Integrated four-point correlators in $\mathcal{N} = 4$ SYM

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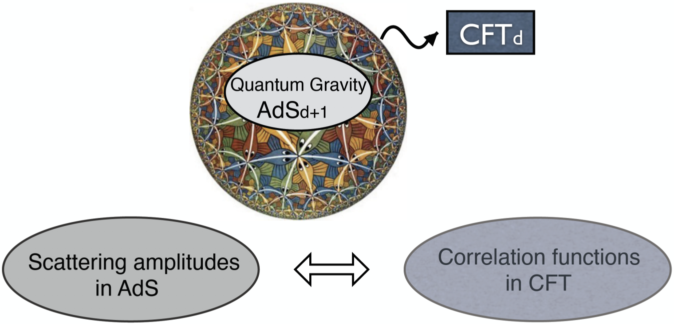
Introduction

- Correlation functions in $\mathcal{N} = 4$ SYM have been a subject of interest for many years, especially since the discovery of AdS/CFT correspondence.



Introduction

- One of the best studied examples of AdS/CFT duality

$$\text{IIB superstring on } \mathrm{AdS}_5 \times \mathrm{S}^5 \quad \Leftrightarrow \quad 4\mathrm{D}\, \mathcal{N}=4\, \mathrm{SYM}$$

- The α' expansion of string amplitudes is mapped to the large-N expansion of CFT correlators; string τ_s (axion & dilaton) is identified with YM coupling τ_{YM} (θ angle & g_{YM}).

- Both type IIB superstring theory and $\mathcal{N}=4$ SYM have $SL(2,\mathbb{Z})$ modular symmetry:

$$\tau \to \frac{a\tau+b}{c\tau+d}, \quad \text{with} \quad \tau=\tau_1+i\tau_2=\frac{\theta}{2\pi}+i\frac{4\pi}{g_{\mathrm{YM}}^2}=\chi+\frac{i}{g_s}.$$

The S-duality of superstring theory and Montonen–Olive duality of the field theory.

Introduction

- Exploring the $SL(2,\mathbb{Z})$ duality symmetry requires to compute amplitudes and correlators at finite couplings, which in general is very hard:
  - Superstring amplitudes at low orders of α' expansion are known to all orders in string coupling. [Green, Gutperle] [Green, Gutperle, Vanhove]...
  - Our aim is to study integrated correlators in $SU(N)$ $\mathcal{N}=4$ SYM with arbitrary N and $\tau_{\rm YM}$.
  - In the large-N expansion, they produce corresponding results of type IIB superstring amplitudes in the α' expansion.
- AdS/CFT duality: beyond supergravity, & finite couplings with non-perturbative effects (matching YM instantons with D-instantons in string theory).

Outline

- A review on general properties of four-point correlation function of superconformal primaries in $\mathcal{N}=4$ SYM.
- The integrated four-point correlators in $\mathcal{N}=4$ SYM, and their relations to supersymmetric localization.
- A novel representation of an integrated four-point correlator.
  - Derive the formula for $SU(2)$ theory.
  - Small-$g_{\rm YM}$ expansion at finite N.
  - Small- and large-λ expansion at large N.
  - Large-N expansion at finite $g_{\rm YM}$ and θ angle.
- Relation to string amplitudes on $\mathrm{AdS}_5 \times S^5$ in α' expansion.
- Conclusion and future directions.

Correlators in $\mathcal{N}=4$ SYM

- We will study the four-point correlator of CPO's,

$$\mathcal{O}_2(x,Y)=\mathrm{tr}(\phi_{I_1}(x)\phi_{I_2}(x))Y^{I_1}Y^{I_2}\,,$$

  where $I_p=1,2,\cdots,6$ and $Y\cdot Y=0$ to subtract trace part.

- Two- and three-point correlators are protected by supersymmetry.

- Supersymmetry and superconformal symmetries imply [Eden et al][Dolan, Osborn]

$$\langle\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle=\langle\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle_{\rm free}+\mathcal{I}_4(x_i,Y_i)\,\mathcal{T}_N(U,V;\tau,\bar{\tau})\,,$$

  here $\mathcal{I}_4(x_i,Y_i)$ is fixed by symmetries, whereas $\mathcal{T}_N(U,V;\tau,\bar{\tau})$ is the dynamic part.

Correlators in $\mathcal{N}=4$ SYM

- This four-point correlator has been extensively studied over past years:
  - Perturbation: known up to 3 loops [Drummond et al]; in the planar limit, the integrand is known up to 10 loops [Bourjaily, Heslop, Tran]; the first non-planar contribution enters at 4 loops [Fleury, Pereira]. Computed from Lagrangian insertion [Intriligator][Eden et al]:

$$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle_\ell = \int d^4x_5 \cdots d^4x_{\ell+4} \langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \mathcal{L}(x_5)\mathcal{L}(x_6)\cdots\mathcal{L}(x_{\ell+4})\rangle_0 \,.$$

  - Exact results in the large-N expansion: known up to sub-sub-subleading term (d^6R^4 in string amplitude) for finite $\tau_{\rm YM}$. The results were derived from the integrated correlators. [Chester, Green, Pufu, Wang, and C.W.]

Correlators in $\mathcal{N}=4$ SYM

- Working in the Mellin space [Mack][Penedones]

$$\mathcal{T}_N(U,V;\tau,\bar\tau)=\int\frac{ds\,dt}{(4\pi i)^2}U^{\frac{s}{2}}V^{\frac{t-4}{2}}\Gamma^2(\frac{4-s}{2})\Gamma^2(\frac{4-t}{2})\Gamma^2(\frac{s+t-4}{2})\mathcal{M}_N(s,t;\tau,\bar\tau)\,.$$

- In the large-N expansion ($c=(N^2-1)/4$),

$$\begin{aligned}\mathcal{M}_N(s,t;\tau,\bar\tau)=&\frac{8c}{(s-2)(t-2)(u-2)}+\frac{15E(\frac{3}{2},\tau,\bar\tau)c^{1/4}}{4\sqrt{2\pi^3}}+\mathcal{M}^{\rm SUGRA}_{\text{1-loop}}(s,t)\\&+\frac{315E(\frac{5}{2},\tau,\bar\tau)}{128\sqrt{2\pi^5}c^{1/4}}\left[(s^2+t^2+u^2)-3\right]\\&+\frac{945\mathcal{E}(3,\frac{3}{2},\frac{3}{2},\tau,\bar\tau)}{64\pi^3c^{1/2}}\left[stu-\frac{1}{4}(s^2+t^2+u^2)-4\right]+\cdots\end{aligned}$$

E is the non-holomorphic Eisenstein series, and $\mathcal{E}$ is its generalisation: contain perturbative and instanton terms.

Supersymmetric Localization

We are interested in the correlator for arbitrary τ_{YM} and N; will use supersymmetric localization [Pestun].... Basic idea:

- Let δ be a Grassmann-odd symmetry of action $S(\phi)$. Now consider the deformed partition function

$$Z(t) = \int \mathcal{D}\phi e^{-S-t\delta V} \Rightarrow \frac{dZ(t)}{dt} = 0\,,$$

  if $\delta^2 V = 0$.

- Therefore $Z(t)$ can be computed at $t=0$ or $t\to\infty$ (if $\delta V > 0$), which localizes the path integral to a submanifold of field space $\delta V = 0$.

Supersymmetric Localization

- We begin with $SU(N)$ $\mathcal{N}=2^*$ SYM on S^4: $\mathcal{N}=4$ SYM on S^4 deformed by mass terms

$$S_m = \int d^4x\sqrt{g}\left(m(iJ/r+K)+m^2L\right)\,,$$

where

$$J=\frac{1}{2}\sum_{i=1}^{2}\mathrm{tr}[(Z_i^2)+(\bar{Z}_i^2)]\,,\quad L=\sum_{i=1}^{2}\mathrm{tr}|Z_i|^2\,,$$

$$K=-\frac{1}{2}\sum_{i=1}^{2}\mathrm{tr}[\chi_i\sigma_2\chi_i+\tilde{x}_i\sigma_2\tilde{x}_i]\,,$$

(Z_i,χ_i) with $i=1,2$ correspond to $\mathcal{N}=2$ hypermultiplet.

- When $m\to 0$, it reduces to $\mathcal{N}=4$ SYM.

Supersymmetric Localization

- The partition function for $\mathcal{N} = 2^*$ SYM on S^4:[Nekorasov][Pestun]...

$$Z(m,\tau,\bar{\tau}) = \int d^{N-1}a \prod_{i<j}(a_i - a_j)^2 \, e^{-\frac{8\pi^2}{g_{\rm YM}^2}\sum_i a_i^2} Z_{\rm pert} \, |Z_{\rm inst}|^2 \, .$$

- $Z_{\rm pert}$ contributes to perturbation,

$$Z_{\rm pert} = H(m) \prod_{i,j} \frac{H(a_{ij})}{H(a_{ij}+m)} \, , \quad H(z) = e^{-(1+\gamma)z^2} \, G(1+iz)\, G(1-iz) \, ,$$

where G is the Barnes-G function.

- The non-perturbative instanton sector $|Z_{\rm inst}|^2 = Z_{\rm inst}\bar{Z}_{\rm inst}$:

$$Z_{\rm inst} = 1 + \sum_{k=1}^{\infty} e^{i2\pi k\tau} Z_k \, .$$

Instanton partition function

- The k-instanton contribution ($\epsilon_1 = \epsilon_2 = 1$ for round S^4)

$$Z_k(m, a_i) = \frac{1}{k!}\left(\frac{2m^2}{m^2+1}\right)^k \oint \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} \prod_{i=1}^{N} \frac{(\phi_I - a_i)^2 - m^2}{(\phi_I - a_i)^2 + 1}$$
$$\times \prod_{I<J}^{k} \frac{\phi_{IJ}^2(\phi_{IJ}^2 + 4)(\phi_{IJ}^2 - m^2)^2}{(\phi_{IJ}^2 + 1)^2((\phi_{IJ} - m)^2 + 1)((\phi_{IJ} + m)^2 + 1)} .$$

- The integration contour is around

$$\phi_I = a_j + (r + s - 1)i,$$

$\{r, s\}$ is the position of a box in a Young diagram.

- A $k = 2$ example:

$$\vec{Y} = \{\ldots, \square\square, \ldots, \}, \quad \vec{Y} = \{\ldots, \begin{smallmatrix}\square\\ \square\end{smallmatrix}, \ldots, \}, \quad \vec{Y} = \{\ldots, \square, \ldots, \square, \ldots\}$$

Integrated correlators from partition function

- No localization for correlators! But, we can extract integrated correlators from the partition function [Binder et al; Chester, Pufu]

$$\mathcal{G}_N(\tau,\bar\tau) = -\frac{8}{\pi}\int dr d\theta\, \frac{r^3\sin^2(\theta)}{U^2}\,\mathcal{T}_N(U,V;\tau,\bar\tau)$$
$$= \tau_2^2\partial_\tau\partial_{\bar\tau}\partial_m^2 \log Z(m,\tau,\bar\tau)\big|_{m=0}\,.$$

  with $U=1+r^2-2r\cos(\theta)$, $V=r^2$.

  - ∂_m^2 brings down operators in the mass terms $(iJ/r+K)$, L, which are integrated over; $\partial_\tau\partial_{\bar\tau}$ brings down CPO's $\sim \mathrm{tr}(Z_3^2)$ at south and north poles.
  - Conformal symmetry maps $S^4\to R^4$; SUSY and superconform relate all the correlators to $\langle\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle$.
  - Other integrated correlators $\partial_m^4\log Z$, $\partial_m^2\partial_b^2\log Z$ (b is the squashed parameter). We will focus on $\mathcal{G}_N(\tau,\bar\tau)$.

Exact results of an integrated correlator: $SU(2)$

For $SU(2)$: from $Z_{\rm pert}$, we find the perturbative term

$$\mathcal{G}_{2,0}(y) = y\int_0^\infty \frac{e^{-t}(6t-9t^2+2t^3)}{2\sinh^2(\sqrt{yt})}dt\,,\quad \text{with}\quad y=4\pi^4/g_{\rm YM}^2\,.$$

- Large-y expansion (loop expansion)

$$\mathcal{G}_{2,0}(y) \sim \sum_{s=2}^{\infty}\frac{(2s-1)\Gamma(2s+1)(-1)^s}{2^{2s-1}\Gamma(s-1)}\zeta(2s-1)y^{1-s}\,.$$

- Small-y expansion (strong coupling)

$$\mathcal{G}_{2,0}(y) \sim \frac{1}{2}+\frac{1}{2}\sum_{s=2}^{\infty}(-1)^s(s-1)(2s-1)^2\Gamma(s+1)\frac{2\zeta(2s)}{\pi^{2s}}y^s\,.$$

Exact results of an integrated correlator: $SU(2)$

- Observation: small & large-y expansion combine to form the perturbative terms of non-holomorphic Eisenstein series,

$$E(s;\tau,\bar\tau) = \frac{1}{\pi^s}\sum_{(m,n)\neq(0,0)}\frac{\tau_2^s}{|m+n\tau|^{2s}}$$
$$= \frac{2\zeta(2s)}{\pi^{2s}}y^s + \frac{2\zeta(2s-1)\Gamma\left(s-\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(s)}y^{1-s} + \text{instantons}\,.$$

This leads to the conjecture for $SU(2)$

$$\mathcal{G}_2(\tau,\bar\tau) \sim \frac{1}{4} + \frac{1}{4}\sum_{s=2}^{\infty}(-1)^s(s-1)(2s-1)^2\Gamma(s+1)E(s;\tau,\bar\tau)\,.$$

- Matches $\mathcal{G}_{2,0}(y)$ by construction. What about instantons ?

Exact results of an integrated correlator: $SU(2)$

- Check instanton terms:
  - One-instanton contribution,

$$\mathcal{G}_{2,1}(\tau,\bar{\tau}) = e^{2\pi i\tau}\left[12y^2 - 3\sqrt{\pi}e^{4y}y^{3/2}(1+8y)\mathrm{erfc}\left(2\sqrt{y}\right)\right] \checkmark$$

  - k instantons. Theorem: only rectangular Young diagrams (p rows & q columns, $pq=k$) contribute to $\partial_m^2 Z_k\big|_{m=0}$. ✓
- The final $SL(2,\mathbb{Z})$-invariant expression (using Schwinger parametrization for $E(s,\tau,\bar{\tau}) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{\pi^s |m+n\tau|^{2s}}$),

$$\mathcal{G}_2(\tau,\bar{\tau}) = \frac{1}{4} + \frac{1}{4}\sum_{(m,n)\neq(0,0)}\int_0^\infty dt\, e^{-t\pi\frac{|m+n\tau|^2}{\tau_2}}\sum_{s=2}^{\infty}(-1)^s(s-1)s(2s-1)^2 t^{s-1}$$
$$= \frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2}\int_0^\infty dt\, e^{-t\pi\frac{|m+n\tau|^2}{\tau_2}}\frac{9t-30t^2+9t^3}{(t+1)^5}.$$

Exact results of an integrated correlator

For $SU(N)$: After study a lot more examples we conjecture,

$$\mathcal{G}_N(\tau,\bar\tau) = \frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2}\int_0^\infty \exp\Big(-t\pi\frac{|m+n\tau|^2}{\tau_2}\Big)B_N(t)\,dt\,,$$

with $B_N(t)=\frac{\mathcal{Q}_N(t)}{(t+1)^{2N+1}}$, & $\mathcal{Q}_N(t)$ is a degree-$(2N-1)$ polynomial in t

$$\mathcal{Q}_N(t) = -\frac{1}{2}N(N-1)(1-t)^{N-1}(1+t)^{N+1}$$
$$\Big\{(3+(8N+3t-6)\,t)\,P_N^{(1,-2)}(z)+\frac{1}{1+t}\Big(3t^2-8Nt-3\Big)P_N^{(1,-1)}(z)\Big\}\,,$$

with $z=\frac{1+t^2}{1-t^2}$, $P_N^{(\alpha,\beta)}$ is the Jacobi polynomial.

Exact results of an integrated correlator

- Some basical properties of $B_N(t)$.
  - Examples: $SU(2)$, $SU(3)$,

$$B_2(t) = \frac{9t - 30t^2 + 9t^3}{(1+t)^5}\,, \quad B_3(t) = \frac{36t^5 - 198t^4 + 252t^3 - 198t^2 + 36t}{(1+t)^7}\,.$$

  - Inversion invariance & integral identity,

$$B_N(t) = \frac{1}{t} B_N(1/t)\,, \qquad \int_0^\infty B_N(t) \frac{1}{\sqrt{t}} dt = 0\,.$$

- $\mathcal{G}_N(\tau, \bar\tau)$ can be formally expanded in terms of non-holomorphic Eisenstein series,

$$\mathcal{G}_N(\tau, \bar\tau) = \frac{N(N-1)}{8} + \frac{1}{2} \sum_{s=2}^{\infty} c_s^{(N)} E(s; \tau, \bar\tau)\,,$$

with $B_N(t) = \sum_{s=2}^{\infty} c_s^{(N)} t^{s-1}/\Gamma(s)$.

Exact results of an integrated correlator

- $B_N(t)$ obeys a differential equation, which leads to a Laplace-difference equation for $\mathcal{G}_N(\tau,\bar\tau)$,

$$\left(\tau_2^2\partial_\tau\partial_{\bar\tau}-2\right)\mathcal{G}_N(\tau,\bar\tau)=N^2\Big[\mathcal{G}_{N+1}(\tau,\bar\tau)-2\mathcal{G}_N(\tau,\bar\tau)+\mathcal{G}_{N-1}(\tau,\bar\tau)\Big]$$
$$-N\Big[\mathcal{G}_{N+1}(\tau,\bar\tau)-\mathcal{G}_{N-1}(\tau,\bar\tau)\Big]\,.$$

- $\mathcal{G}_1=0$, once $\mathcal{G}_2$ is provided, the Laplace-difference equation determines $\mathcal{G}_N(\tau,\bar\tau)$ for all N.
- We will now study properties of $\mathcal{G}_N(\tau,\bar\tau)$ in various limits.

Exact results of an integrated correlator: perturbation

- Weak-coupling expansion at finite N

$$\mathcal{G}_{N,0}(\tau_2) = 4c\Bigg[\frac{3\,\zeta(3)a}{2} - \frac{75\,\zeta(5)a^2}{8} + \frac{735\,\zeta(7)a^3}{16} - \frac{6615\,\zeta(9)\left(1+\frac{2}{7}N^{-2}\right)a^4}{32} + \frac{114345\,\zeta(11)\left(1+N^{-2}\right)a^5}{128} - \frac{3864861\,\zeta(13)\left(1+\frac{25}{11}N^{-2}+\frac{4}{11}N^{-4}\right)a^6}{1024} + \cdots\Bigg]$$

with $a=\lambda/(4\pi^2)$.

  - The One- and two-loop terms are proved to agree with known results.
  - The non-planar contributions start to enter at four loops, in agreement with known results.
  - It gives all-loop predictions for any N.

Exact results of an integrated correlator: large-N expansion

- Large-N expansion with small-λ:

$$\mathcal{G}_N(\tau,\bar\tau) \sim \sum_{g=0}^{\infty} N^{2-2g}\, \mathcal{G}^{(g)}(\lambda)\,,$$

$$\mathcal{G}^{(0)}(\lambda) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}\zeta(2n+1)\Gamma\left(n+\frac{3}{2}\right)^2}{\pi^{2n+1}\Gamma(n)\Gamma(n+3)}\lambda^n\,,$$

$$\mathcal{G}^{(1)}(\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^n(n-5)(2n+1)\zeta(2n+1)\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{24\,\pi^{2n+1}\Gamma(n)^2}\lambda^n\,,$$

$$\ldots\ldots$$

They are all convergent with a finite radius $|\lambda| < \pi^2$. After resummation (agree with [Binder et al; Chester])

$$\mathcal{G}^{(0)}(\lambda) = \lambda \int_0^\infty dw\, w^3 \frac{{}_1F_2\left(\frac{5}{2};2,4;-\frac{w^2\lambda}{\pi^2}\right)}{4\pi^2 \sinh^2(w)}\,,$$

$$\ldots\ldots$$

Exact results of an integrated correlator: large-N expansion

- Large-N expansion with large-λ:

$$\mathcal{G}^{(0)}(\lambda) \sim \frac{1}{4} + \sum_{n=1}^{\infty} \frac{\Gamma\left(n-\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)\Gamma(2n+1)\zeta(2n+1)}{2^{2n-2}\pi\,\Gamma(n)^2\,\lambda^{n+1/2}}\,,$$

$$\mathcal{G}^{(1)}(\lambda) \sim -\frac{\sqrt{\lambda}}{16} - \sum_{n=1}^{\infty} \frac{n^2(2n+11)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)^2\zeta(2n+1)}{24\,\pi^{\frac{3}{2}}\Gamma(n+2)\,\lambda^{n+1/2}}\,,$$

$$\cdots\cdots$$

- They are all asymptotic & not Borel summable; require non-perturbative (world-sheet instanton) completions

$$\Delta\mathcal{G}^{(0)}(\lambda) = i\left[8\mathrm{Li}_0(e^{-2\sqrt{\lambda}}) + \frac{18\mathrm{Li}_1(e^{-2\sqrt{\lambda}})}{\lambda^{1/2}} + \frac{117\mathrm{Li}_2(e^{-2\sqrt{\lambda}})}{4\lambda} + \cdots\right].$$

Exact results of an integrated correlator: large-N expansion

- Large-N expansion with finite $\tau_{\rm YM}$ can be obtained by expanding $B_N(t)$

$$B_N(t) = \sum_{\ell=0}^{\infty} N^{\frac{1}{2}-\ell} p_\ell(t)$$

with

$$p_0(t) = -\frac{3}{8\sqrt{\pi}}(t^{1/2}+t^{-3/2}), \quad p_1(t) = \frac{15}{64\sqrt{\pi}}(t^{3/2}+t^{-5/2}),$$

$$p_2(t) = \frac{315}{4096\sqrt{\pi}}(t^{5/2}+t^{-7/2}) - \frac{39}{4096\sqrt{\pi}}(t^{1/2}+t^{-3/2}),$$

$$p_3(t) = \frac{945}{16384\sqrt{\pi}}(t^{7/2}+t^{-9/2}) - \frac{375}{16384\sqrt{\pi}}(t^{3/2}+t^{-5/2}),$$

$$\cdots\cdots$$

Exact results of an integrated correlator: large-N expansion

- Large-N in terms of $E(s;\tau,\bar\tau)$ (with half integer s):

$$
\begin{aligned}
\mathcal{G}_N(\tau,\bar\tau) \sim{}& \frac{N^2}{4} - \frac{3N^{\frac12}}{2^4}E(\tfrac32;\tau,\bar\tau) + \frac{45}{2^8 N^{\frac12}}E(\tfrac52;\tau,\bar\tau) \\
&+ \frac{3}{N^{\frac32}}\left[\frac{1575}{2^{15}}E(\tfrac72;\tau,\bar\tau) - \frac{13}{2^{13}}E(\tfrac32;\tau,\bar\tau)\right] + \frac{225}{N^{\frac52}}\left[\frac{441}{2^{18}}E(\tfrac92;\tau,\bar\tau) - \frac{5}{2^{16}}E(\tfrac52;\tau,\bar\tau)\right] \\
&+ \frac{63}{N^{\frac72}}\left[\frac{3898125}{2^{27}}E(\tfrac{11}{2};\tau,\bar\tau) - \frac{44625}{2^{25}}E(\tfrac72;\tau,\bar\tau) + \frac{73}{2^{22}}E(\tfrac32;\tau,\bar\tau)\right] \\
&+ \frac{945}{N^{\frac92}}\left[\frac{31216185}{2^{31}}E(\tfrac{13}{2};\tau,\bar\tau) - \frac{41895}{2^{26}}E(\tfrac92;\tau,\bar\tau) + \frac{1639}{2^{27}}E(\tfrac52;\tau,\bar\tau)\right] + \cdots,
\end{aligned}
$$

  It reproduces those in [1912.13365] and leads to new results.

- The result can also be obtained from the Laplace-difference equation, after input the initial data.

Comparison with IIB string amplitudes on $\mathrm{AdS}_5 \times S^5$

- From $\partial_m^2 \partial_\tau \partial_{\bar\tau} \log Z$
  - $\frac{N^2}{4}$ is the integrated supergravity term.
  - $-\frac{3N^{\frac{1}{2}}}{2^4} E(\frac{3}{2}; \tau, \bar\tau)$ is the integrated R^4 on $AdS_5 \times S^5$.
  - $\frac{45}{2^8 N^{\frac{1}{2}}} E(\frac{5}{2}; \tau, \bar\tau)$ is the integrated $d^4 R^4$ on $AdS_5 \times S^5$.
  - All $d^{4n+2} R^4$ terms integrated to 0 with this measure.
- $\partial_m^4 \log Z$ is more difficult to evaluate. First a few orders in large-N expansion were determined in [2008.02713], which give non-zero integrated $d^6 R^4, d^{10} R^4, \cdots$ on $AdS_5 \times S^5$.

Comparison with IIB string amplitudes on $\mathrm{AdS}_5 \times S^5$

- One can reconstruct the un-integrated correlator in the large-N expansion by an ansatz for the Mellin amplitude,

$$\mathcal{M}(s,t) = \frac{\rho\, c}{(s-2)(t-2)(u-2)} + c^{1/4}\alpha + \mathcal{M}^{\rm SUGRA}_{\text{1-loop}}(s,t) + \frac{\beta_2(s^2+t^2+u^2)+\beta_1}{c^{1/4}} + \frac{\gamma_3 stu + \gamma_2(s^2+t^2+u^2)+\gamma_1}{c^{1/2}} + \cdots .$$

- They are higher-derivative terms on $AdS_5 \times S^5$:

$$\{\rho, \mathcal{M}^{\rm SUGRA}_{\text{1-loop}}\} \to R; \quad \alpha \to R^4;$$
$$\{\beta_1,\beta_2\} \to d^4R^4; \quad \{\gamma_1,\gamma_2,\gamma_3\} \to d^6R^4 .$$

- The unknown coefficients are fixed using two integrated correlators, and also the flat-space limit in the case of d^6R^4.

Conclusion

- Integrated four-point correlators can be computed using supersymmetric localisation.
- We focused on one integrated correlator, which is presented as a Lattice sum for any N and $\tau_{\rm YM}$, and manifests the $SL(2,\mathbb{Z})$ invariance of $\mathcal{N} = 4$ SYM.
- The integrated correlator obeys a Laplace-difference equation that relates the correlator of $SU(N)$ with those of $SU(N+1)$ and $SU(N-1)$.
- Various limits: small-$g_{\rm YM}$ at finite N; large-N expansion with small & large λ; large-N expansion with finite $\tau_{\rm YM}$; relation to string amplitudes on $AdS_5 \times S^5$.

Future directions

- Prove the conjectured formula for the integrated correlator.
- Generalisation to the other integrated correlator $\partial_m^4 \log Z$. In the large-N expansion, it is given by generalised non-holomorphic Eisenstein series. [Chester, Green, Pufu, Wang, C.W.]
- Higher-point correlators, they are modular forms with non-zero $SL(2, \mathbb{Z})$ weights. [Green, C.W.]
- Correlators of higher-weight CPO's. They have interesting hidden relations to the lowest-weight correlator. [Caron-Huot, Trinh][Abl, Heslop, Lipstein]...
- Any connections with integrability?

Thank you!