Around a quantum chaos

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Based on 2406. 02782 with A. Valov(Jerusalem) and S. Nechaev(Paris), JHEP(2024)

2411.11968 with A.Tyutyakina(Paris), R.Sharipov(Lyubliana), V.Gritsev(Amsterdam) and A. Polkovnikov(Boston)

In progress with A. Valov , in progress with I.Lyubimov(MIPT)

Outline of the talk

- Comments of quantum chaos diagnostics
- Late-time KPZ scaling via the Krylov chain
- Towards the DSSYK-gravity duality via Krylov complexity fluctuation
- Geometry of the 2-dimensional parameter space and transition ergodic-> extended non-ergodic->localized phases

Diagnostics for a quantum chaos

- Late-time behaviour of correlators. Spectral formfactors. Generalization of Lyapunov exponents.OTOC
- Fractal dimensions of eigenfunctions
- Late-time behaviour of Krylov complexity. Asymptotic behaviour of Lanzcos coefficients
- Induced geometry of the parameter space. Response of the system on a perturbation

Historical remarks

--- Late time OTOC   Larkin-Ovchinnikov , Formulation of operator growth picture, scrambling
Black holes etc   Shenker,Susskind, Cotler, Stanford,Swingle,Sonner, Barbon,Caputa
and many others

---- Chaotic- extended non-ergodic, localized(integrable) phases. Via evaluation of the
Fractal dimension of eigenmodes. NEE phase — since 2015 Amini, Kravtsov,
Khaymovich. Many recent examples

---- Krylov space approach   Parker, Cao, Avdoshkin, Suffidi, Altman 2019,
Barbon-Rabinovich-Sonner (finite systems), Dymarsky-Smolkin(CFT),
Balasubramian, Caputa, Magan, Wu (survival amplitude as «order parameter»)

– Induced geometry of parameter space, Phase transitions-singularity in geometry
Kolodrubetz,Gritzev, Polkovnikov 13, Gritsev-Polkovnikov-Liska
Integrable point is attarctive Kim-Polkovnikov 23' . Geodesic motion in the
Information metrics.

Surprizes with late-time behaviour

Unexpected late-time behaviour for several models. First found
In integrable spin chains (finite) Liubotina, Znidaric,Prosen 19'. Confirmed
later in several other many-body systems

$$\langle S^z(x,t)S^z(0,0)\rangle \propto (\Gamma t)^{-\frac{2}{3}})f_{kpz}((\Gamma t)^{-\frac{2}{3}}x) \qquad z = 3/2,\ t = x^z\ \text{s}$$

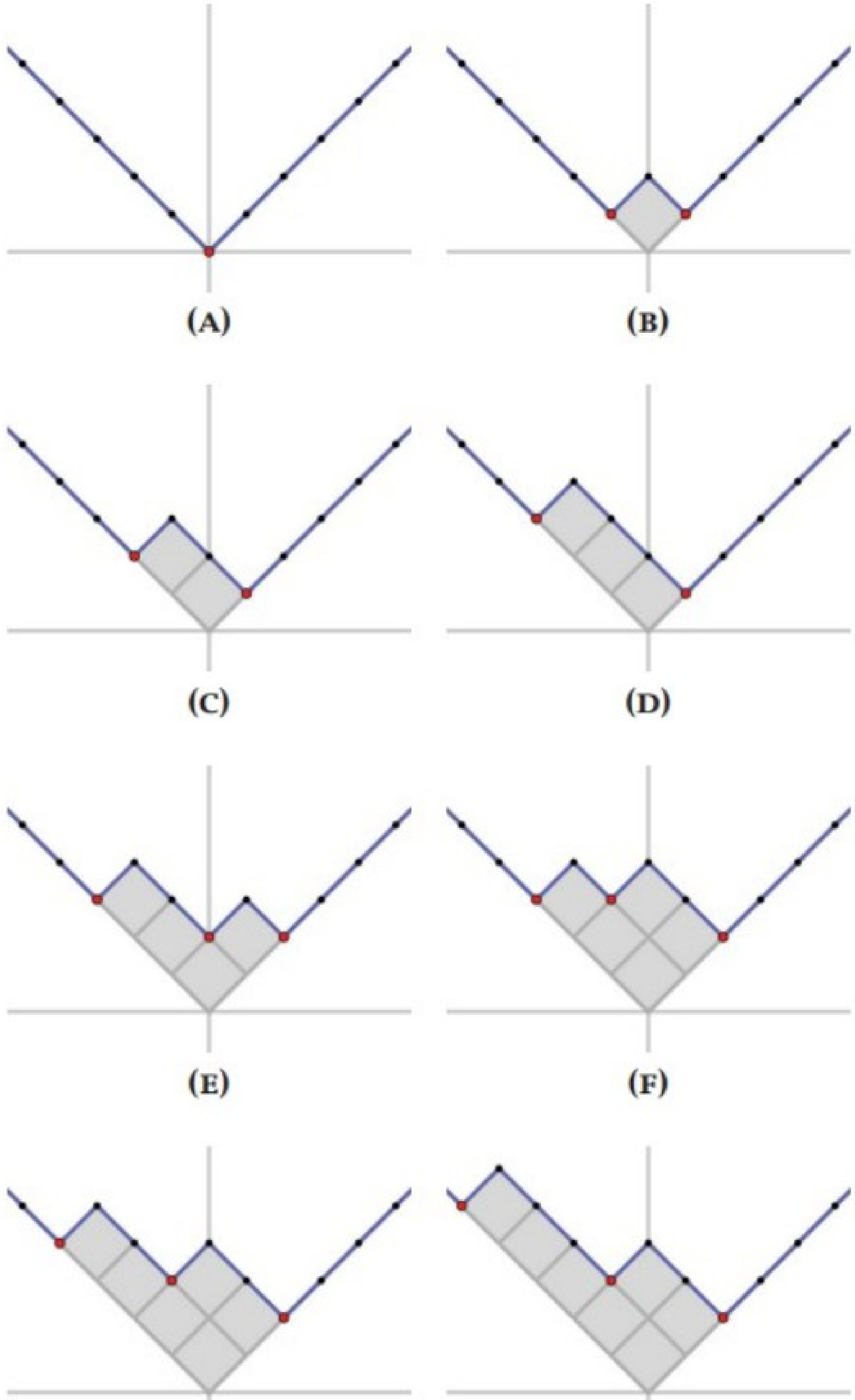
$$\langle S_i^z(t)S_i^z(0)\rangle \propto (\Gamma t)^{-\frac{2}{3}}$$

Dynamical KPZ critical scaling

What is the origin of this phenomena?
Is it specific for integrable model or takes place for chaotic systems as well?

Seems that for non-integrable deformations works as well.
Several explanations suggested. Goldstone mode for the effective U(1) symmetry
Formation of the giant soliton(in Landau-Lifshitz chain) . No consensus
concerning explanation yet.

Our strategy: 1. consider the Krylov space for the quite generic operators.
2. Take into account the generic behaviour of the Lanzcos coefficients for the
finite system 3. Focus at the Heisenberg time-scale T~dim H



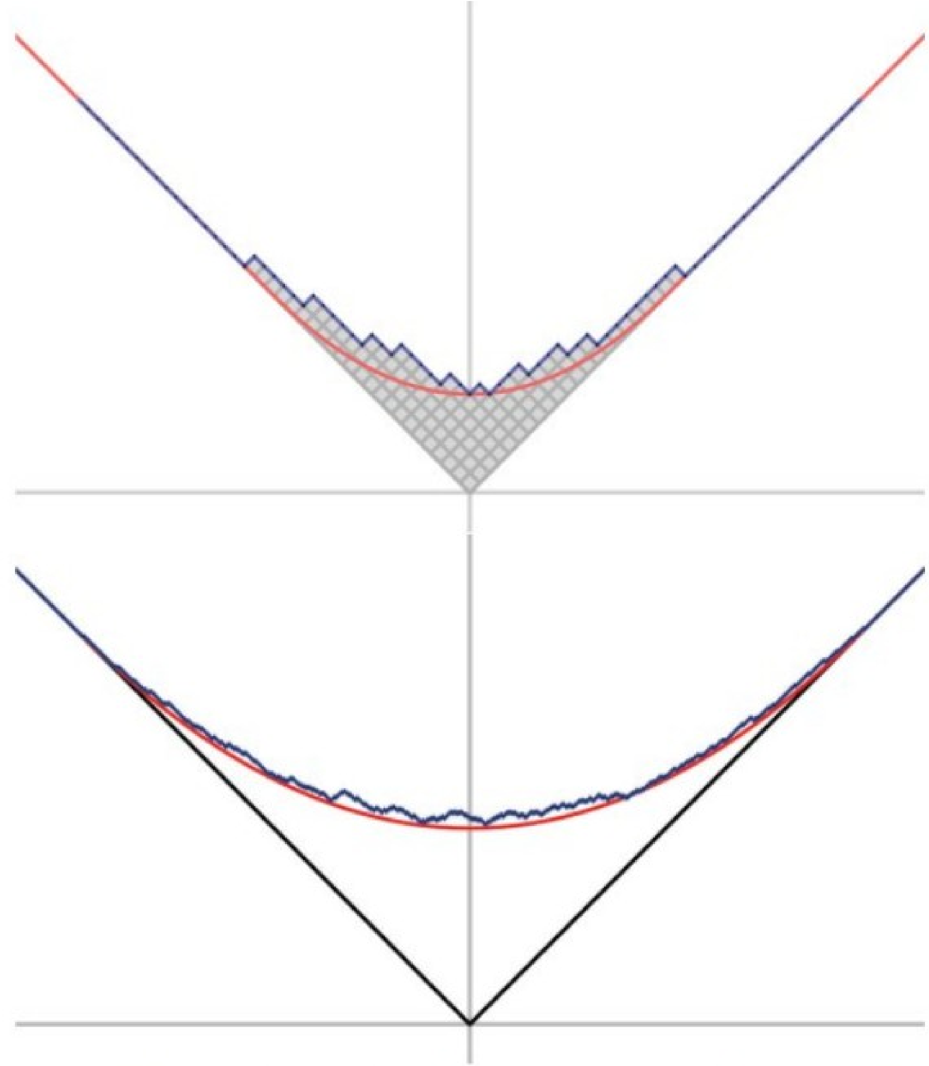


Figure 6. Simulation of the corner growth model. The top shows the model after a medium amount of time and the bottom shows it after a longer amount of time. The blue interface is the simulation, while the red curve is the limiting parabolic shape. The blue curve has vertical fluctuations of order $t^{1/3}$ and decorrelates spatially on distances of order $t^{2/3}$.

-40 0 40 80 120 160 200

KPZ universality class

Kardar-Parisi-Zhang 86'

$$\partial_t h(x,t) = \nu\partial_{xx}h(x,t) - \lambda(\partial_x h(x,t))^2 + \xi$$

$$\partial_t u(x,t) + \partial_x(-\lambda u^2 - \nu\partial_x u - \xi) = 0$$

$$u = \partial_x h,$$

$$Z(x,t) = e^{\lambda h(x,t)}$$

$$\partial_t Z(x,t) = \tfrac{1}{2}\partial_{xx}Z(x,t) + \xi(x,t)Z(x,t)$$

Partition function of polymer

$$h(x,0) = 0$$

$$h(0,t) \sim vt + (\Gamma t)^{1/3}\chi_{flat}$$

Late time 1-point function

where $\chi_{flat} = \chi_{GOE}$ with the probability distribution

$$P(\chi_{GOE} \leq s) = \det\left(1 - K_{\mathrm{Ai}}(s)\right)$$

and $K_{\mathrm{Ai}}(s)$ in the Fredholm determinant is the Airy kernel.

KPZ universality class

$$h(x,0) = |x|$$

$$P\left(\frac{x^2}{2t}h(x,t) - \frac{t}{24} \geq -s\right) = F_t(s)$$

$$F_t(2^{-1/3}t^{1/3}s) \to F_{GUE}(s)$$

$$F_{GUE}(s) = \det(I - K_{\mathrm{Ai}}(s)) = \exp\left(-\int_s^\infty (x-s)^2 q^2(x)dx\right)$$

where $q(x)$ obeys the Painleve II equation

$$q''(x) = (x + 2q^2(x))q(x)$$

subject to the boundary condition $q(x) \sim \mathrm{Ai}(x)$ as $x \to \infty$.

$$\langle u(x,t)u(0,0)\rangle \sim (\Gamma t)^{-2/3} f_{kpz}\left((\Gamma t)^{-2/3}x\right)$$

Late time correlator

Tabulated function

Krylov basis. Propeties

$$O(t) = e^{iHt} O(0) e^{-iHt}, \qquad \mathcal{L} = [H, \ldots]. \qquad \mathcal{L}^n |O\rangle$$

Tower of operator for some seec

$$H|K_n\rangle = a_n|K_n\rangle + b_{n-1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle$$

where

$$a_n = \langle K_n|H|K_n\rangle, \qquad b_n = \langle A_n|A_n\rangle^{1/2}, \quad |A_{n+1}\rangle = (H - a_n)|K_n\rangle - b_n|K_{n-1}\rangle$$

Krylov basis $\qquad |O_n\rangle = b_n^{-1}|K_n\rangle$

$$L_{nm} = \langle O_n|\mathcal{L}|O_m\rangle = \begin{pmatrix} a_0 & b_1 & 0 & 0 & 0 & \ldots \\ b_1 & a_1 & b_2 & 0 & 0 & \\ 0 & b_2 & a_2 & b_3 & 0 & \\ 0 & 0 & b_3 & a_3 & b_4 & \\ 0 & 0 & 0 & b_4 & 0 & \\ \vdots & & & & & \ddots \end{pmatrix}$$

Matrix of Lanzcos coefficients

Gram-Scmidt orthogonalization in the operator space. Seed operator O is arbitrary

Krylov basis.Propeties

$$\frac{\partial \phi_n}{\partial t} = -b_{n+1}\phi_{n+1} + a_n\phi_n + b_n\phi_{n-1} \qquad \phi_n(t) = i^{-n}\langle O_n(0)|O(t)\rangle, \; \phi_n(0) = \delta_{n,0}.$$

Hopping problem on the Krylov chain. Equivalent to the open Toda(Dymarsky A.G 20)

$$C_O(t) = \langle O|e^{i\mathcal{L}t}|O\rangle$$

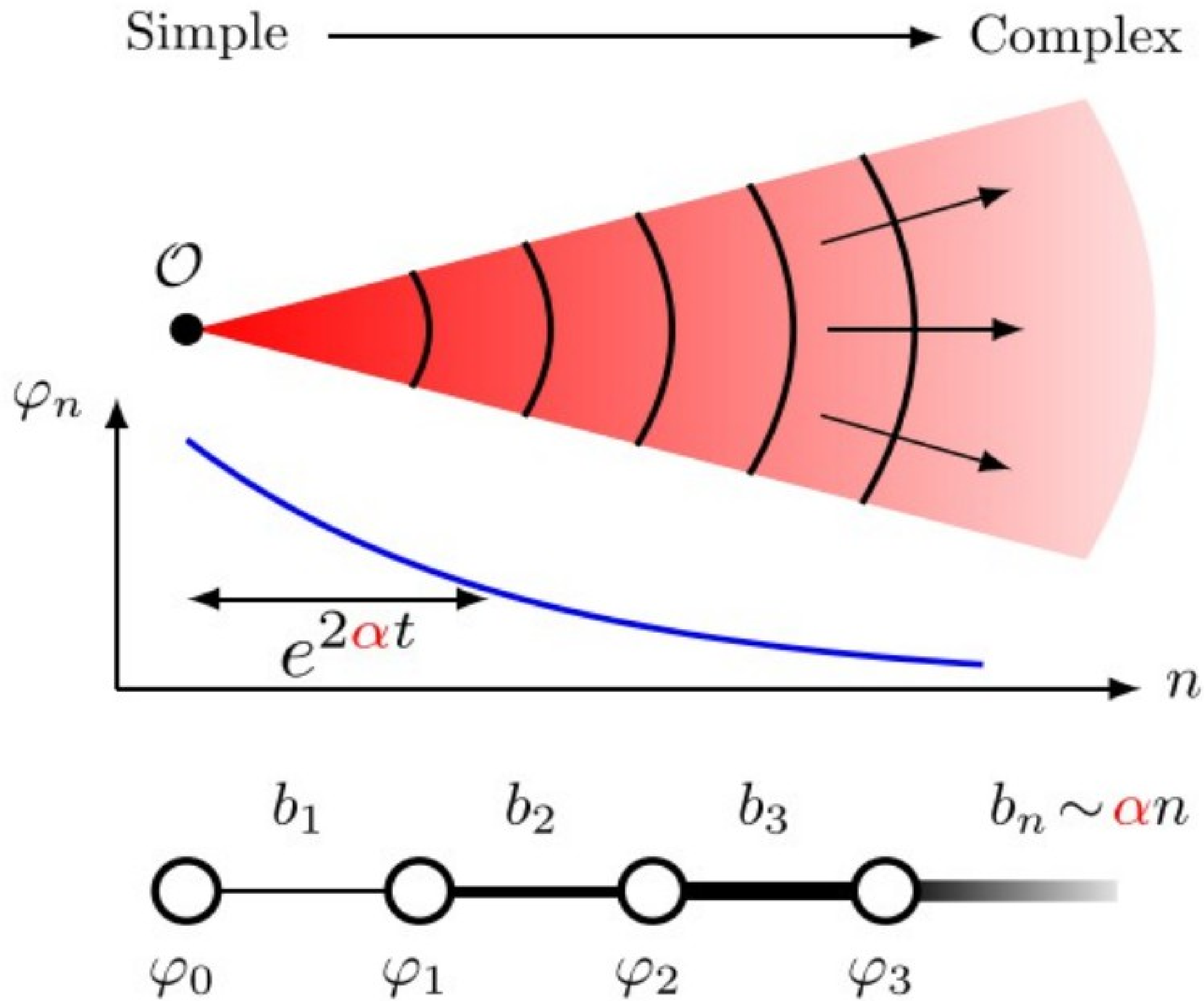
$$\mu_{2n} = \langle O|\mathcal{L}^{2n}|O\rangle = \frac{d^{2n}}{dt^{2n}}C(t)|_{t=0}; \quad C(t) = \sum_n \frac{\mu_n t^n}{n!}$$

$$\mu_{2n} = \sum_{h_n \in D_n} \prod_k b_{h_k + h_{k+1}/2}$$

Properly defined Dick paths

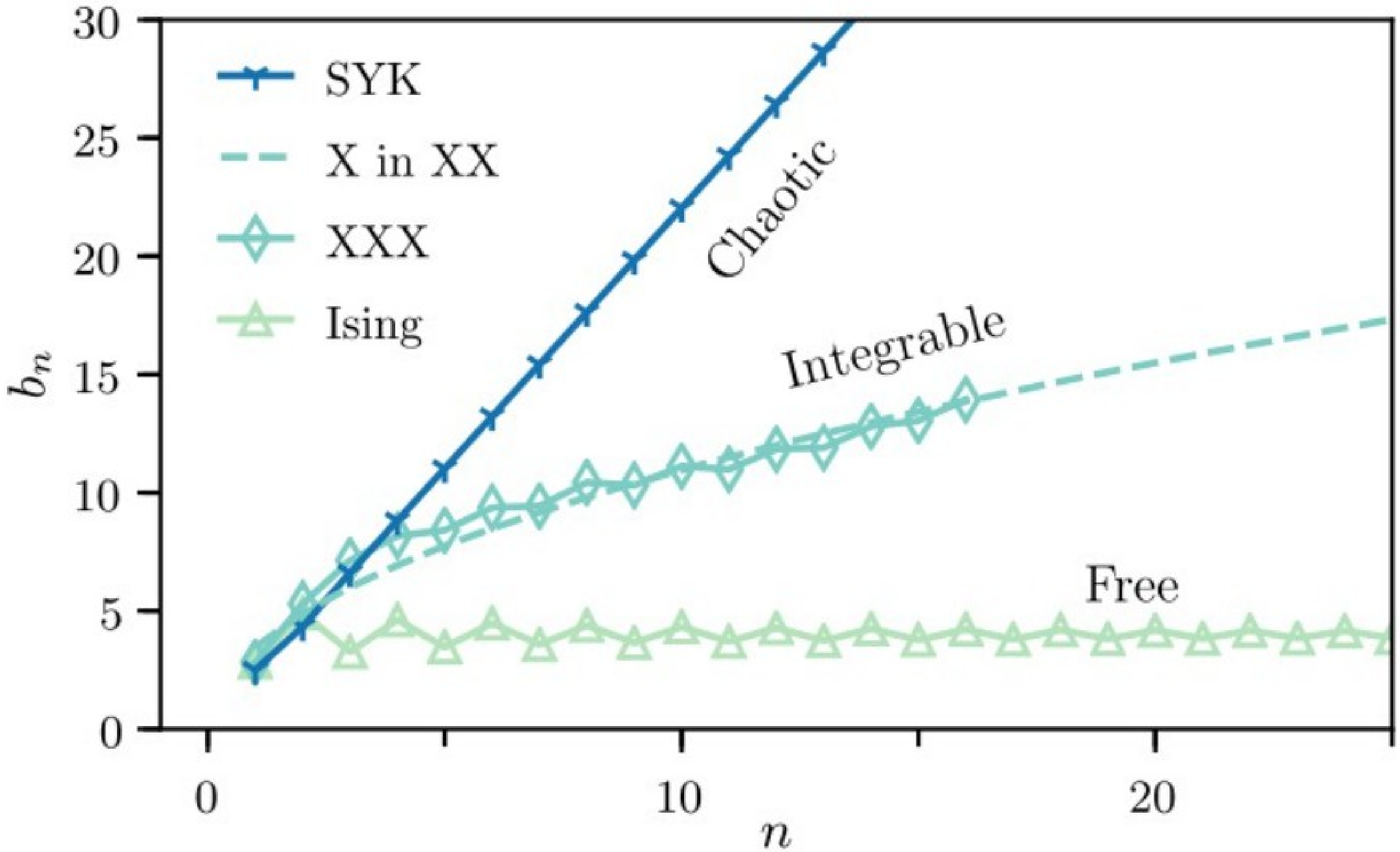
$$\mathcal{K}(t) = \sum_n n|\phi_n|^2 \qquad \mathcal{K}(t) = e^{\rho t}$$

Krylov complexity   measures the operator spreading



Parker et al 19

Krylov basis properties. Asymptotic behavior of b(n)



Krylov basis in matrix beta-ensemble

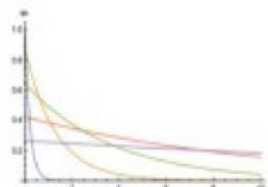
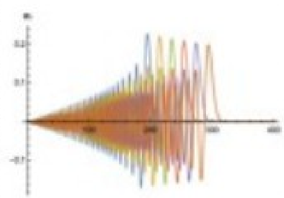
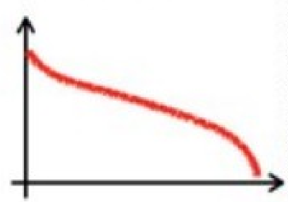
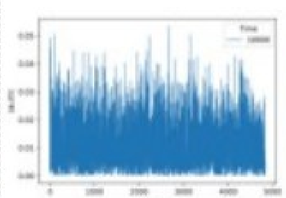
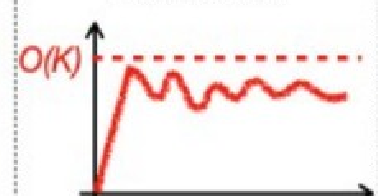
$$p(\lambda_1,\ldots,\lambda_n) = Z_{\beta,N} e^{-\frac{\beta N}{4}\sum_k \lambda_k^2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta,$$

Distribution of the eigenvalues

$$H_\beta = \frac{1}{\sqrt{\beta N}} \begin{pmatrix} N(0,2) & \chi_{(N-1)\beta} & & & \\ \chi_{(N-1)\beta} & N(0,2) & \chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix},$$

Balasubramian et al 2022, Dimitru-Edelman 2006

Krylov chains in systems with finite number degrees of freedom

n	**Lanczos coefficients**	**wavefunction**	**K-complexity**	**time scales**
$1 \ll n < S$	Linear growth in *n* $b_n \sim \alpha n$		Exponential growth in time 	$0 \lesssim t \lesssim \log S$
$n \gg S$	Plateau, constant in *n* $b_n \sim \Lambda S$		Linear growth in time 	$t \gtrsim \log S$
$n \sim e^{2S}$	Descent 		Saturation 	$t \sim e^{2S}$

Rabinovici,Barbon,Sonner 21-22

Random walks on the Krylov chains

$$K \leq D^2 - D + 1\,,$$

D -dimension of the Hilbert space

K-length of the Krylov chain

The problem of evaluation of autocorrelator of the seed operator and correlators of two operators on the Krylov chain is provided by the evaluation of the transition amplitudes on the Krylov chain with inhomogeneous and in general random hopping

We shall be interested in the transition time of order

Of the length of the chain T~cK, c=O(1)

Growing Lanzcos coefficients

Consider the Krylov chain with the finite UV-like cut-off. Example — finite Gaussian Matrix model. Consider the random walk on such chain at large Euclidean time-Heisenberg.time T~cK, K- length of the chain
Take use some analytic results from the probability theory

Random matrix in Krylov basis

Seems to be overlooked in 2d gravity

$$M_K \sim -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} W'(x)$$

W- Gaussian white noise. Stochastic Airy operator

Ramirez et al 2011

$$N^{1/6}(\lambda_{j,N} - 2\sqrt{N}))_{j=1}^{\infty} \to_{N\to\infty} (\lambda_1 > \lambda_2 \dots)_{Airy}$$

Asymptotic behavior of tridiagonal matrix near the Spectral edge for any beta.

$$e^{-TM_k} \to \exp\left(-\frac{T}{2} SAO_\beta\right)$$

At large T the spectrum of the «Hamiltonian M» coincides with the spectrum of SAO

Exact result for Gaussian potential

Gorin,Sodin,Borodin 16-18

$$\lim_{N\to\infty}\frac{1}{2}\left(\left(\frac{M_N}{2\sqrt{N}}\right)^{[TN^{2/3}]}+\left(\frac{M_N}{2\sqrt{N}}\right)^{[TN^{2/3}]-1}\right)=\exp\left(-\frac{T}{2}SAO_\beta\right)$$

$$Z(0,2t^3)e^{\frac{t^3}{12}}=\lim_{K\to\infty}K\left[\left(\frac{M_K}{2\sqrt{K}}\right)^{2[tK^{2/3}]}+\left(\frac{M_K}{2\sqrt{K}}\right)^{2[tK^{2/3}]+1}\right]_{1,1}$$

Return probability
On the Krylov chain
Autocorrelator!

Solution to KPZ equation with wedge initial condition

The KPZ scaling is quite universal for autocorrelators
at the Heisenberg time for systems with finite Hilbert space.
If t^3~K we have identification of Lanzcos time and KPZ time.

Growing Lanzcos's. Numerics

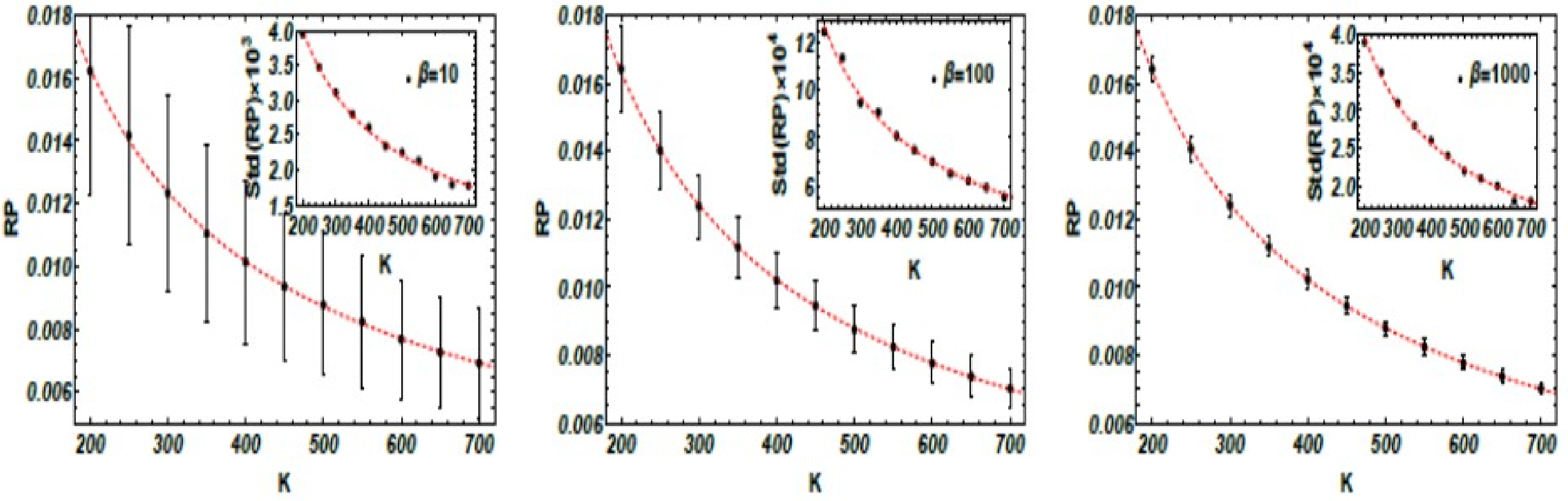


Figure 9. Scaling dependence of the return probability $RP = C(0, K)$ averaged over 1000 realizations of Krylov chains of length $K = 700$ for different β's: black dots and error bars represent mean and standard deviations of the return probabilities; insets show the scaling dependence of the standard deviations. All red dashed curves are proportional to $K^{-2/3}$

KPZ scaling is seen numerically
- both random and deterministic Lanzcos coefficients

--both integrable and chaotic b(n) asymptotics

Descending Lanzcos b(n)

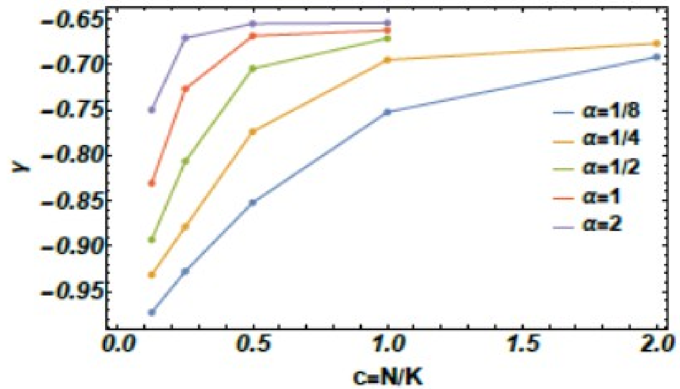


Figure 6. Scaling dependence of the return probability of N-step random walks on a Krylov chain, where the hopping parameters are the Lanczos coefficients $b_n = ((K-n)/K)^\alpha$.

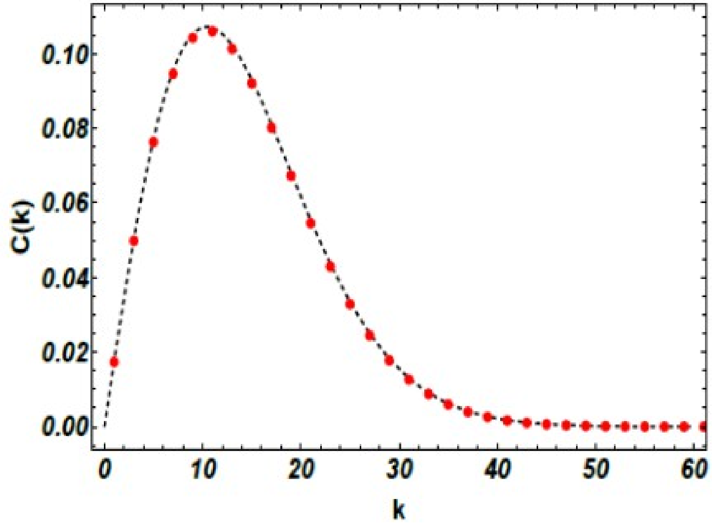


Figure 7. Two-point correlator $C(k,t)$ (red dots) and its approximation $2\alpha^{1/3}K^{-1/3}\mathrm{Ai}[(2\alpha)^{1/3}K^{-1/3}k + a_1]$ (black dashed line) for a Krylov chain, where the hopping parameters $b_k = ((K-k)/K)^\alpha$, $\alpha = 1/2$.

3-rd order phase
Transition at
Critical Euclidean time

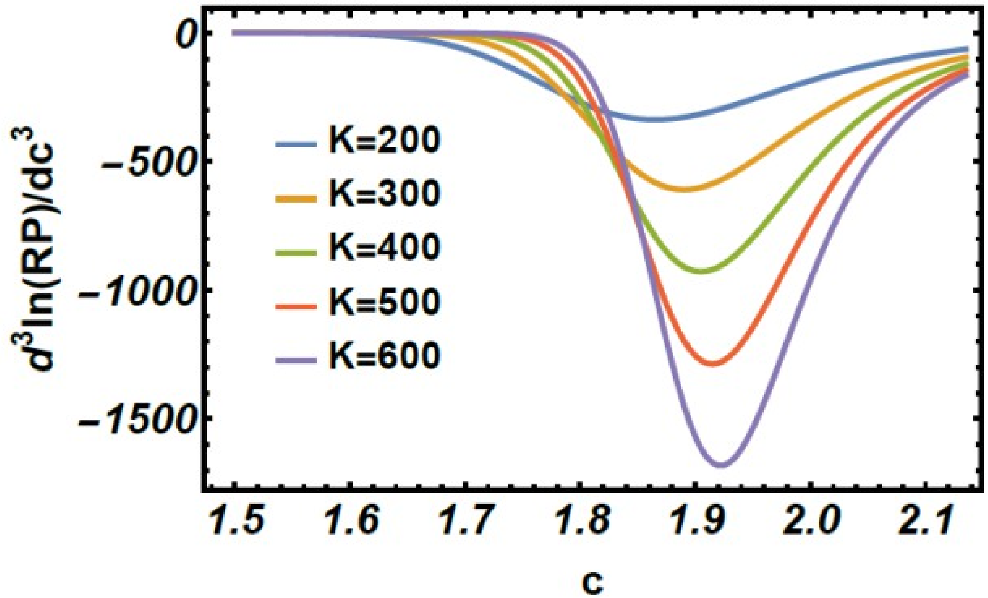


Figure 11. The third derivative of the logarithm of the return probability, $\frac{d^3 \ln C(0,cK)}{dc^3}$, on the Krylov chain of size K with growing Lanczos coefficients $b(n) = \sqrt{n}$.

DQPT in Euclidean
Time !

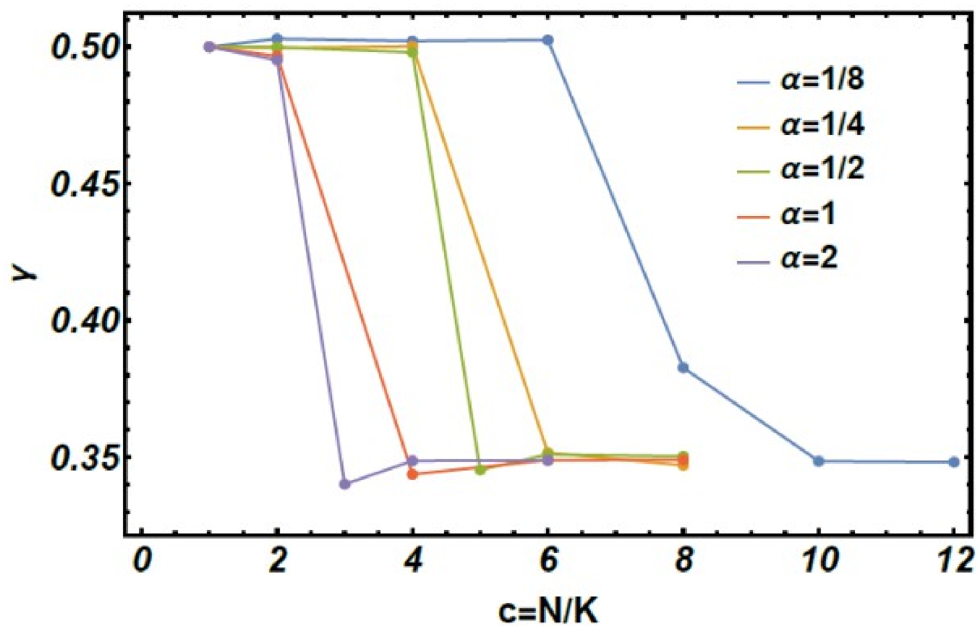


Figure 12. Scaling of fluctuations of the middle point of the N-step Brownian bridge on a Krylov chain of size K, where the hopping parameters are the Lanczos coefficients $b(n) = n^{\alpha}$.

Full Krylov chain for finite system. Gauss-KPZ transition

- There is the Gauss-KPZ transition at the descending part at the Heisenberg time. For random and deterministic b(n), for integrable and chaotic asymptotics
- It is not the 3-rd order phase transition. Crossover
- Possible explanation of experimental data
- Is not useful for chaos diagnostics

Scaling of fluctuations in DSSYK

(Valov, A.G. In progress)

The Lanczos coefficients for the DSSYK are known exactly hence we can perform similar analysis for fluctuations of the Krylov complexity

$$H_{SYK} = i^{p/2} \sum_{i_1 \dots i_p} \Psi_{i_1} \dots \Psi_{i_p}, \qquad < J^2_{i_1 \dots i_p} >= \frac{J^2}{\lambda C_p^N} \qquad N \to \infty, \qquad p \to \infty, \qquad \lambda = \frac{2p^2}{N}\ fixed$$

$$H = \frac{J}{\sqrt{\lambda}}(\alpha + \alpha^+)$$

$$b(n) = J\sqrt{\frac{n}{1-q}}, \qquad n \leq \frac{1}{\lambda}$$

$$b(n) = \frac{J}{\sqrt{\lambda(1-q)}}(1 - \frac{q^n}{2}), \quad n \geq \frac{1}{\lambda}$$

$$b(n) = J\sqrt{\frac{(1-q^n)}{\lambda(1-q)}} \qquad q = e^{-\lambda}$$

$$\langle (K(t) - \bar{K}(t))^2 \rangle = \sum_n (n - \bar{n})^2 P_{0n}(t) \qquad P_{0n}(t) = |\phi_n(t)|^2 \qquad |O(t)\rangle = e^{it\mathcal{L}}|O\rangle = \sum_n \phi_n(t)|O_n\rangle$$

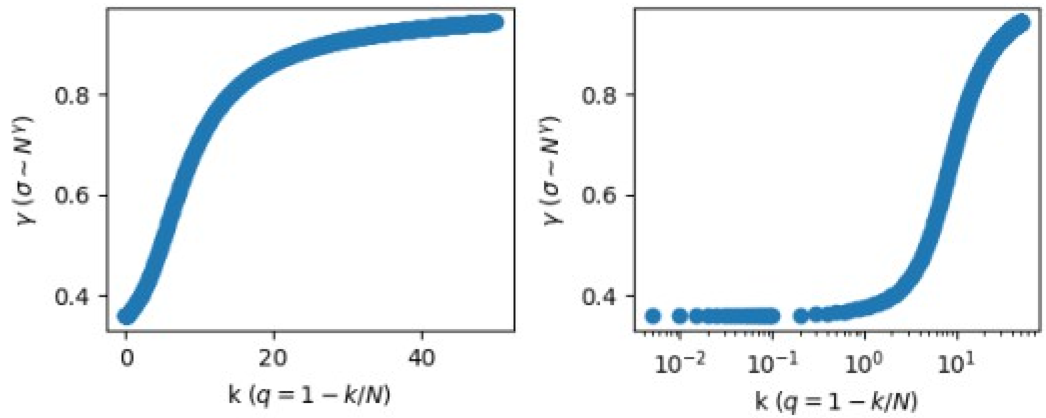


Figure 1: q-dependence of the scaling exponent of the Krylov variance (evolution operator - $e^{cNH_{DSSYK}}$). KPZ-ballistic transition at q^*

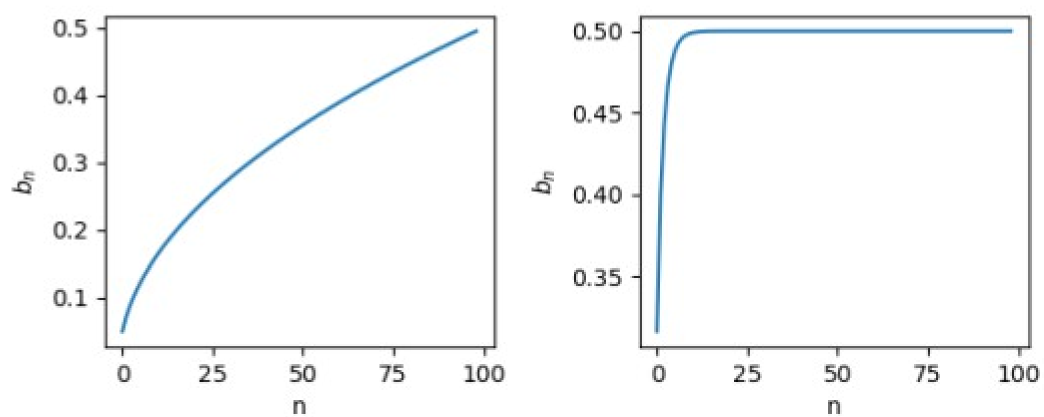


Figure 2: q-dependence $(q = 1 - k/N)$ of the Lanczos coefficients b_n on the Krylov chain of length $N = 100$ for $k = 0.01$ (left) and $k = 40$ (right).

KPZ-ballistic transition at the Euclidean Heisenberg time at critical q

Fluctuation of Krylov complexity versus subdiffusion-superdiffusion

DSSYK → Sin-dilaton JT gravity

DSSYK → 3d de Sitter geometry

Krylov complexity in DSSYK → Length of geodesics

Fluctuation of the Krylov complexity corresponds to the fuctuation of length in gravity models. Transition from subdiffusion(KPZ) to superdiffusion

$< \delta x > \propto t^{\alpha}, \quad \alpha = 1$

$\partial_t \rho = \sqrt{-\Delta} \rho$

Superdiffusion in de Sitter (Milekhin-Xu)

Possible transition to de Sitter at large q (Susskind,Verlinde....

Hilbert space geometry

2411.11968

– Distinguish quantum chaos via response to a perturbation. The quantum tensor quantifies the response- it involves quantum metric(real part) and Berry curvature(imaginary part)

$$g_{\alpha\beta}^{(n)} = \sum_{m \neq n} \frac{\langle n|\partial_\alpha H|m\rangle \langle m|\partial_\beta H|n\rangle}{(E_n - E_m)^2}, \qquad G_{\alpha\beta} = \frac{1}{N}\sum_{n=1}^{N} g_{\alpha\beta}^{(n)},$$

$$ds^2 \equiv 1 - \left|\left\langle n(\vec{\lambda})\middle| n(\vec{\lambda} + d\vec{\lambda})\right\rangle\right|^2.$$

low the induced geometry of parameter space tells about criticality/integrability/chaoticity?

Several conjectures supported by the examples

1. Quantum metric is singular at quantum critical point,Zanardi et al 07, Gritsev,Polkovnikov,Liska 13'

2. The system selects the geodesic motion in the quantum metric under adiabatic evolution - Sugiura,Clayes,Dymarsky,Polkovnikov 21'

3. Integrable points act as attractors in the parameter space Kim-Polkovnikov 23'

2-dimensional parameter space

Geometry of Interpolation between quantum integrability and quantum chaos with matrix Hamiltonians=finite-dimensional Hilbert space. Combination of analytical and numerical tools

1. Start with the chaotic system and perturb it in chaotic direction

$$H = H_0 + xH_x + yH_y, \qquad \rho(H_a) = e^{-\frac{1}{2}NTr(H_a^2)}, \quad a = 0, x, y.$$

Berry-Schukla (2020) evaluated the metric at the origin- fidelity susceptibility

2. Start with integrable model and perturb it in chaotic direction

$H = \Lambda_0 + xH_x + yH_y,$ where $\Lambda_0 = \mathrm{diag}(\lambda_i)$ with $\rho(\lambda_i) = e^{-\frac{1}{2}\lambda_i^2}$, and H_x and H_y are drawn from GUE

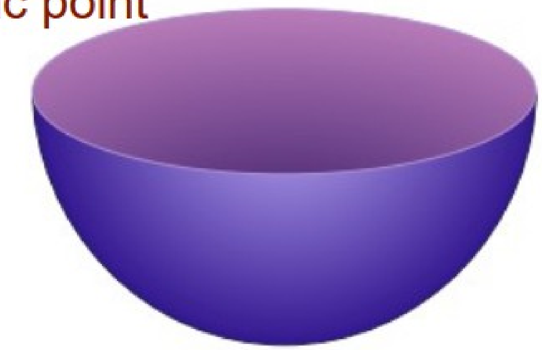
In family of Rozentzweig-Porter models

Independent diagonal matrix — according Berry-Tabor is integrable system.
No interaction between eigenvalues

Chaotic point

$$\left(\frac{dZ}{dr}\right)^2 + \left(\frac{dR}{dr}\right)^2 = \overline{G}_{rr}(r),$$

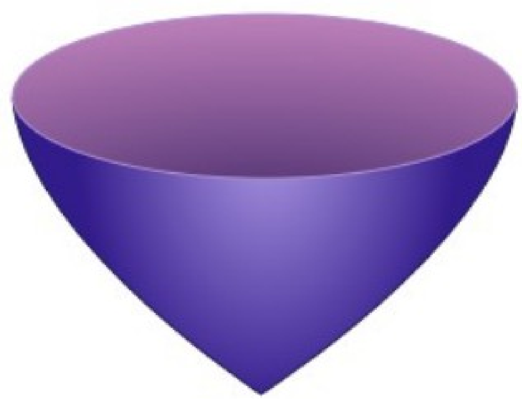
$$R(r)^2 = \overline{G}_{\phi\phi}(r).$$



Exact large N result

$$(Z - \sqrt{N-1})^2 + R^2 = N - 1,$$

Integrable point

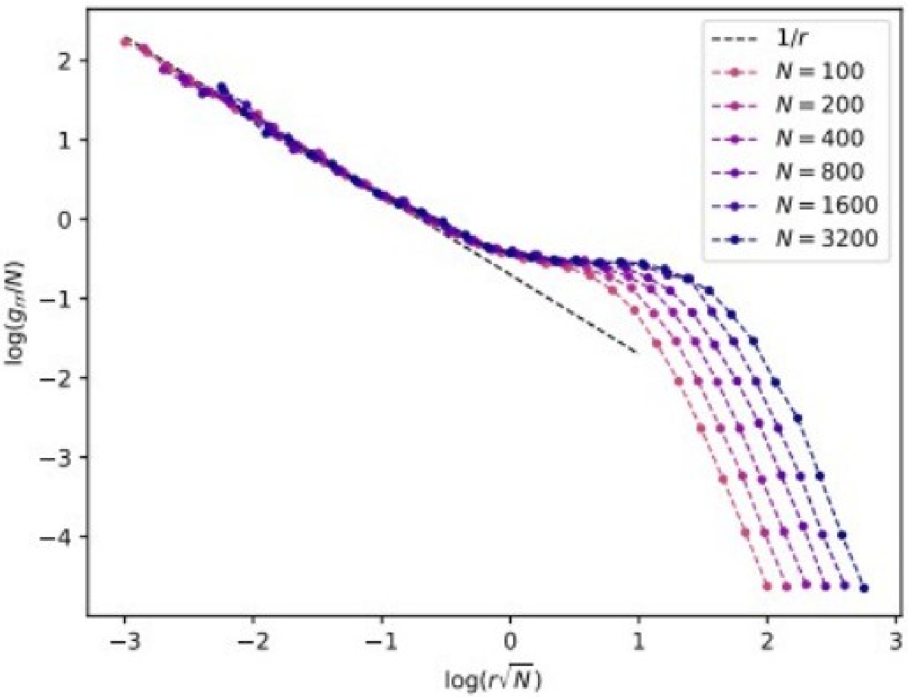


$$\overline{G}_{\phi\phi} = r \frac{1}{2\sqrt{2}} \arctan\left(\frac{\sqrt{2}}{r}\right)$$

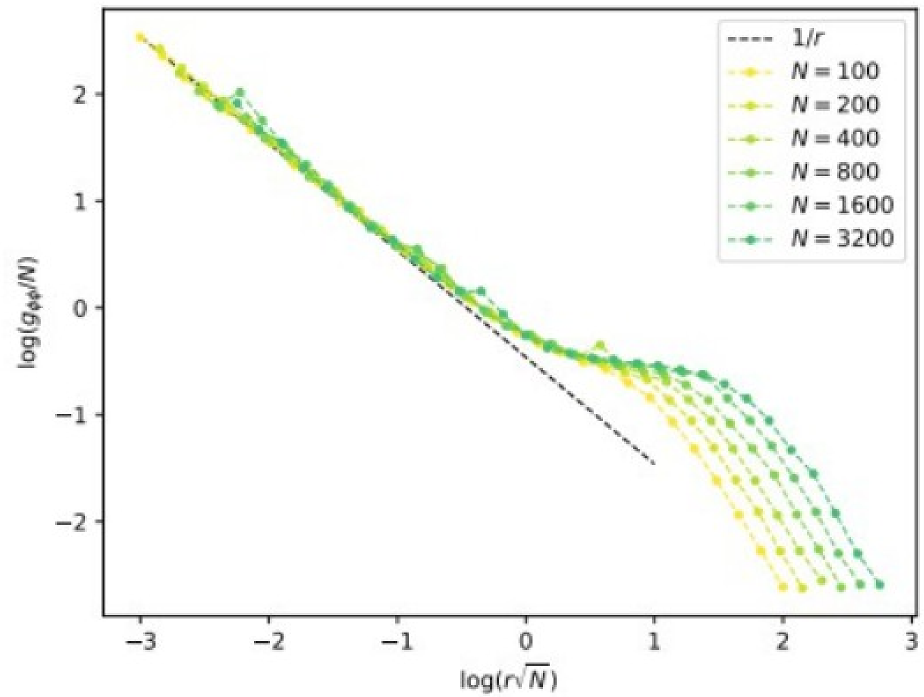
Exact result for metric in N=2 case

$$\overline{G}_{rr} = \frac{1}{4}\left(\frac{\operatorname{arccot}\left(\frac{r}{\sqrt{2}}\right)}{\sqrt{2}r} - \frac{1}{2+r^2}\right)$$

Integrable point at large N



(a) G_{rr} component of QGT as a function of rescaled parameter $r \to r\sqrt{N}$.



(b) G_{pp} component of QGT as a function of rescaled parameter $r \to r\sqrt{N}$.

3 regimes : integrable - KAM-like — chaotic

Universal behaviour 1/r near the integrable point. Precise example of «Integrability is attractive» scenario. r^{-2} behaviour in the chaotic regime

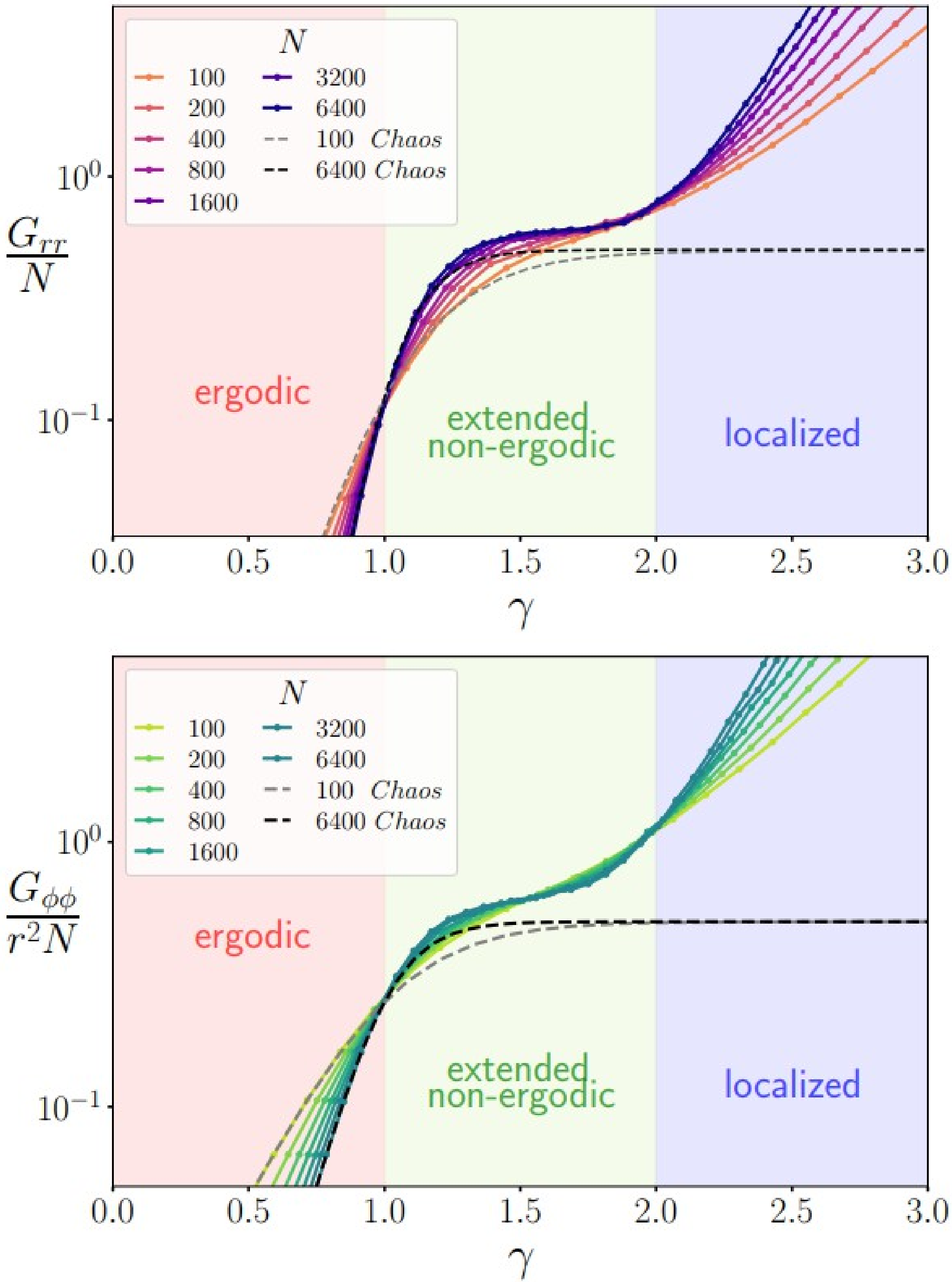


FIG. 4: Components of QGT as a function of scaling parameter

$$r = N^{(1-\gamma)/2}$$

Parametrization used in RP model

Remarkably the geometric tensor captures all three regimes :

The transition points are reproduced Correctly.

The extended non-ergodic phase found by Amini,Kravtsov,Khaymovich 2015 in RP model

In 2-parameter Russian Doll model Motammari,Khaymovich, A.G. 2022

Extended non-ergodic phase Fractal dimension 0<D_q<1

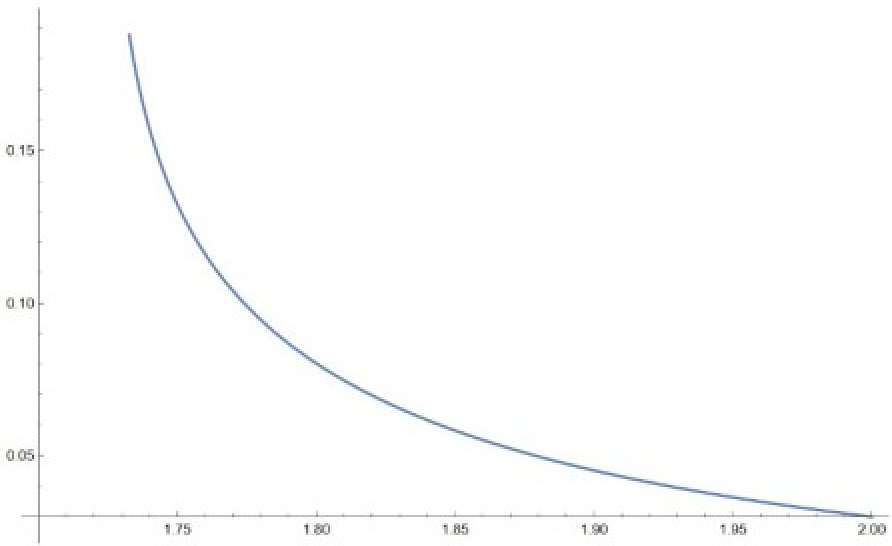
Dependence on distributions in random energy model

(Lyubimov,A.G. In progress)

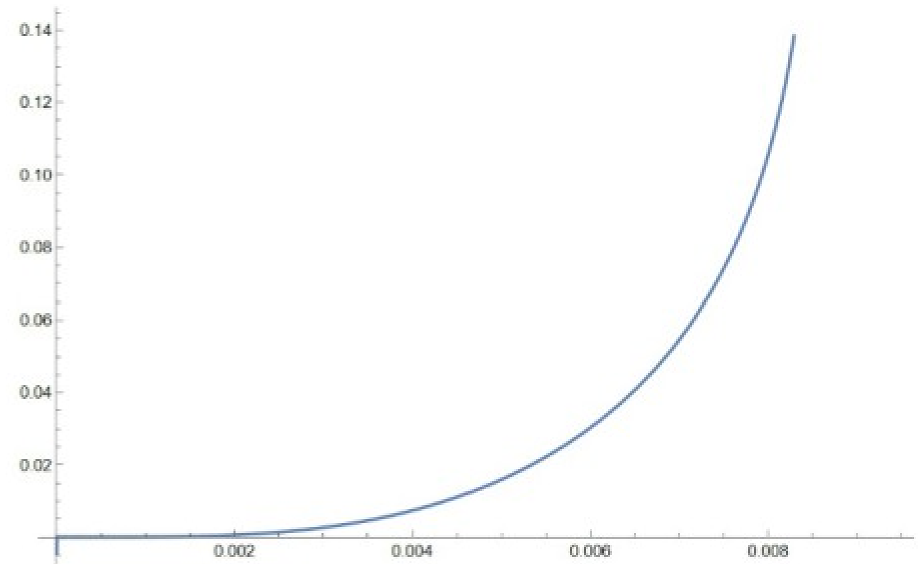
$$H = \begin{pmatrix} \lambda_1 & -re^{i\theta} \\ -re^{-i\theta} & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & -(x+iy) \\ -(x-iy) & \lambda_2 \end{pmatrix}$$

$\lambda \in (-W/2, W/2).$ $\lambda_1 \in (-W/2+\delta, W/2+\delta)$

Shifted distribution. We can control the presense or absence of the level crossing



$G_{rr}(r)$ for $W=1$ and $\delta=0.99$



$G_{rr}(r)$ for $W=1$ and $\delta=1.01$

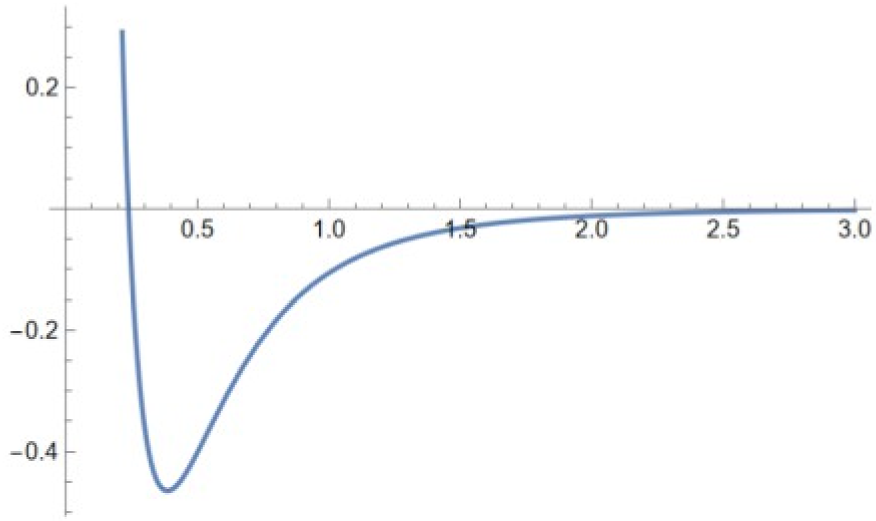
Singularity disappears when there is no level crossing in distributions

Berry ring. Surprise

$$\Omega_{xy} = \int \frac{d\lambda_1 d\lambda_2}{W^2} \frac{\frac{1}{8}(\zeta - \sqrt{\zeta^2 + 4r^2})^4 - 2r^4}{(\zeta^2 + 4r^2)(4r^2 + \zeta^2 - \zeta\sqrt{\zeta^2 + 4r^2})^2}$$

$$\Omega_{xy} = \frac{1}{2W^2} \ln \frac{(\sqrt{4r^2 + (\delta - W)^2} + W - \delta)(\sqrt{4r^2 + (\delta + W)^2} - W - \delta)}{(\sqrt{\delta^2 + 4r^2} - \delta)^2}$$

$$\int \Omega_{xy} dx dy = \begin{cases} \frac{\pi\delta}{2}(2W - \delta) & \delta < W \\ \frac{\pi W^2}{2} & \delta > W \end{cases}$$



$\rho(r)$ for $W = 1$ and $\delta = 1$

Distribution of «vortexes» which generates the corresponding Berry curvature.

A kind of charged Berry ring

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Conclusion and open questions

- Possible explanation of the KPZ scaling in late-time autocorrelators in spin chains. Seems to be generic for finite systems
- KPZ-ballistic transition at some q-- additional argument in favor of duality DSSYK-de Sitter
- Difference between the geometry of parameter space near chaotic and integrable points
- Non-ergodic extended phase is seen in the information Fisher metric

Open questions

- More general geometry around the chaotic,integrable and intermediate regimes for higher dimensional parameter space
- Double scaling limit in 2d gravity, Anderson localization and KPZ in gravity in the Krylov picture
- More about the DSSYK-de Sitter duality via Krylov space and chord basis?
- Chord basis and knot invariants(in progress)
- Correlation of the Krylov chains

Thank you for attention!