Classical solutions and semiclassical expansion of non-relativistic strings in $AdS_5 \times S^5$

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AF, J. Nieto arXiv:2109.13240 (classical solutions)
AF, J. Nieto, A. Torrielli arXiv:2102.00008 (perturbative expansion)

Motivations

- Holographic principle was first formulated between type IIB superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathcal{N} = 4$ SYM on flat spacetime.

  J. Maldacena (1997)

- Extensively studied, both for BPS and non-BPS operators (integrability).

  Reviews: O. Aharony, S. Gubser, J. Maldacena, H. Ooguri, Y. Oz (1999)
  Integrability - Beisert et al. (2010)

Generalisations:

- $\mathrm{AdS}_d/\mathrm{CFT}_{d-1}$, with $d = 3, 4$.

  Review: T. Klose (2010)
  G. Arutyunov, S. Frolov (2008)
  B. Stefanski (2009)
  O. Aharony, O. Bergman, D. Jafferis, J. Maldacena (2008)
  Review: A. Sfondrini (2014)
  S. Gukov, E. Martinec, G. Moore, A. Strominger (2005)
  L. Eberhardt, M. Gaberdiel, W. Li (2017)

- (deformed AdS_d)/(deformed CFT_{d-1}), with $d = 5$.

  F. Delduc, M. Magro, B. Vicedo (2013)
  Ö. Gürdogan, V. Kazakov (2015)

- We do not know much about non-AdS holography

M. Taylor (2016)
R.N. Caldeira Costa, M. Taylor (2011)
M. Guica, K. Skenderis, M. Taylor, B.C. van Rees (2011)
J. Hartong, B. Rollier (2014)
M.H. Christensen, J. Hartong, N. Obers, B. Rollier (2014)
J. Hartong, E. Kiritsis, N. Obers (2015)
V.K. Dobrev (2014)
M. Sakaguchi, K. Yoshida (2008)

- Non-relativistic strings propagate on String Newton-Cartan / Torsional Newton-Cartan spacetimes (therefore non-AdS)

J. Gomis, H. Ooguri (2000)
J. Gomis, J. Gomis, K. Kamimura (2005)
T. Harmark, J. Hartong, N. Obers (2017)
T. Harmark, J. Hartong, L. Menculini, N. Obers, Z. Yan (2018)
T. Harmark, J. Hartong, L. Menculini, N. Obers, G. Oling (2019)
E. Bergshoeff, J. Gomis, Z. Yan (2018)
E. Bergshoeff, J. Gomis, J. Rosseel, C. Simsek, Z. Yan (2020)
J. Gomis, J. Oh, Z. Yan (2019)

- NR strings potentially interesting for new examples of non-AdS holography!

Relativistic $\mathrm{AdS}_5 \times \mathrm{S}^5$ holography

- Focus on the string side of relativistic $\mathrm{AdS}_5/\mathrm{CFT}_4$
- $\mathrm{AdS}_5 \times \mathrm{S}^5$ string action is highly non trivial
- However, it is classically integrable (i.e. its e.o.m. admit a Lax pair)
  I. Bena, J. Polchinski, R. Roiban (2004)
- The perturbative ($T \gg 1$) world-sheet S-matrix is known
  T. Klose, T. McLoughlin, R. Roiban, K. Zarembo (2006)
- Quantum integrability of the model is hard to prove
- Top-down approach:
  - *assume* the model is quantum integrable.
  - Compute the *exact* S-matrix. N. Beisert (2005), R. Janik (2006)
  - At perturbative level, check it matches the known result.
    G. Arutyunov, S. Frolov (2006)
- With exact S-matrix, compute the string spectrum (via Bethe ansatz) $\Longleftrightarrow$ scaling dimensions of operators on dual CFT

In this talk:

- Focus on the string side of NR AdS/CFT duality, in particular on the topic of perturbative S-matrix
- NR strings in $AdS_5 \times S^5$: review the action construction, $\mathbb{Z}_2$ orbifold symmetry
- Classical NR string solutions (BMN- , GKP-like, with closed and $\mathbb{Z}_2$ twisted b.c.)
- Light-cone gauge fixing and semiclassical expansion in $T \gg 1$

Review NR action construction

Relativistic strings in *flat* spacetime

$$S = -\frac{T}{2}\int \mathrm{d}^2\sigma\, \gamma^{\alpha\beta}\, \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \;, \qquad \gamma^{\alpha\beta} \equiv \sqrt{-h}h^{\alpha\beta}$$

NR limit:

$$X^0 \to cX^0 \;, \qquad X^1 \to cX^1 \;, \qquad X^i \to X^i \quad (i = 2, ..., D-1)$$

J. Gomis, H. Ooguri (2000)

instead of X^1, can pick any other spatial direction, and get same answer.

Relativistic strings in *curved* spacetime

$$S = -\frac{T}{2}\int \mathrm{d}^2\sigma\, \gamma^{\alpha\beta}\, g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$$

which directions X^μ need to be rescaled for the NR limit?

For $\mathrm{AdS}_5 \times \mathrm{S}^5$ (but works also for other coset spaces)

$$AdS_5 = \frac{SO(4,2)}{SO(5)} \qquad S^5 = \frac{SO(6)}{SO(5)} \qquad AdS_5 \times S^5 = \frac{G}{H}$$

String action on this coset

R. Metsaev, A. Tseytlin (1998)

$$S = -\frac{T}{2}\int \mathrm{d}^2\sigma\, \gamma^{\alpha\beta}\langle A_\alpha, A_\beta\rangle \ , \qquad A_\alpha = g^{-1}\partial_\alpha g \ , \qquad g \in G$$

A specific choice of g corresponds to a choice of coordinates

$\mathrm{Lie}(G)$ has generators $P_a, J_{ab}, P_{a'}, J_{a'b'}$, where $a \in \mathrm{AdS}_5$, $a' \in \mathrm{S}^5$.

Take e.g. GGK coset representative

J. Gomis, J. Gomis, K. Kamimura (2005)

$$g = g_{\mathrm{AdS}} g_{\mathrm{S}} \ , \qquad g_{\mathrm{AdS}} = e^{x^1 P_1} e^{x^0 P_0} e^{x^m P_m} \ , \qquad g_{\mathrm{S}} = e^{y^{a'} P_{a'}}$$

Coordinates $\Longleftrightarrow$ Generators

NR limit: contraction $\mathfrak{so}(4,2) \to$ 5d Newton-Hooke

This gives us a recipe how to rescale coordinates

However, sometimes this is not enough.

Contraction requires $P_0 \to cP_0$, $P_i \to cP_i$, any $i \in \{1,2,3,4\}$.

In GGK coset representative, only $i = 1$ gives known GGK theory.

In this talk, we use a different coset representative. Coordinates:

$$\begin{array}{ll} AdS_5: & t,\ z_1,\ z_2,\ z_3,\ z_4 \\ S^5: & \phi,\ y_1,\ y_2,\ y_3,\ y_4 \end{array}$$

t time-like isometry (non-compact), ϕ space-like isometry

NR algebra contraction implies rescaling of coordinates:

$$\underbrace{t \to t\ ,\quad z_1 \to z_1}_{\text{longitudinal}}\ ,\quad \underbrace{z_m \to \frac{1}{c} z_m\ ,\quad \phi \to \frac{1}{c}\phi\ ,\quad y_i \to \frac{1}{c} y_i}_{\text{transverse}}\ ,\quad T \to c^2 T$$

- Assuming c large, expand relativistic vielbeine $\hat{E}_\mu{}^{\hat{A}}$
- Read off SNC vielbeine from the power series expansion

$$\hat{E}_\mu{}^A = c\tau_\mu{}^A + \frac{1}{c} m_\mu{}^A + \mathcal{O}(c^{-2}) \qquad \hat{E}_\mu{}^a = e_\mu{}^a + \mathcal{O}(c^{-1})$$

- Take the $c \to \infty$ and neglect $\mathcal{O}(c^{-1})$ terms
- the term $c^2 \tau_\mu{}^A \tau_\nu{}^B \eta_{AB}$ needs to be eliminated from the action
- fine tune a closed B-field: $B_{\mu\nu} = c^2 \tau_\mu{}^A \tau_\nu{}^B \varepsilon_{AB}$
- total divergent term is a perfect square = finite + subleading. Price: introduce 2 extra world-sheet scalar fields (λ_+, λ_-).

$$S^{\mathrm{NR}} = -\frac{T}{2} \int \mathrm{d}^2\sigma \Big(\gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu H_{\mu\nu} + \varepsilon^{\alpha\beta} (\lambda_+ e_\alpha{}^+ \tau_\mu{}^+ + \lambda_- e_\alpha{}^- \tau_\mu{}^-) \partial_\beta X^\mu \Big)$$

$$e_\alpha{}^\pm \equiv e_\alpha{}^0 \pm e_\alpha{}^1 \qquad [\, h_{\alpha\beta} = -e_{(\alpha}{}^+ e_{\beta)}{}^-] \qquad \tau_\alpha{}^\pm \equiv \tau_\alpha{}^0 \pm \tau_\alpha{}^1$$

E. Bergshoeff, J. Gomis, Z. Yan (2018)

Comments:

- Local symmetry: string Newton-Cartan algebra (= Galilei + n.c. extensions). Requires a torsion-free constraint
  E. Bergshoeff, J. Gomis, J. Rosseel, C. Simsek, Z. Yan (2019)
- Map to strings on TNC background from null-reduction.
  T. Hartmark, J. Hartong, L. Menculini, N. Obers, Z. Yan (2018)
  T. Hartmark, J. Hartong, L. Menculini, N. Obers, G. Oling (2018)
- New reformulation of SNC strings, no torsion constraint required
  L. Bidussi, T. Hartmark, J. Hartong, N. Obers, G. Oling (2021)

$\mathbb{Z}_2$ orbifold symmetry

Observation 1

- Bergshoeff, Gomis, Yan action is not invariant under ${e_\alpha}^\pm \to {e_\alpha}^\mp$
- GGK action is invariant (depends only through $h_{\alpha\beta}$)
- they are equivalent under the condition: ${e_1}^+{e_0}^- - {e_0}^+{e_1}^- \geq 0$
- Bergshoeff, Gomis, Yan action is invariant under

$$ {e_\alpha}^\pm \to {e_\alpha}^\mp \,, \qquad \lambda_\pm \to \lambda_\mp \,, \qquad {\tau_\alpha}^\pm \to {\tau_\alpha}^\mp \,. $$

Observation 2

- $\{{\tau_\mu}^A, {m_\mu}^A, {e_\mu}^a\}$ depend *quadratically* on z_1 (invariant $z_1 \to -z_1$)
- term linear in derivative ${\tau_\mu}^+\partial_\alpha X^\mu \to {\tau_\mu}^-\partial_\alpha X^\mu$
- $\mathbb{Z}_2$ symmetry: $\qquad z_1 \to -z_1 \,, \qquad {e_\alpha}^\pm \to {e_\alpha}^\mp \,, \qquad \lambda_\pm \to \lambda_\mp$
- use it to construct string solutions with twisted b.c.

Classical string solutions

- Fix conformal gauge (with ${e_\alpha}^\pm$ satisfying positivity constraint)
- Solve e.o.m. for the fields $(t, z_i, \phi, y_i, \lambda_\pm)$ + Virasoro constraints
- Impose b.c. to cancel surface term (closed / twisted).

$\lambda_\pm$ e.o.m.

$$\left(\dot{t} + \frac{z_1'}{1 + (\frac{z_1}{2R})^2}\right) = 0 \qquad \left(t' + \frac{\dot{z}_1}{1 + (\frac{z_1}{2R})^2}\right) = 0$$

change of variable $z_1 = 2R\tan(\frac{Z}{2R})$

$$\dot{t} + Z' = 0 \qquad t' + \dot{Z} = 0$$

$\Longrightarrow$ t, Z satisfy a wave equation

$$t = f_-(\tau - \sigma) - f_+(\tau + \sigma) \qquad z_1 = 2R\tan\left[\frac{f_+(\tau + \sigma) + f_-(\tau - \sigma)}{2R} + c\right]$$

- Residual $\mathrm{Diff}_+ \oplus \mathrm{Diff}_-$ symmetry acting on $\tau \pm \sigma$
- Fix it by choosing f_+, f_- linear in their arguments

$$t = \kappa\tau \qquad\qquad z_1 = 2R\tan\left[-\frac{\kappa}{2R}(\sigma - \sigma_0)\right]$$

- Should be general behaviour for all NR strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$
- winding around z_1 though the Tangent: go to $+\infty$, back from $-\infty$
- get solutions with non-trivial spectrum. Exception/different realization of Gomis-Ooguri statement?

[Gomis-Ooguri (2000): *In flat space, NR strings can have non-trivial spectrum only if they have non-zero winding around a spatial longitudinal compact direction.*]

Closed string solutions

b.c.

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi)\ , \qquad \lambda_\pm(\tau,\sigma) = \lambda_\pm(\tau,\sigma+2\pi)$$

- **BMN-like** ($\kappa = nR$, $n \in \mathbb{Z}$)

$$t = nR\,\tau \qquad z_1 = 2R\tan\left(-\frac{n}{2}\sigma\right) \qquad \phi = w\tau \qquad \lambda_\pm = \pm\frac{w^2}{2nR}\cos(n\sigma)$$

other coords $= 0$.

Noether charges $J = -w\pi T$, $E = -\frac{w^2\pi T}{nR}$, dispersion relation

$$E - \frac{w}{nR}J = 0 \qquad \text{or} \qquad E - \frac{1}{\pi nRT}J^2 = 0$$

To compare with the BMN of *relativistic* theory:

$$t = nR\,\tau \qquad \phi = nR\,\tau \qquad \text{other coords} = 0$$

Dispersion relation: $E - J = 0$

- **Generalised BMN-like**: static profile on AdS $z_m = z_m(\sigma)$
- $z_m(\sigma)$ must satisfy a generalised Lamé equation, solution: $z_m \propto z_1$
- same conserved charges as for the BMN-like

- **GKP-like**: spinning string on AdS

$$z_2 = \rho(\sigma)\cos(\omega\tau) \qquad z_3 = \rho(\sigma)\sin(\omega\tau) \qquad z_4 = 0$$

- $\rho(\sigma)$ must satisfy an equation which is not a generalised Lamé
- Dispersion relation is more complicated to write in a closed form

BMN-like solution has also been confirmed by studying the recent new formulation of the NR action J. Kluson (2021)

Twisted string solutions

twisted b.c.

$$z_1(\tau,\sigma) = -z_1(\tau,\sigma+2\pi) \qquad \lambda_\pm(\tau,\sigma) = -\lambda_\mp(\tau,\sigma+2\pi)$$

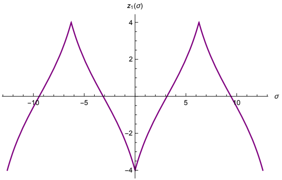
+ closed (or also twisted) b.c. on other X^μ fields.

Twisted BMN-like: when $\sigma \in [0, 2\pi]$

$$t = \kappa\tau \qquad z_1 = 2R\tan\left[-\frac{\kappa}{2R}(\sigma-\pi)\right] \qquad \phi = w\tau$$

$$\lambda_\pm = \pm\frac{w^2}{2\kappa}\cos\left[\frac{\kappa}{R}(\sigma-\pi)\right]$$

when $\sigma \in \mathbb{R}$: glue many copies of the above



- Linearisation in large R expansion:

$$z_1 = 2R\tan\left[-\frac{\kappa}{2R}(\sigma-\pi)\right] = -\kappa(\sigma-\pi) - \frac{\kappa^3}{12R^2}(\sigma-\pi)^3 + \ldots$$

- this does not happen for the BMN-like with closed b.c.

- $\exists$ twisted version of generalised BMN-like and GKP-like

AF, J. Nieto (2021)

Uniform light-cone gauge fixing

This has been studied for NR strings in J. Kluson (2017, 2018)

In first-order formalism

$$S^{NR} = \int \mathrm{d}^2\sigma \left(p_\mu \dot{X}^\mu + \frac{\gamma^{01}}{\gamma^{00}} V_1 + \frac{1}{2T\gamma^{00}} V_2 \right) \equiv \int \mathrm{d}^2\sigma (p_\mu \dot{X}^\mu - \mathcal{H})$$

conjugate momenta $p_\mu \equiv \delta S^{NR}/\delta \dot{X}^\mu$, Virasoro constraints:

$$\begin{aligned} V_1 &= p_\mu X'^\mu \\ V_2 &= H^{-1}_{\mu\nu} p_\mu p_\nu + T^2 H_{\mu\nu} X'^\mu X'^\nu + T\lambda_A {\tau_\mu}^A p_\nu H^{-1}_{\mu\nu} \\ &+ \frac{T^2}{4} \lambda_A \lambda_B {\tau_\mu}^A {\tau_\nu}^B H^{-1}_{\mu\nu} - T^2 \lambda_A \varepsilon^{AB} \eta_{BC} {\tau_\mu}^C X'^\mu \end{aligned}$$

Observation: $H_{\mu\nu}$ needs to be invertible.

Need two isometry directions (time + space). For us: t, ϕ

Introduce light-cone coordinates ($0 \le a \le 1$)

$$X_+ = (1-a)t + a\phi \;, \qquad X_- = \phi - t \;,$$
$$p_+ = (1-a)p_\phi - a p_t \;, \qquad p_- = p_\phi + p_t \;,$$

Fix light-cone gauge

$$X_+ = \tau \;, \qquad p_+ = \text{const}$$

The model now has d.o.f. $(p_-, X_-, p_I, X_I, \lambda_A)$. $p_{\lambda_A} \approx 0$:

$$\partial_\tau p_{\lambda_A} = \{p_{\lambda_A}, H\} = \mathcal{G}_A(\lambda_B) \approx 0 \qquad \Longrightarrow \qquad 2 \text{ eqn. to solve for } \lambda_A$$

Solve $V_1 \approx 0$ and $V_2 \approx 0$, the model reduces to

$$S^{NR}_{l.g.} = \int \mathrm{d}^2\sigma \, (p_I \dot{X}^I - \mathcal{H}) \qquad \mathcal{H} = -p_-(X^I, X'^I, p_I) \qquad X^I = (z_i, y_i)$$

$\mathcal{H}$ is a very complicated expression!

Advantage: this is a complicated dynamics, but on a cylinder (flat)

It may be expanded about free fields in flat spacetime

Semi-classical expansion

Expand fields about twisted BMN-like solution in large T

$$z_1 \to 2R\tan\left[-\frac{\kappa}{2R}(\sigma-\pi)\right] + \frac{1}{\sqrt{T}}\tilde{z} \qquad z_m \to \frac{1}{\sqrt{T}}z_m \qquad y_i \to \frac{1}{\sqrt{T}}y_i$$

$$p_{z_i} \to \sqrt{T}p_{z_i} \qquad p_{y_i} \to \sqrt{T}p_{y_i}$$

After expanding also in large R

$$S^{NR}_{l.g.} = \int \mathrm{d}\tau\mathrm{d}\sigma\left(\mathcal{L}_2 + \frac{1}{R^2}\hat{\mathcal{L}}_2 + \frac{1}{\sqrt{T}}\mathcal{L}_3 + \frac{1}{R^2\sqrt{T}}\hat{\mathcal{L}}_3 + ...\right)$$

- $\mathcal{L}_2$: 8 free massless scalar fields on cylinder
- $\mathcal{L}_n$ with $n \geq 3$: corrections independent of σ.
- $\hat{\mathcal{L}}_n$ with any n: corrections *dependent* of σ.

Decompactification limit

To define asymptotic states for S-matrix:

- take the cylinder radius to be infinite
- cut the cylinder

$\Longrightarrow$ theory on 2d Minkowski (now $\sigma \in \mathbb{R}$)

Obstacle: correction terms $\hat{\mathcal{L}}_n$ contains terms e.g.

$$\frac{(\sigma-\pi)^2}{R^2} F(X^I, X'^I, p_I)$$

In decomp. limit: $\sigma \sim R \qquad \Longrightarrow \qquad \hat{\mathcal{L}}_n$ becomes leading

Static gauge

Take the Bergshoeff, Gomis, Yan action, e.o.m. for $\lambda_\pm$

$$\varepsilon^{\alpha\beta} e_\alpha{}^\pm \tau_\beta{}^\pm = 0 \quad \Longrightarrow \quad e_\alpha{}^\pm = \psi(\tau,\sigma)\tau_\alpha{}^\pm \quad \Longrightarrow \quad \sqrt{-h}h^{\alpha\beta} = \sqrt{-\tau}\tau^{\alpha\beta}$$

where $\tau_{\alpha\beta} \equiv \tau_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$, and $\tau_{\mu\nu}$ is the AdS_2 metric

Fix static gauge:

$$t = \tau \qquad z_1 = \sigma$$

Action becomes:

$$S^{NR}_{s.g.} = -\frac{T}{2}\int \mathrm{d}^2\sigma \sqrt{-\tau}\left(\tau^{\alpha\beta}\partial_\alpha x^a \partial_\beta x^a + \frac{2}{R^2}x^a x^a + \tau^{\alpha\beta}\partial_\alpha y^m \partial_\beta y^m\right)$$

Simple dynamics: 8 free fields (x^a, y^m), x^a massive, y^m massless

Disadvantage: they live in AdS_2, difficult to define asymptotic states

Summary

- Focused on classical string solutions of the GGK theory
- E.o.m. for λ_A impose non-trivial profile on *both* longitudinal coords (t, z_1)
- Found NR analogue of BMN, GKP solutions with closed and twisted b.c.
- Fixed uniform light-cone gauge
- Expanded about twisted BMN-like solution ($T \gg 1, R \gg 1$)
- Expansions about solutions with closed b.c. (or around $z_1 = 0$) not meaningful

Open questions

- for a full understanding of NR AdS/CFT, rôle of supersymmetry?
- dual field theory on the boundary? "NR limit" of $\mathcal{N} = 4$ SYM?
- Interpretation of BMN-like dispersion relations for operators on dual field theory?
- connection between our solutions and the spinning strings for NR limit in target + world-sheet ($\sim$Landau-Lifshitz)? D. Roychowdhury (2019-2020)
- classical integrability?
- connection with leading term in the expansion of relativistic string action about AdS_2 minimal surfaces of straight Wilson lines?

  S. Giombi, R. Roiban, A. Tseytlin (2017)

Thank you for your attention!