# Polyvector deformations in supergravity and the string/M-theory dynamics

Ilya Bakhmatov

ITMP Moscow

Based on joint works with
E. Musaev, E. Ó Colgáin, M. Sheikh-Jabbari and others

# Statement

For any supergravity solution there exists a deformation, such that the field equations reduce to the classical Yang-Baxter equation.

# $d = 10$ type II supergravity solution

**Initial data**:

- $g_{\mu\nu}$, $b_{\mu\nu}$, $\phi$; RR fields ($F_\mu$, $F_{\mu_1\mu_2\mu_3}$, $F_{\mu_1\ldots\mu_5}$ for IIB)
- isometries $[k_a, k_b] = f_{ab}{}^c k_c$

**Deformation**:

- Choose a bi-vector $\beta^{\mu\nu} = \frac{1}{2} r^{ab} k_a^\mu k_b^\nu$
- New NSNS fields: $G + B = (g + b)(1 + \beta(g + b))^{-1}$
- Dilaton transformation $e^{-2\Phi} |\det G_{\mu\nu}|^{1/2} = e^{-2\phi} |\det g_{\mu\nu}|^{1/2}$

Deformed background $G_{\mu\nu}$, $B_{\mu\nu}$, $\Phi$, etc. is a solution **if** the $r$-matrix satisfies the classical Yang-Baxter equation:

$$f_{de}{}^{[a} r^{b|d|} r^{c]e} = 0.$$

# Deformation map

$$G + B = (g + b)(1 + \beta(g + b))^{-1}$$

- $\beta \to 0$: $g, b$ invariant: $\beta$ is the deformation parameter
- $b \to 0$: open/closed string map: $G + B = (g^{-1} + \beta)^{-1}$

## Open/closed string map

- $G, B$: closed string fields
- $g, \beta$: effective open string description
- $\beta$ non-commutativity parameter for the dual SYM

# AdS/CFT and deformations

| | | |
|---|---|---|
| Strings/supergravity on $\mathrm{AdS}_5 \times \mathbb{S}^5$ | $\Longleftrightarrow$ | $\mathcal{N} = 4$ SYM |
| $\downarrow$ | | $\downarrow$ |
| Deformations of $\mathrm{AdS}_5 \times \mathbb{S}^5$ | $\Longleftrightarrow$ | Field theory deformations |

- field theory deformations: less supersymmetry
- field theory deformations: non-commutativity (Moyal product)
- supergravity deformations : TsT, YB, generalization to $d = 11$;
- moving beyond $\mathrm{AdS}_5 \times \mathbb{S}^5$.

# TsT: a prototype deformation

U(1)×U(1) isometry: adapted coordinates $(x^1, x^2)$
[Lunin, Maldacena; Frolov'05]

$$\boxed{\begin{array}{c}\text{T-duality}\\ x^1\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{shift}\\ x^2 \to x^2 + \eta x^1\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{T-duality}\\ x^1\end{array}}$$

$$ds^2 = \frac{-dt^2 + dx_3^2 + dz^2}{z^2} + \frac{z^2}{z^4 + \eta^2}(dx_1^2 + dx_2^2) + ds^2(\mathbb{S}^5),$$

$$B = \frac{\eta}{z^4 + \eta^2}\, dx_1 \wedge dx_2, \qquad e^{2(\Phi - \Phi_0)} = \frac{z^4}{z^4 + \eta^2}.$$

Dual to non-commutative deformation of SYM
[Hashimoto, Itzhaki; Maldacena, Russo'99]

# Maldacena-Russo geometry from bi-vector deformation

$$\boxed{\begin{matrix}\text{T-duality}\\ x^1\end{matrix}} \Longrightarrow \boxed{\begin{matrix}\text{shift}\\ x^2 \to x^2 + \eta x^1\end{matrix}} \Longrightarrow \boxed{\begin{matrix}\text{T-duality}\\ x^1\end{matrix}}$$

Shifts of $x^{1,2}$ are generated by the momenta: $\beta = \frac{1}{2} r^{ab} k_a k_b = \eta P_1 \wedge P_2$

The $r$-matrix picks out $P_{1,2}$ from the full isometry algebra of $AdS_5$
$k_a \in \{P_\mu, M_{\mu\nu}, D, K_\mu\}$

The open/closed string map generates the required deformation terms

$$G + B = (g^{-1} + \beta)^{-1}$$

$$G_{\mu\nu} dx^\mu dx^\nu = \frac{-dt^2 + dx_3^2 + dz^2}{z^2} + \frac{z^2}{z^4 + \eta^2}(dx_1^2 + dx_2^2) + ds^2(\mathbb{S}^5),$$
$$B = \frac{\eta}{z^4 + \eta^2} dx_1 \wedge dx_2.$$

# Yang-Baxter deformations: generalizing TsT

Yang-Baxter deformed $AdS_5 \times S^5$ $\sigma$-model in the supercoset formulation:
[Klimčík'02; Delduc, Magro, Vicedo'13]

$$S = -\frac{1}{4}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \,\mathrm{Str}\left[A_\alpha \,\mathrm{d} \circ \frac{1}{1 - \eta R_g \circ \mathrm{d}} A_\beta\right],$$

$A_\alpha = g^{-1}\partial_\alpha g, \ g \in SO(4,2);$

$\mathrm{d}$ projects onto the certain graded component of $\mathfrak{so}(4,2)/\mathfrak{so}(4,1)$.

$\exists$ Lax pair $\implies$ classical integrability
$\kappa$-symmetry preserved by the deformation

$$R_g(X) = g^{-1}R(gXg^{-1})g, \quad X \in \mathfrak{so}(4,2)$$

$R$ satisfies the homogeneous classical Yang-Baxter equation,

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = 0, \quad X, Y \in \mathfrak{so}(4,2).$$

# $r$-matrix parameterisation of the YB deformation

Parameterise $R$ in terms of the **r-matrix**:

$$R(X) = \mathrm{Tr}_2[r(1 \otimes X)] = \sum_{a,b} r^{ab} k_a \mathrm{Tr}[k_b X],$$

$$r = \frac{1}{2} \sum_{a,b} r^{ab} k_a \wedge k_b, \quad \{k_a\} = \mathrm{bas}\, \mathfrak{so}(4,2), \quad [k_a, k_b] = f_{ab}{}^c k_c.$$

classical Yang-Baxter equation in terms of $r^{ab}$:

$$f_{de}{}^{[a} r^{b|d|} r^{c]e} = f_{de}{}^a r^{bd} r^{ce} + f_{de}{}^c r^{ad} r^{be} + f_{de}{}^b r^{cd} r^{ae} = 0.$$

# Two views of Yang-Baxter deformations

Standard narration:

- $r$-matrix solving the CYBE is **an input**;
- $\sigma$-model in **coset formalism** is deformed by an ($r$-matrix-dependent) operator;
- deformed background is a solution to (generalised) supergravity;
- deformed backgrounds can be obtained via TsT or nonabelian T-duality.

Our narration:

- Start with a bi-Killing $r \sim k \wedge k$; **not fixing the CYBE**;
- Deformation is the **open-closed string map**, essentially matrix inversion;
- CYBE is sufficient for the deformed background to solve supergravity;
- procedure works for **non-coset geometries**, no (obvious) relation to integrability;
- supergravity solution generation.

# Example: non-abelian YB deformation $\beta = \eta D \wedge P_1$

The associated YB deformed geometry is given by

$$ds^2 = \frac{z^2(dt^2 + dx_1^2 + dz^2) + \eta^2(dt - tz^{-1}dz)^2}{z^4 + \eta^2(z^2 + t^2)} + \frac{t^2(-d\phi^2 + \cosh^2\phi d\theta^2)}{z^2},$$

$$B = -\eta \frac{tdt \wedge dx^1 + zdz \wedge dx^1}{z^4 + \eta^2(t^2 + z^2)}, \quad \Phi = \frac{1}{2}\log\left[\frac{z^4}{z^4 + \eta^2(t^2 + z^2)}\right],$$

$$x_0 = t\sinh\phi, \qquad x_2 = t\cosh\phi\cos\theta, \qquad x_3 = t\cosh\phi\sin\theta.$$

This is a solution to generalized type IIB supergravity, with the vector $I^\mu$ describing the modification:

$$\nabla_\mu \beta^{\mu\nu} = I^\nu, \qquad I = -\eta\partial_1$$

# Generalised IIB supergravity

Unimodularity condition $r^{ab}[k_a, k_b] = r^{ab} f_{ab}{}^c k_c = 0$

For non-unimodular $r$-matrices, deformed background is not a solution of supergravity.

$$R_{\mu\nu} = \frac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} - \nabla_\mu X_\nu - \nabla_\nu X_\mu,$$
$$R = \frac{1}{12} H^2 - 4\nabla_\mu X^\mu + 4 X_\mu X^\mu,$$
$$\frac{1}{2}\nabla^\rho H_{\rho\mu\nu} = X^\rho H_{\rho\mu\nu} + \nabla_\mu X_\nu - \nabla_\nu X_\mu.$$

There are also deformed field equations for the RR sector.

**$I^\mu$ is a Killing vector**

$$X_\mu = \partial_\mu \Phi + I^\nu (G_{\nu\mu} + B_{\nu\mu})$$

# Yang-Baxter deformations and DFT

**Double Field Theory**: manifestly T-duality covariant form of supergravity

**Doubled coordinates**: $X^M = (x^\mu, \tilde{x}_\mu)$

**Dynamical variable**: generalized metric $\mathcal{H}_{MN} \in \frac{O(d,d)}{O(d)\times O(d)}$

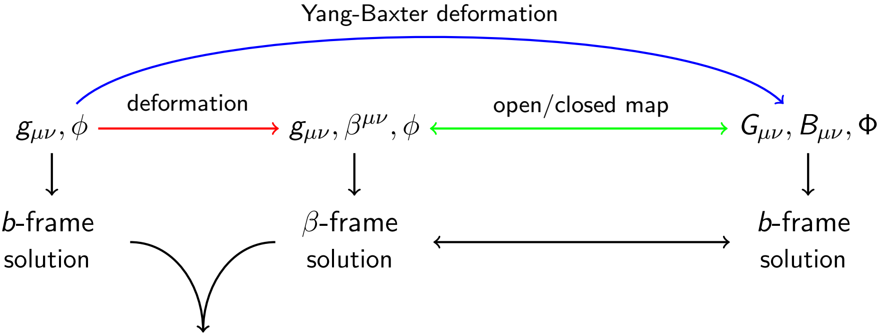
$$\begin{bmatrix} g & \beta g^{-1} \\ g^{-1}\beta & g^{-1} + \beta g \beta \end{bmatrix} = \mathcal{H}_{MN} = \begin{bmatrix} G + BG^{-1}B & GB \\ BG & G^{-1} \end{bmatrix}$$

**Action** $(d = \phi + \frac{1}{4}\log g_{\mu\nu})$:

$$S = \int dx\, d\tilde{x}\, e^{-2d} \left( \frac{1}{8}\mathcal{H}^{MN}\partial_M \mathcal{H}^{KL}\partial_N \mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{KL}\partial_L \mathcal{H}^{MN}\partial_N \mathcal{H}_{KM} - \right.$$
$$\left. -2\partial_M d \partial_N \mathcal{H}^{MN} + 4\mathcal{H}^{MN}\partial_M d \partial_N d \right).$$

The open-closed string map $G_{\mu\nu} + B_{\mu\nu} = (g^{\mu\nu} + \beta^{\mu\nu})^{-1}$ can be realized as a frame change in DFT.

# Yang-Baxter deformations and DFT




constraints on $\beta^{\mu\nu} \Longrightarrow$ CYBE(r)=0, if $\beta^{\mu\nu} = r^{ab} k_a^\mu k_b^\nu$

## Equations of motion in the $\beta$-frame [Andriot, Betz'13]

$$\boxed{\mathcal{R} - 4(\partial\Phi)^2 + 4\nabla^2\Phi} - \frac{1}{2}R^2 = 4(\beta^{mr}\partial_r\Phi + I^m)^2 - \hat{\mathcal{R}} + G_{mn}\hat{\nabla}^m(\beta^{nr}\partial_r\Phi + I^n);$$

$$\boxed{\mathcal{R}_{pq} + 2\nabla_p\partial_q\Phi} + \frac{1}{4}R_p{}^{mn}R_{qmn} = \hat{\mathcal{R}}_{(pq)} - 2\hat{\nabla}_{(p}\big(\beta_{q)r}\nabla^r\Phi + I_{q)}\big);$$

$$e^{2\Phi}\hat{\nabla}^m(e^{-2\Phi}R_{mrp}) + 2I^mR_{mrp} = 2e^{-2\Phi}\nabla^q(e^{2\Phi}\nabla_{[p}\beta_{r]q}) - 4\mathcal{R}_{[p}{}^s\beta_{r]s} - e^{2\Phi}\nabla^m(e^{-2\Phi}\nabla_m\beta_{rp}) + 8G_{n[p}\nabla_{r]}(\beta^{nq}\partial_q\Phi)$$

Notation:

$$\hat{\mathcal{R}} = G_{mn}\hat{\mathcal{R}}^{mn}, \quad \hat{\Gamma}_p^{mn} = \nabla^{(m}\beta^{n)}{}_p - \frac{1}{2}\nabla_p\beta^{mn} + \beta^{mq}\Gamma^n{}_{pq},$$

$$\hat{\mathcal{R}}^{mn} = -\beta^{pq}\partial_q\hat{\Gamma}_p^{mn} + \beta^{mq}\partial_q\hat{\Gamma}_p{}^{pn} + \hat{\Gamma}_p{}^{mn}\hat{\Gamma}_q{}^{qp} - \hat{\Gamma}_p{}^{qm}\hat{\Gamma}_q{}^{pn},$$

$$I^m = \nabla_k\beta^{km} \equiv -\hat{\Gamma}_k{}^{km}, \quad R^{mnp} = 3\beta^{q[m}\nabla_q\beta^{np]},$$

$$\hat{\nabla}^mV^p = -\beta^{mn}\partial_nV^p - \hat{\Gamma}_n{}^{mp}V^n, \quad \hat{\nabla}^mV_p = -\beta^{mn}\partial_nV_p + \hat{\Gamma}_p{}^{mn}V_n.$$

# $d = 11$ supergravity

Is there anything similar?

$$\text{strings} \to \text{membranes}$$
$$(g_{\mu\nu}, b_{\mu\nu}) \to (g_{\mu\nu}, C_{\mu\nu\rho})$$

Obvious obstacles:

- no integrability in membrane worldvolume theory setup;
- no clear "third rank anticommutativity" for the dual gauge theory;

**However**

- **Exceptional field theory** (ExFT) provides geometrical setup that parallels the $d = 10$ DFT case;
- We propose a concrete $d = 11$ solution deformation method based on a frame change in the $SL(5)$ ExFT.

# $SL(5)$ ExFT setup

**Exceptional Field Theory**: manifestly U-duality covariant form of $d = 11$ supergravity

**Extended coordinates**: $(x^\alpha, Y^M)$, $\alpha \in \{1, \dots, 7\}$, $M \in \{1, \dots, 10\}$
$SL(5)$ U-duality group: $11 = 7 + 4$ split.

The **generalized metric** in two frames:

$$e^{-\frac{\Phi}{2}} \begin{bmatrix} G^{-\frac{1}{2}} G_{\mu\nu} & V_\mu \\ V_\nu & G^{\frac{1}{2}}(1 + V^2) \end{bmatrix} = m_{mn} = e^{-\frac{\phi}{2}} \begin{bmatrix} g^{-\frac{1}{2}}(g_{\mu\nu} + W_\mu W_\nu) & W_\mu \\ W_\nu & g^{\frac{1}{2}} \end{bmatrix}$$

encodes degrees of freedom

$$V^\mu = \frac{1}{3!} \frac{1}{\sqrt{G}} \epsilon^{\mu\nu\rho\lambda} C_{\nu\rho\lambda}, \qquad W_\mu = \frac{1}{3!} \sqrt{g}\, \epsilon_{\mu\nu\rho\lambda} \Omega^{\nu\rho\lambda}.$$

We propose a tri-vector deformation $\Omega^{\mu\nu\lambda} = \frac{1}{3!} \rho^{abc} k_a{}^\mu k_b{}^\nu k_c{}^\lambda$

# Deformation prescription in $d = 11$

**Initial data**:

- background fields $g_{\mu\nu}$, $g_{\alpha\beta}$, $c_{\mu\nu\rho}$, $c_{\alpha\beta\gamma}$ **($4+7$ split)**
- isometry algebra $[k_a, k_b] = f_{ab}{}^c k_c$

**Procedure**:

- Specify a tri-vector deformation $\Omega^{\mu\nu\lambda} = \frac{1}{3!}\rho^{abc} k_a{}^\mu k_b{}^\nu k_c{}^\lambda$, $W = \star_4 \Omega$:
- New external metric $G_{\alpha\beta} = (1 + W_\mu W^\mu)^{1/3} g_{\alpha\beta}$;
- New internal metric $G_{\mu\nu} = (1 + W_\lambda W^\lambda)^{-2/3}(g_{\mu\nu} + W_\mu W_\nu)$;
- New 3-form $c_{\mu\nu\lambda} = (1 + W_\rho W^\rho)^{-1} \Omega_{\mu\nu\lambda}$.

Note:

- This replaces the open-closed string map of $d = 10$ theory;
- Powers of $1/3$ and $-2/3$ agree with uplift of a TsT.

# Examples

Deformation agrees with the sequence:
(reduction to $d = 10$) — (TsT) — (uplift back to $d = 11$).

Geometry with a flat 3-torus:

$$\mathrm{d}s^2_{(11)} = \mathrm{d}s^2(M_7) + G_{zz}\mathrm{d}z^2 + \delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j, \qquad \Omega = \gamma\,\partial_{x^1}\wedge\partial_{x^2}\wedge\partial_{x^3},$$

$$\mathrm{d}\tilde{s}^2_{(11)} = \left(1+\gamma^2\right)^{1/3}\left[\mathrm{d}s^2(M_7) + G_{zz}\mathrm{d}z^2\right] + \left(\frac{1}{1+\gamma^2}\right)^{2/3}\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j,$$

$$C = \frac{\gamma}{1+\gamma^2}\,dx^1\wedge dx^2\wedge dx^3.$$

$\mathcal{N}=2$ supersymmetry preserving deformation of $\mathrm{AdS}_4\times\mathbb{S}^7$
[Lunin, Maldacena'05]
deform along the $U(1)^3 < U(1)^4 < SO(8)$

# Non-abelian deformation in $d = 11$

$$\Omega = \frac{2}{R^3}\,\rho_a\epsilon^{abc}\,D\wedge P_b\wedge P_c = \frac{4}{R^3}\,\rho_a x^a\,\partial_0\wedge\partial_1\wedge\partial_2 - \frac{2}{R^3}\,z\,\rho_a\epsilon^{abc}\,\partial_b\wedge\partial_c\wedge\partial_z,$$

$$ds^2 = \frac{R^2}{4}\left(z^3+\rho_a x^a - \frac{\rho^2}{4z}\right)^{-\frac{2}{3}}\Big[-(dx^0)^2+(dx^1)^2+(dx^2)^2 + \left(1+\frac{\rho_a x^a}{z^3}\right)dz^2 - \frac{1}{z^2}\rho_a dx^a dz\Big] + \frac{R^2}{z}\left(z^3+\rho_a x^a - \frac{\rho^2}{4z}\right)^{\frac{1}{3}} d\Omega^2_{(7)},$$

$$F = -\frac{3R^3z^2}{8}\left(1+\frac{\rho^2}{12z^4}\right)\left(z^3+\rho_a x^a - \frac{\rho^2}{4z}\right)^{-2} dx^0\wedge dx^1\wedge dx^2\wedge dz.$$

The field equations of $d = 11$ supergravity imply an algebraic constraint for the $\rho$-matrix:

$$\rho^2 = -\rho_0^2+\rho_1^2+\rho_2^2 = 0.$$

# Conclusions

- T-duality covariant description of $d = 10$ supergravity
- open-closed string map

$\Big\} \Longrightarrow$ Supergravity knows about S-matrix symmetries in the string $\sigma$-model

- U-duality covariant description of $d = 11$ supergravity
- frame change map for $d = 11$

$\Big\} \Longrightarrow$ Supergravity might know about symmetries of the membrane worldsheet theory

? general form of constraints for $\Omega$ from ExFT equations of motion;

? algebraic picture of tri-vector deformations;

? the role of higher simplex equations;

? generalized $d = 11$ supergravity and $\kappa$-symmetry of the membrane;

? study deformations that are not uplifts of YB or TsT from $d = 10$.