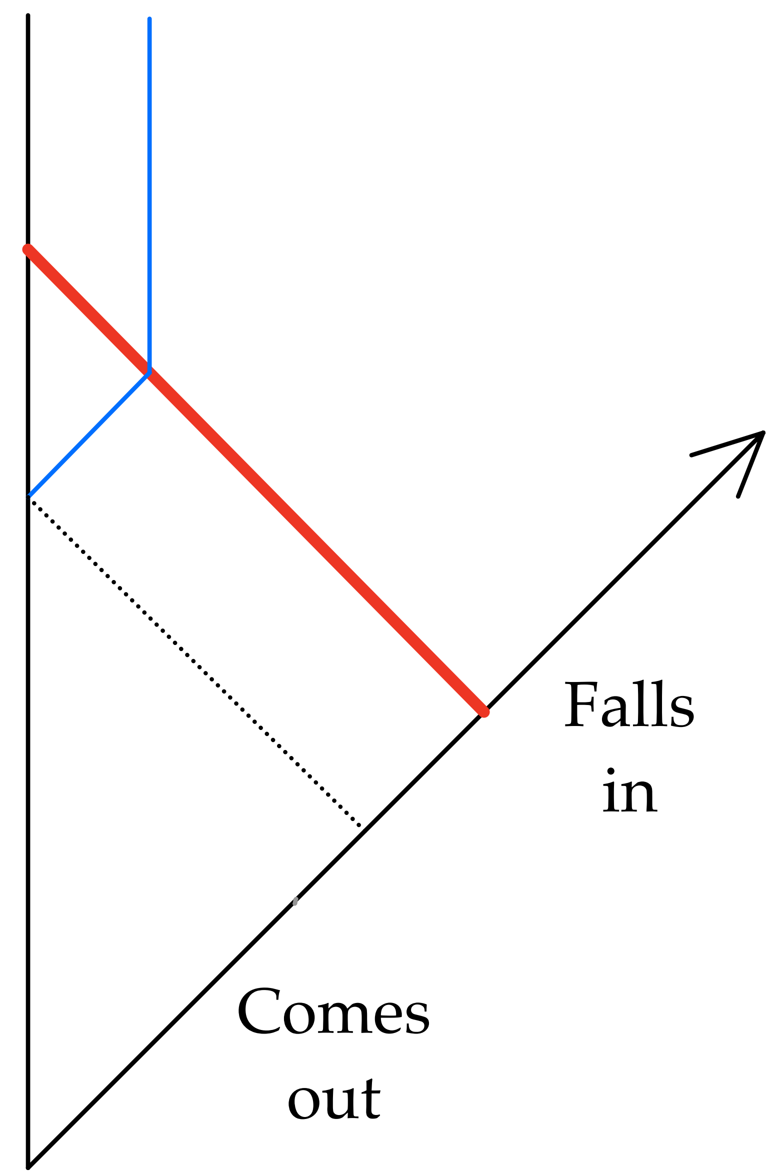
Not fully inclusive

$$\mathcal{A} = \mathcal{N} \int \mathrm{d}v \, e^{i(E' - E_0)v} e^{-4iGM\,E' \log(-v/\mu)}$$

Horizon

$$\boxed{\mathcal{A} = \mathcal{N} \int_{-\infty}^{0} \mathrm{d}v \, e^{i(E' - E_0)v} e^{-4iGM\,E' \log(-v/\mu)}}$$

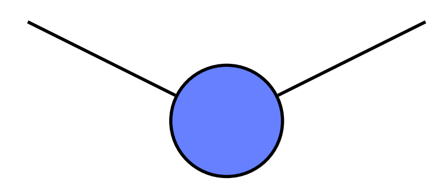
Hawking's amplitude



Pair Production

Simply cross in-state to out-state

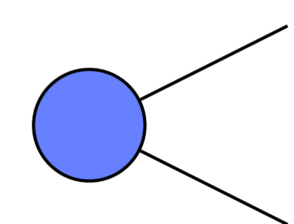
$$\mathcal{A} = \mathcal{N} \int_{-\infty}^{0} \mathrm{d}v \, e^{i(E'-E_0)v} e^{-4iGM\,E' \log(-v/\mu)}$$



Crossing

$$\mathcal{B} = \mathcal{N} \int_{-\infty}^{0} \mathrm{d}v \, e^{i(E'+E_0)v} e^{-4iGM\,E' \log(-v/\mu)}$$

$$\sim e^{-2\pi GME'} \Gamma(1 - 4iGME')$$



Pair Production

Count outgoing states

$$|\mathcal{B}|^2 = (\text{factor})\frac{1}{e^{8\pi GME'} - 1}$$

Thermal distribution

$$T = \frac{1}{8\pi GM}$$

Outlook

Higher Orders

Analog of PM expansion

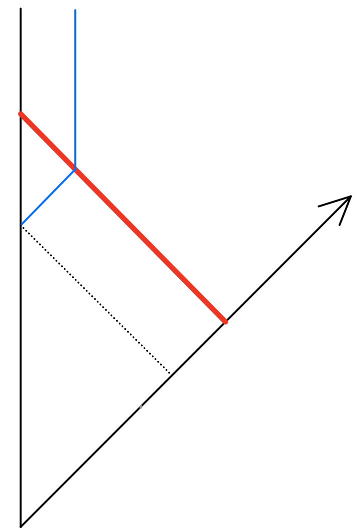
$$\exp\left(i\frac{S}{\eta}\right) = \exp\left(i\frac{S^{(0)} + S^{(1)} + \cdots}{\eta}\right)$$

Also computed $S^{(1)}$

Finite size effect

$$S^{(0)} + S^{(1)} = -4GME' \log\big(-v/\mu\big) - \frac{16G^2M^2E'}{v}$$

$$\simeq -4GME' \log\big(-(v + 4GM)/\mu\big)$$



Finite radius of horizon

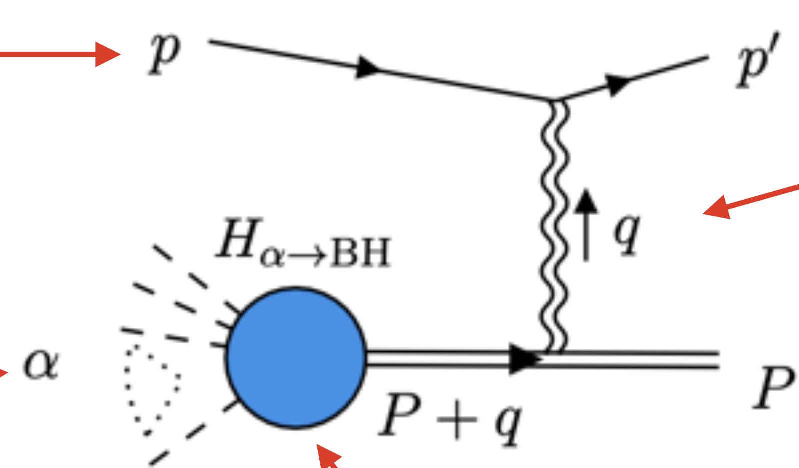
Outlook

Hawking computation: closely related to PM methods of amplitudes

- Remove background?

Pair production: crossing
One pair enough
(Squeezing)

Single graviton
exchange: enough



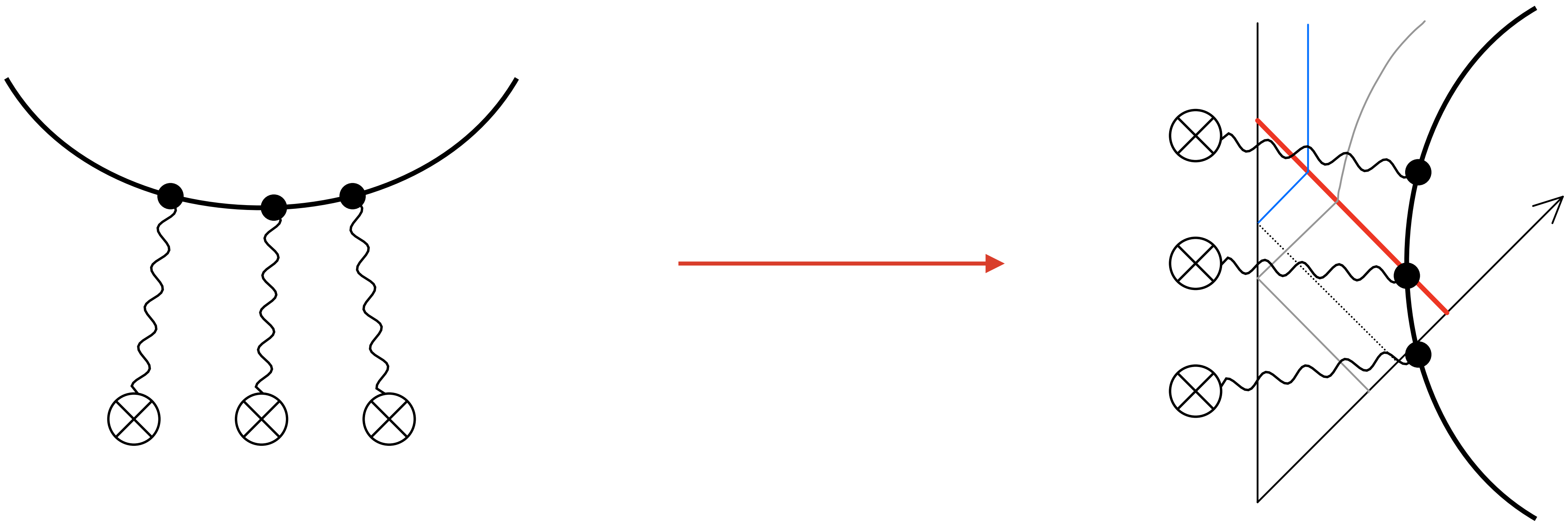
Infalling radiation:
coherent state

Matter to BH:
form factor?

Outlook

Hawking computation: closely related to PM methods of amplitudes

- Remove background?
- Non-local pair production?



Outlook

Hawking computation: closely related to PM methods of amplitudes

- Remove background?
- Non-local pair production - tunneling? *Parikh & Wilczek*
- Exponentiate with more care: stationary phase
- Radiation reaction?
- Page curve?