Introduction to the $\mathrm{T\overline{T}}$ deformation

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based on works in collaboration with R. Tateo and S. Negro

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Plan of the talk

- **Introduction:** QFTs and deformations in the Renormalization Group (RG) framework
- **Motivations**
- **Part I:** definition of the $\mathrm{T}\overline{\mathrm{T}}$ deformation
- **Part II:** integrable structure of the $\mathrm{T}\overline{\mathrm{T}}$ deformation
- **Part III:** generalizations and further directions

Introduction

What are QFTs?

- **Path integral formulation**:
  - basic object is the action $\mathcal{A}[\Phi] = \int d\mathbf{x}\, \mathcal{L}\big[\Phi(\mathbf{x}), \partial_\mu \Phi(\mathbf{x}), \dots\big]$ and the partition function $\mathcal{Z} = \int \mathcal{D}\Phi\, e^{-\mathcal{A}[\Phi]}$
  - observables are correlation functions $\langle \mathcal{O}_{a_1}(\mathbf{x}_1) \cdot \dots \cdot \mathcal{O}_{a_n}(\mathbf{x}_n) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi\, \mathcal{O}_{a_1}(\mathbf{x}_1) \cdot \dots \cdot \mathcal{O}_{a_n}(\mathbf{x}_n)\, e^{-\mathcal{A}[\Phi]}$

What are deformations of QFTs?

- **Renormalization Group (RG)**:
  - set a cut-off scale Λ_0 and integrate over the modes with $\Lambda > \Lambda_0 \quad \longrightarrow \quad \frac{\partial \mathcal{A}}{\partial l} = B(\{\mathcal{A}\})$ with $l = \log \Lambda$
  - parametrize $\mathcal{A}$ with $\{\alpha_k\}_{k \geq 0}$ as $\mathcal{A} = \mathcal{A}^\star + \sum_{k \geq 0} \alpha_k \int d\mathbf{x}\, \mathcal{O}_k(\mathbf{x})$

$$\frac{\partial \alpha_k}{\partial l} = B_k(\{\alpha_i\}) \quad \xrightarrow[\text{linearize+diagonalize}]{} \quad \frac{\partial \sigma_k}{\partial l} = (\Delta_k - d)\sigma_k + \mathcal{O}\big(\sigma^2\big)$$

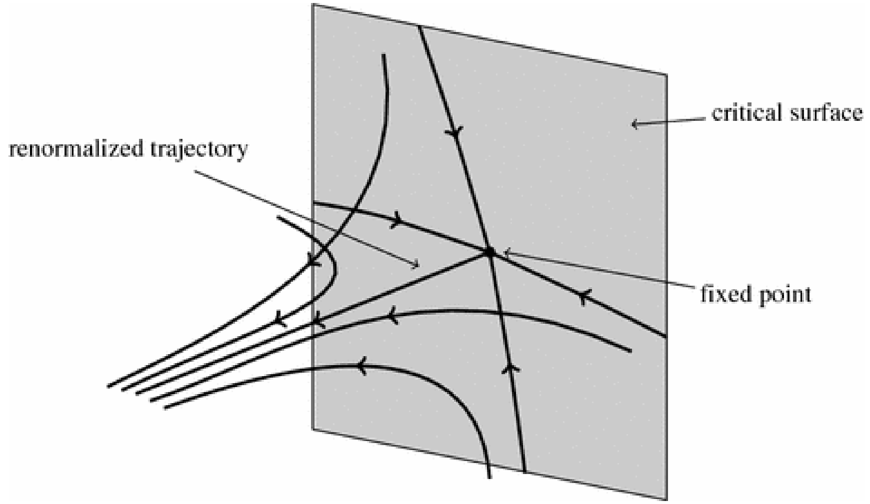
Linearized RG flow: $\sigma_k(\Lambda) = \left(\frac{\Lambda}{\Lambda_0}\right)^{\Delta_k - d} \sigma_k(\Lambda_0)$

As Λ is lowered from Λ_0 we distinguish between:

- $\Delta_k - d > 0$ (**irrelevant operators**): drive the theory back to the fixed point
- $\Delta_k - d < 0$ (**relevant operators**): drive the theory away from the fixed point along an RG *trajectory*

RG flow in $d = 2$

- Fixed points of the RG flow are *Conformal Field Theories* (CFTs);
- Relevant deformations of a $\mathrm{CFT_{UV}}$ flow, along an RG *trajectory*, into a $\mathrm{CFT_{IR}}$ with smaller central charge (*C-theorem*); intermediate points are *non-scale invariant* QFTs.
- Irrelevant deformations of QFTs usually shatter UV-*completness;* lead to *Effective Field Theories* (EFTs) with finite UV cut-off.



Motivations

Why should we study the $\mathrm{T}\overline{\mathrm{T}}$ deformation?

- **Practical reasons**:
  - is under a high degree of control, i.e. exact flow equations for some quantities
  - preserve existing symmetries of the seed theory, e.g. integrable structures

  $\longrightarrow$ **easy to handle**

  - give access to new integrable models $\longrightarrow$ **tool to generate new integrable models**

- **Conceptual reasons**:
  - non-trivial and very unusual UV behaviour which admits analysis $\longrightarrow$ **explore new kinds of QFTs**

    Cardy (2016) Tolley (2019) Dubovsky, Gorbenko, Mirbabayi (2017)
  - relation to random geometry, ghost-free massive gravity and Jackiw-Teitelboim (JT) gravity
  - insight into the nature of Holography $\longrightarrow$ **extend Holography beyond the AdS/CFT paradigm**

McGough, Mezei, Verlinde (2016) & Giveon, Itzhaki, Kutasov (2017)

Part I
definition of the $\mathrm{T}\overline{\mathrm{T}}$ deformation

Setup and convention

- $d=2$ QFT in Euclidean space-time with UV controlled by a CFT: $\mathcal{A}=\mathcal{A}_{\mathrm{CFT}}+\mu\int d\mathbf{x}\,\mathcal{O}_\Delta(\mathbf{x})$ with $\Delta<2$

convention: cartesian coordinates $\mathbf{x}=(x^1,x^2)$ VS complex coordinates $\mathbf{z}=(z,\bar{z})$: $\begin{cases} z=x^1+\mathbb{i}x^2 \\ \bar{z}=x^1-\mathbb{i}x^2 \end{cases}$

- conserved current: local translational and rotational symmetry $\Longrightarrow$ $\exists \boldsymbol{T}^{\mu\nu}(\mathbf{x})$ (stress-energy tensor) s.t.
  - $\boldsymbol{T}^{\mu\nu}=\boldsymbol{T}^{\nu\mu}$;
  - $\frac{\partial}{\partial x^\mu}\boldsymbol{T}^{\mu\nu}(\mathbf{x})=0$ (continuity equation);

convention: cartesian components $\{\boldsymbol{T}^{11},\boldsymbol{T}^{12},\boldsymbol{T}^{22}\}$ VS complex components $\{\mathrm{T},\bar{\mathrm{T}},\Theta\}$

$$\boldsymbol{T}^{11}=-\left(\mathrm{T}+\bar{\mathrm{T}}-2\Theta\right)\ , \qquad \boldsymbol{T}^{12}=\boldsymbol{T}^{21}=\mathbb{i}\mathcal{P}=-\mathbb{i}\left(\mathrm{T}-\bar{\mathrm{T}}\right)\ , \qquad \boldsymbol{T}^{22}=-\mathcal{H}=\mathrm{T}+\bar{\mathrm{T}}+2\Theta$$

- conserved charges: $\frac{\partial}{\partial x^\mu}\boldsymbol{T}^{\mu\nu}(\mathbf{x})=0 \quad\Longrightarrow\quad \exists Q^\mu=\int_C \boldsymbol{T}^{2\mu}(\mathbf{x})\,dx^1$ s.t. $\frac{\partial}{\partial x^2}Q^\mu=0$

  energy and momentum: $E=-Q^2$ and $P=-\mathbb{i}Q^1$

The $\mathrm{T\overline{T}}$ operator

Theorem (A. B. Zamolodchikov, 2004)

For any $d = 2$ QFT with global translational symmetry, there exists *a local operator* $\mathrm{T\overline{T}}(\mathbf{z})$ s.t.

i. $\lim_{\mathbf{z}'\to\mathbf{z}} \mathrm{T}(\mathbf{z})\overline{\mathrm{T}}(\mathbf{z}') - \Theta(\mathbf{z})\Theta(\mathbf{z}') = \frac{1}{4\pi^2}\mathrm{T\overline{T}}(\mathbf{z}) + \text{derivatives}$

ii. $\frac{1}{4\pi^2}\langle n|\mathrm{T\overline{T}}|n\rangle = \langle n|\mathrm{T}|n\rangle\,\langle n|\overline{\mathrm{T}}|n\rangle - \langle n|\Theta|n\rangle^2$ (factorization property)

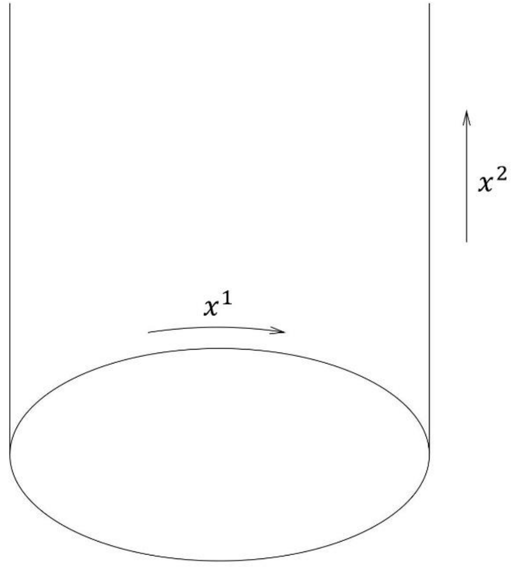
where $\{|n\rangle\}_{n\in\mathbb{N}}$ is a basis of eigenstates of the energy and momentum operators.

Remarks:

- $\mathrm{T\overline{T}}(\mathbf{z})$ is an irrelevant operator corresponding (in the $\mu \to 0$ limit) to $\mathrm{T\overline{T}}(\mathbf{z}) = \lim_{\mathbf{z}'\to\mathbf{z}} \mathrm{T}(\mathbf{z})\overline{\mathrm{T}}(\mathbf{z}')$ in the CFT.
- global translational symmetry $\Longrightarrow$ the QFT lives either on an infinite plane or an infinitely long cylinder.
- in cartesian coordinates $\mathrm{T\overline{T}}(\mathbf{x}) = -\pi^2\det[\boldsymbol{T}^{\mu\nu}(\mathbf{x})]$ and $\langle n|\det[\boldsymbol{T}^{\mu\nu}]|n\rangle = \det[\langle n|\boldsymbol{T}^{\mu\nu}|n\rangle]$

Toward the $\mathrm{T\overline{T}}$ deformation

- $d = 2$ QFT on an infinitely long cylinder with circumference $R \quad \Longrightarrow \quad (x^1, x^2) \sim (x^1 + R, x^2)$
- Hilbert space defined at constant$-x^2$ slices:
  - $E_n(R) = R\langle n|\mathcal{H}|n\rangle = -R\langle n|\boldsymbol{T}^{22}|n\rangle$
  - $\partial_R E_n(R) = -\langle n|\boldsymbol{T}^{11}|n\rangle$
  - $P_n(R) = R\langle n|\mathcal{P}|n\rangle = -\mathbb{i}R\langle n|\boldsymbol{T}^{12}|n\rangle = \frac{2\pi k_n}{R}$ (momentum quantization)



- factorization property: $\langle n|\det[\boldsymbol{T}^{\mu\nu}]|n\rangle = \det[\langle n|\boldsymbol{T}^{\mu\nu}|n\rangle] = \frac{1}{2R}\partial_R\left(E_n^2(R) - P_n^2(R)\right)$
- infinitesimal transformation on the energy levels, keeping the momentum unchanged

$$E_n(R, \delta\tau) = E_n(R) + \delta\tau\, R\langle n|\det[\boldsymbol{T}^{\mu\nu}]|n\rangle \quad \text{and} \quad P_n(R, \delta\tau) = P_n(R)$$

<u>Remark</u>: $E_n(R, \delta\tau)$ is associated to a new eigenstate $|n(\delta\tau)\rangle$

The $\mathrm{T}\overline{\mathrm{T}}$ deformation

- from infinitesimal to finite transformation: $\boxed{\partial_\tau E_n(R,\tau) = R\,\langle n(\tau)|\det[\boldsymbol{T}^{\mu\nu}]|n(\tau)\rangle}$
- assumption: the factorization property holds at finite τ $\;\Longrightarrow\;$ $\langle n(\tau)|\det[\boldsymbol{T}^{\mu\nu}]|n(\tau)\rangle = \det[\langle n(\tau)|\boldsymbol{T}^{\mu\nu}|n(\tau)\rangle]$

$$\boxed{\partial_\tau E_n(R,\tau) = \frac{1}{2}\partial_R\big(E_n^2(R,\tau) - P_n^2(R)\big)} \quad \text{(inviscid inhomogeneous Burgers equation)}$$

Remark: each level is deformed independently from the others $\longrightarrow$ drop the subscript n

- general solution by the method of characteristics:

$$\boxed{E^2(R,\tau) - P^2(R) = E^2(\mathcal{R}_0) - P^2(\mathcal{R}_0)} \quad \text{with} \quad \boxed{\mathcal{R}_0^2 = \big(R + \tau E(R,\tau)\big)^2 - \big(\tau P(R)\big)^2}$$

Example: zero momentum case $(P = 0)$: $E(R,\tau) = E(\mathcal{R}_0)$ with $\mathcal{R}_0 = R + \tau E(R,\tau)$

the deformed theory lives on a cylinder with radius $\mathcal{R}_0$, which depend on the energy and momentum!

Example: the $\mathrm{T}\overline{\mathrm{T}}$ deformation of a CFT

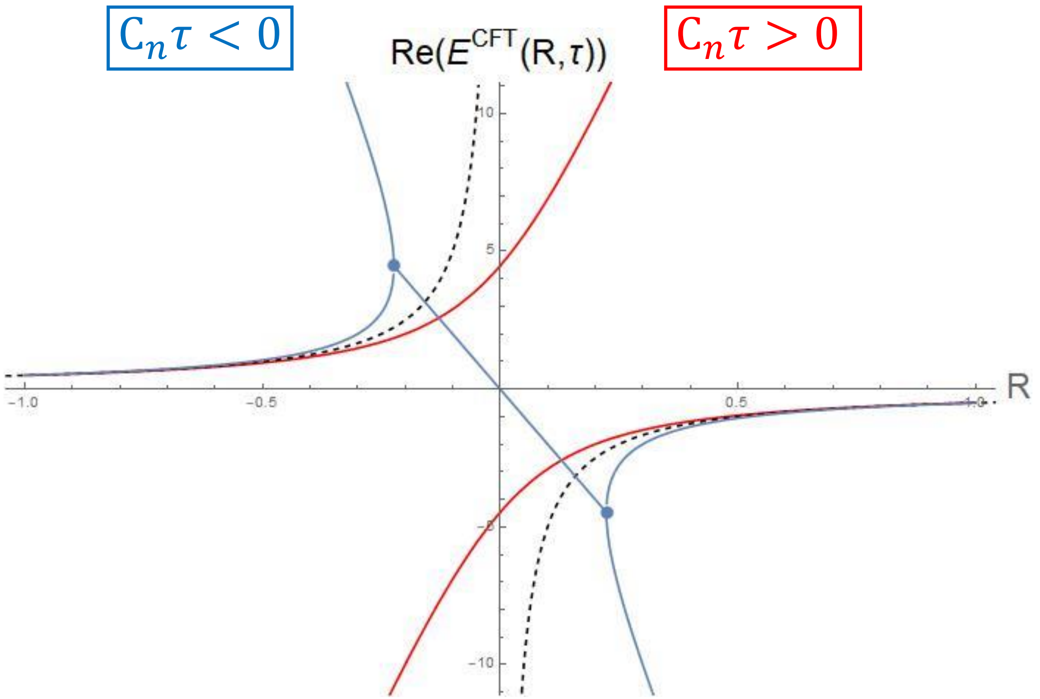
- Energy and momentum of a CFT with central charge c

$$E^{\mathrm{CFT}}(R) = \frac{2\pi}{R}\left(h^+ + h^- - \frac{c}{12}\right)$$

$$P^{\mathrm{CFT}}(R) = \frac{2\pi}{R}(h^+ - h^-) \quad \text{with} \quad h^\pm = h_0^\pm + n^\pm$$

- Energy levels of the deformed theory:

$$E^{\mathrm{CFT}}(R,\tau) = -\frac{R}{2\tau} + \frac{R}{2\tau}\sqrt{1 + \frac{4\tau}{R}E^{\mathrm{CFT}}(R) + \frac{4\tau^2}{R^2}\left(P^{\mathrm{CFT}}(R)\right)^2}$$



Remark: $E^{\mathrm{CFT}}(R,\tau) + \frac{R}{2\tau}$ energy levels of a Nambu-Goto string in critical dimension $(d = 26)$

- zero momentum states $(h^+ = h^- = h)$:

$$E_n^{\mathrm{CFT}}(R,\tau) = -\frac{R}{2\tau} + \frac{R}{2\tau}\sqrt{1 + \frac{8\pi^2}{R^2}\mathrm{C}_n\tau} \quad \text{with} \quad \mathrm{C}_n = 2(h_0 + n) - \frac{c}{12}$$

The classical $\mathrm{T\overline{T}}$ flow equations

How does the deformation affects the classical action?

- The classical Hamiltonian density fulfils

$$\partial_\tau E_n(R,\tau) = R\langle n(\tau)|\det[\boldsymbol{T}^{\mu\nu}]|n(\tau)\rangle \quad \Longrightarrow \quad \boxed{\partial_\tau \mathcal{H}(\mathbf{x},\tau) = -\det[\boldsymbol{T}^{\mu\nu}(\mathbf{x},\tau)]}$$

  where $\mathcal{H}(\mathbf{x},\tau)$ depends on $\mathbf{x}$ through $\{\phi(\mathbf{x}),\phi'(\mathbf{x}),\pi(\mathbf{x})\}$ and $\frac{d}{d\tau}\mathcal{H}(\mathbf{x},\tau) = \partial_\tau\mathcal{H}(\mathbf{x},\tau)$

What about the evolution of the Lagrangian density $\mathcal{L}(\mathbf{x},\tau)$?

- $\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} \;\longrightarrow\; \dot\phi(\mathbf{x},\tau)$ depends on $\mathbf{x}$ through $\{\phi(\mathbf{x}),\phi'(\mathbf{x}),\pi(\mathbf{x})\}$ and explicitly on τ
- $\mathcal{L}(\mathbf{x},\tau)$ depends on $\mathbf{x}$ through $\{\phi(\mathbf{x}),\phi'(\mathbf{x}),\dot\phi(\mathbf{x},\tau)\}$ and $\frac{d}{d\tau}\mathcal{L}(\mathbf{x},\tau) = \partial_\tau\mathcal{L}(\mathbf{x},\tau) + \left(\partial_\tau\dot\phi(\mathbf{x},\tau)\right)\frac{\partial\mathcal{L}}{\partial\dot\phi}$
- $\mathcal{H} = \pi\dot\phi - \mathcal{L} \;\longrightarrow\; \partial_\tau\mathcal{H}(\mathbf{x},\tau) = -\partial_\tau\mathcal{L}(\mathbf{x},\tau) \quad\Longrightarrow\quad \boxed{\partial_\tau\mathcal{L}(\mathbf{x},\tau) = \det[\boldsymbol{T}^{\mu\nu}(\mathbf{x},\tau)]}$

The action fulfils $\boxed{\frac{d}{d\tau}\mathcal{A}(\tau) = \int d\mathbf{x}\,\det[\boldsymbol{T}^{\mu\nu}(\mathbf{x},\tau)]}$ and $\det[\boldsymbol{T}^{\mu\nu}(\mathbf{x},\tau)]$ is the perturbing operator.

Solving the flow equation

How to solve the equation $\partial_\tau \mathcal{L}(\mathbf{x},\tau) = -\det[\boldsymbol{T}^{\mu\nu}(\mathbf{x},\tau)]$ for $\mathcal{L}(\mathbf{x},\tau)$?

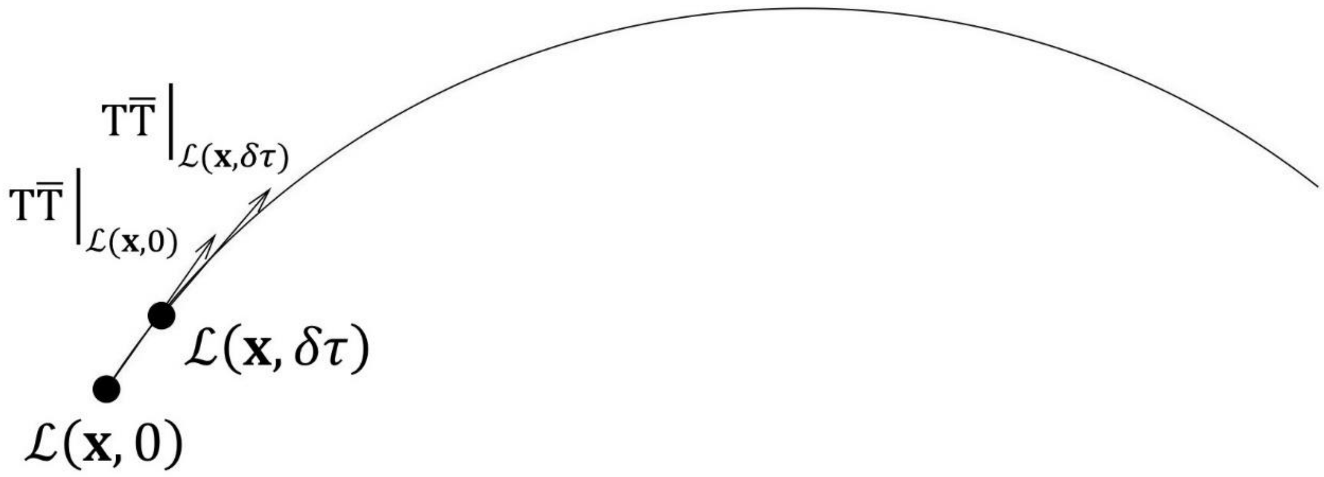
- geometrically $\det[\boldsymbol{T}^{\mu\nu}(\mathbf{x},\tau)]$ is the tangent vector to the curve $\mathcal{L}(\mathbf{x},\tau)$ as τ varies
- $\boldsymbol{T}^{\mu\nu}(\mathbf{x},\tau)$ depends on $\mathbf{x}$ through $\mathcal{L}(\mathbf{x},\tau)$, e.g. $\boldsymbol{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial_\mu \phi}\partial^\nu \phi - \delta^{\mu\nu}\mathcal{L}$ for a scalar theory

<u>Strategy</u>: expand $\mathcal{L}(\mathbf{x},\tau)$ around $\tau = 0$: $\mathcal{L}(\mathbf{x},\tau) = \mathcal{L}^{(0)}(\mathbf{x}) + \sum_{n\geq 1} \mathcal{L}^{(n)}(\mathbf{x})\,\tau^n$

- $\mathcal{L}^{(0)}(\mathbf{x}) = \mathcal{L}(\mathbf{x},0)$ is the original Lagrangian.
- $\mathcal{L}^{(1)}(\mathbf{x}) = -\det[\boldsymbol{T}^{\mu\nu}(\mathbf{x},0)]$

$\vdots$

Reconstruct $\mathcal{L}(\mathbf{x},\tau)$ order by order in τ



Example: $\mathrm{T\overline{T}}$ −deformed Lagrangians

- **Free scalar theory**: $\mathcal{L}(\mathbf{x}) = \frac{1}{4}\delta^{\mu\nu}\partial_\mu\underline{\phi}\cdot\partial_\nu\underline{\phi}$ with $\underline{\phi} = (\phi_1, \dots, \phi_N)$

$$\boxed{\mathcal{L}(\mathbf{x},\tau) = \frac{1}{2\tau}\left(-1 + \sqrt{\det\left[\delta_{\mu\nu} + \tau h_{\mu\nu}\right]}\right)} \quad \text{with} \quad h_{\mu\nu} = \partial_\mu\underline{\phi}\cdot\partial_\nu\underline{\phi}$$

<u>**Remark**</u>: $\mathcal{L}(\mathbf{x},\tau) + \frac{1}{2\tau} = \frac{1}{2\alpha'}\sqrt{\det\left[\eta_{\alpha\beta}\partial_\mu X^\alpha \partial_\nu X^\beta\right]}$ is the Nambu Goto Lagrangian in $d = (N+2)$ in static gauge

$$\begin{cases} X^\mu = x^\mu \quad , & \mu = 1,2 \\ X^{i+2} = \sqrt{\tau}\phi_i \quad , & i = 1, \dots, N \end{cases}$$

- **Interacting scalar theory**: $\mathcal{L}^V(\mathbf{x}) = \frac{1}{4}\delta^{\mu\nu}\partial_\mu\underline{\phi}\cdot\partial_\nu\underline{\phi} + V(\underline{\phi})$

$$\boxed{\mathcal{L}^V(\mathbf{x},\tau) = \frac{V}{1-\tau V} + \frac{1}{2\tilde{\tau}}\left(-1 + \sqrt{\det\left[\delta_{\mu\nu} + \tilde{\tau} h_{\mu\nu}\right]}\right)} \quad \text{with} \quad \tilde{\tau} = \tau(1 - \tau V)$$

Part II
integrable structure of the $\mathrm{T}\overline{\mathrm{T}}$ deformation

Study case: the sine-Gordon model

- sG equation first emerged as compatibility condition between the I and II fundamental forms of pseudo-spherical surfaces embedded in $\mathbb{R}^3$

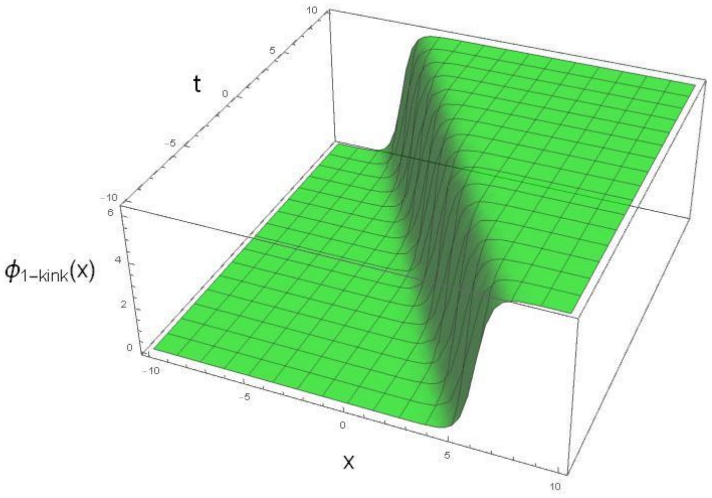
$$\partial_z \partial_{\bar{z}} \phi = \sin(\phi)$$

- auto-Bäcklund transform: $\begin{cases} \partial_z \varphi = \partial_{\bar{z}} \phi + 2a \sin\left(\frac{1}{2}(\varphi + \phi)\right) \\ \partial_{\bar{z}} \varphi = -\partial_z \phi + \frac{2}{a} \sin\left(\frac{1}{2}(\varphi - \phi)\right) \end{cases}$ with φ and ϕ solution to the sG equation

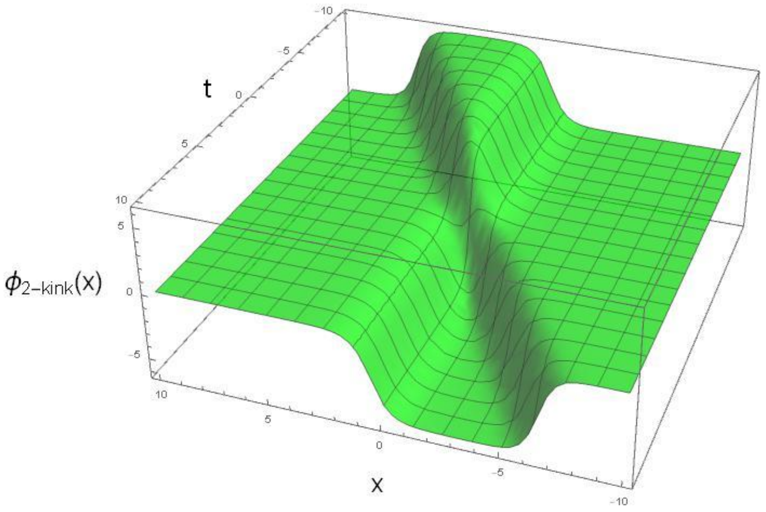
- field theoretical description: $\mathcal{L}_{\mathrm{sG}}(\mathbf{z}) = \partial_z \phi \, \partial_{\bar{z}} \phi + 4 \sin^2\left(\frac{\phi}{2}\right)$

- soliton solutions:

1-kink



2-kink



Classical integrability in sG model

- **Lax pair formulation**

  $\mathfrak{su}(2)$-valued connection $\Omega = L(\mathbf{z},\lambda)dz + \bar{L}(\mathbf{z},\lambda)d\bar{z}$ (λ spectral parameter) such that

$$\mathrm{d}\Omega = \Omega \wedge \Omega \quad \Leftrightarrow \quad \text{sG equation}$$

- Conserved currents and charges are obtained from Ω using standard techniques

  - **conserved currents** $\{\mathrm{T}_{k+1}(\mathbf{z}), \Theta_{k-1}(\mathbf{z}), \bar{\mathrm{T}}_{k+1}(\mathbf{z}), \bar{\Theta}_{k-1}(\mathbf{z})\}_{k\geq 1}$ which fulfil the continuity equations

$$\partial_{\bar{z}}\mathrm{T}_{k+1} = \partial_z \Theta_{k-1} \quad \text{and} \quad \partial_z \bar{\mathrm{T}}_{k+1} = \partial_{\bar{z}} \bar{\Theta}_{k-1}$$

  - **conserved charges** $\{I_k^{\pm}(R)\}_{k\geq 1}$ independent and in involution

$$I_k^+(R) = -\int \left[\mathrm{T}_{k+1}(\mathbf{x}) + \Theta_{k-1}(\mathbf{x})\right] dx^1 \quad \text{and} \quad I_k^-(R) = -\int \left[\bar{\mathrm{T}}_{k+1}(\mathbf{x}) + \bar{\Theta}_{k-1}(\mathbf{x})\right] dx^1$$

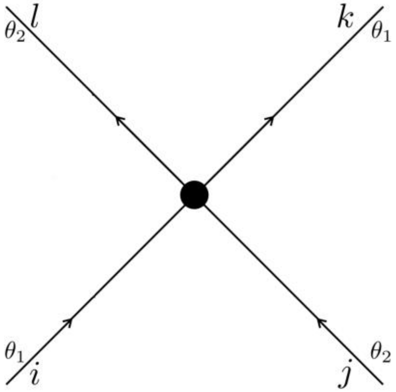
Remark: the $k=1$ current is the stress-energy tensor and $E = I_1^+(R) + I_1^-(R)$, $P = I_1^+(R) - I_1^-(R)$

Quantum integrability in sG model

- **Exact S-matrix**
  - n-particle scattering is factorized into 2-particle scattering:

Yang-Baxter equation

  - scattering is elastic: $S_{ij}^{kl}(\theta_1 - \theta_2) =$



- **Castillejo-Dalitz-Dyson (CDD) ambiguity**: $S_{ij}^{kl}(\theta)$ determined up to an overall factor $\Phi(\theta)$ such that
  1. $\Phi(\theta)\Phi(-\theta) = 1$ (unitarity)
  2. $\Phi(\mathbb{i}\pi + \theta)\Phi(\mathbb{i}\pi - \theta) = 1$ (crossing symmetry)

- generic CDD factor: $\boxed{\Phi(\theta) = \exp\{\mathbb{i} \sum_{s \geq 1} \alpha_s \sinh(s\theta)\}}$

- **Quantum charges**, i.e. eigenvalues of $\{\hat{I}_k^{\pm}(R)\}_{k\geq 1}$, are obtained using the *Thermodynamic Bethe Ansatz* (TBA) or the *non-Linear Integral Equation* (NLIE)

- **Example**: NLIE

  - solve the NLIE for the unknown function $f_\nu(\theta)$ (counting function)

    $$f_\nu(\theta) = \nu(R, \alpha_0|\theta) - \int_{\mathcal{C}_1} d\theta' \, \mathcal{K}(\theta - \theta') \log\left(1 + e^{-f_\nu(\theta')}\right) + \int_{\mathcal{C}_2} d\theta' \mathcal{K}(\theta - \theta') \log\left(1 + e^{f_\nu(\theta')}\right)$$

    with

    - $\mathcal{K}(\theta) = \frac{1}{2\pi i} \partial_\theta \log S(\theta)$ (kernel)
    - $\nu(R, \alpha_0|\theta) = 2\pi i \alpha_0 - imR \sinh\theta$ (driving term)
    - $\mathcal{C}_1 = \mathbb{R} + \mathbb{i}0^+$ and $\mathcal{C}_2 = \mathbb{R} - \mathbb{i}0^+$ for the ground state

  - compute the quantum charges from $f_\nu(\theta)$

    $$I_k^{\pm}(R) \propto \int_{\mathcal{C}_1} \frac{d\theta}{2\pi i} e^{\pm k\theta} \log\left(1 + e^{-f_\nu(\theta')}\right) - \int_{\mathcal{C}_2} \frac{d\theta}{2\pi i} e^{\pm k\theta} \log\left(1 + e^{f_\nu(\theta')}\right)$$

$\mathrm{T\overline{T}}$ deformation as a CDD deformation

Proposition: the $\mathrm{T\overline{T}}$ −deformed sG model is obtained through a CDD deformation of the S-matrix

$$\boxed{S(\theta,\tau) = S(\theta)\exp\{\mathbb{i}\tau m^2 \sinh(\theta)\}}$$

Recall the general CDD factor

$$\Phi(\theta) = \exp\left\{\mathbb{i}\sum_{s\geq 1}\alpha_s \sinh(s\theta)\right\}$$

To prove it, use the NLIE formalism

- The transformation $S(\theta) \to S(\theta,\tau)$ affects only $\mathcal{K}(\theta)$: $\mathcal{K}(\theta-\theta') \to \mathcal{K}(\theta-\theta') + \tau\frac{m^2}{2\pi}\cosh(\theta-\theta')$
- The additional term $\tau\frac{m^2}{2\pi}\cosh(\theta-\theta')$ can be reabsorbed as

$$\nu(R,\alpha_0|\theta) \to \nu(\mathcal{R}_0,\alpha_0|\theta-\theta_0) \text{ with } \mathcal{R}_0\cosh\theta_0 = R + \tau E(R,\tau) \text{ and } \mathcal{R}_0\sinh\theta_0 = \tau P(R)$$

$$\boxed{\mathcal{R}_0^2 = \left(R+\tau E(R,\tau)\right)^2 - \left(\tau P(R)\right)^2}$$

- The deformed energy and momentum fulfil

$$\begin{pmatrix} E(R,\tau) \\ P(R) \end{pmatrix} = \begin{pmatrix} \cosh\theta_0 & \sinh\theta_0 \\ \sinh\theta_0 & \cosh\theta_0 \end{pmatrix}\begin{pmatrix} E(\mathcal{R}_0) \\ P(\mathcal{R}_0) \end{pmatrix} \implies \boxed{E^2(R,\tau) - P^2(R) = E^2(\mathcal{R}_0) - P^2(\mathcal{R}_0)}$$

$\mathrm{T}\overline{\mathrm{T}}$ deformation as a coordinate transformation

- Lagrangian of $\mathrm{T}\overline{\mathrm{T}}$ −deformed sG: $\mathcal{L}_{\mathrm{sG}}(\mathbf{z},\tau) = \frac{V_{\mathrm{sG}}}{1-\tau V_{\mathrm{sG}}} + \frac{1}{2\tilde{\tau}}\left(-1+\sqrt{1+4\tilde{\tau}\partial_z\phi\partial_{\bar{z}}\phi}\right)$
- there exists an $\mathfrak{su}(2)$-valued connection $\Omega = L(\mathbf{z},\lambda,\tau)dz + \bar{L}(\mathbf{z},\lambda,\tau)d\bar{z}$ such that

$$\mathrm{d}\Omega = \Omega \wedge \Omega \quad \Leftrightarrow \quad \mathrm{T}\overline{\mathrm{T}} \text{ −deformed sG EoM}$$

- there exists a coordinate transformation $\mathbf{\Psi}_\tau$: $\mathbb{C} \to \mathbb{C} : \mathbf{z} \to \mathbf{w} = \mathbf{\Psi}_\tau(\mathbf{z})$ such that

$$\Omega = L(\mathbf{z},\lambda,\tau)\, dz + \bar{L}(\mathbf{z},\lambda,\tau)\, d\bar{z} = L(\mathbf{w},\lambda)\, dw + \bar{L}(\mathbf{w},\lambda)\, d\bar{w}$$

- the Jacobian of $\mathbf{\Psi}_\tau$ is

$$\mathcal{J}^{-1}(\mathbf{w}) = \begin{pmatrix} 1+2\tau\,\Theta(\mathbf{w}) & 2\tau\,\mathrm{T}(\mathbf{w}) \\ 2\tau\,\overline{\mathrm{T}}(\mathbf{w}) & 1+2\tau\,\Theta(\mathbf{w}) \end{pmatrix} \quad \text{and} \quad \mathcal{J}(\mathbf{z}) = \begin{pmatrix} 1-2\tau\,\Theta(\mathbf{z},\tau) & -2\tau\,\mathrm{T}(\mathbf{z},\tau) \\ -2\tau\,\overline{\mathrm{T}}(\mathbf{z},\tau) & 1-2\tau\,\Theta(\mathbf{z},\tau) \end{pmatrix}$$

Remarks:

- the explicit expression of $\mathbf{w} = \mathbf{\Psi}_\tau(\mathbf{z})$ depend on the solution
- the hessian of $\mathbf{\Psi}_\tau$ is symmetric on shell: $\partial_{\bar{w}}(\partial_w z) = \partial_w(\partial_{\bar{w}} z) \quad \Leftrightarrow \quad \partial_{\bar{w}}\Theta(\mathbf{w}) = \partial_w\overline{\mathrm{T}}(\mathbf{w})$

The coordinate transformation

- The coordinate transformation is valid for any theory

EoMs of the original theory $\rightleftarrows$ EoMs of the $\mathrm{T\overline{T}}$ −deformed theory

- In cartesian coordinates $\boldsymbol{\Psi}_\tau : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \to \mathbf{y} = \boldsymbol{\Psi}_\tau(\mathbf{x})$

$$\left(\mathcal{J}^{-1}\right)^\mu_\nu = \frac{\partial x^\mu}{\partial y^\nu} = \delta^\mu_\nu - \tau\, g^{\mu\delta}\epsilon_{\delta\rho}\epsilon_{\sigma\nu}\boldsymbol{T}^{\rho\sigma}(\mathbf{y}) \quad \text{and} \quad \mathcal{J}^\mu_\nu = \frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu + \tau\, g^{\mu\delta}\epsilon_{\delta\rho}\epsilon_{\sigma\nu}\boldsymbol{T}^{\rho\sigma}(\mathbf{x},\tau)$$

- It provides an alternative definition of the $\mathrm{T\overline{T}}$ deformation at classical level
  - construct the $\mathrm{T\overline{T}}$ −deformed classical action $\boxed{\int \mathcal{L}(\mathbf{x},\tau)\, d\mathbf{x} = \int \left(\mathcal{L}(\mathbf{y}) - \tau \det[T^{\mu\nu}(\mathbf{y})]\right) d\mathbf{y}}$
  - construct the local currents and charges of a $\mathrm{T\overline{T}}$ −deformed integrable theory

$$\mathfrak{I}_k = \mathrm{T}_{k+1}(\mathbf{w})\, dw + \Theta_{k-1}(\mathbf{w})\, d\bar{w} = \mathrm{T}_{k+1}(\mathbf{z},\tau)\, dz + \Theta_{k-1}(\mathbf{z},\tau)\, d\bar{z} \quad \text{with} \quad \mathrm{d}\mathfrak{I}_k = 0$$

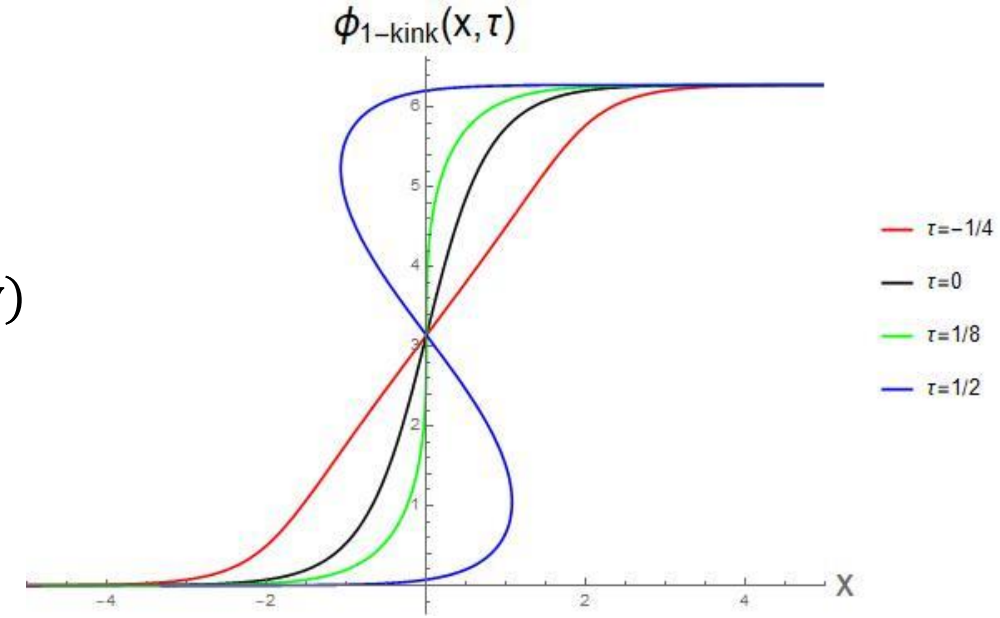
  - construct $\mathrm{T\overline{T}}$ −deformed classical solutions

Deformed classical solutions

- Relation between deformed and original solutions: $\phi_0(\mathbf{x},\tau) = \phi_0(\mathbf{y})$
- **Strategy** :
  - start from $\phi_0(\mathbf{y})$ and integrate

$$\frac{\partial x^\mu}{\partial y^\nu} = \delta^\mu_\nu - \tau\, g^{\mu\delta}\epsilon_{\delta\rho}\epsilon_{\sigma\nu}\boldsymbol{T}^{\rho\sigma}(\mathbf{y})|_{\phi=\phi_0} \;\longrightarrow\; \mathbf{x} = \boldsymbol{\Psi}_\tau^{-1}(\mathbf{y})$$

  - invert the latter relation as $\mathbf{y} = \boldsymbol{\Psi}_\tau(\mathbf{x})$



- **Simple example** : 1-kink solution in sG

$$2(y^1 - v y^2) = \log\tan\frac{\phi(\mathbf{y})}{4} \quad\longrightarrow\quad 2(x^1 - v x^2) = \log\tan\frac{\phi(\mathbf{x},\tau)}{4} + 8\tau\cos\frac{\phi(\mathbf{x},\tau)}{2}$$

- Emergence of shock-wave phenomena in the deformed solutions

Remark: not all the solutions can be obtained analytically

Part III
generalizations and further directions

Other deformations

So far we learned that the $\mathrm{T}\overline{\mathrm{T}}$ deformation is equivalent to

- a coordinate transformation at classical level (true for any theory);
- a CDD deformation at quantum level (true at least for integrable theories).

Idea: use alternatively these tools to generate new deformations

- **Example 1**:
  - coordinate transformations involving local conserved currents of an integrable theory: $\boldsymbol{T}^{\mu\nu} \to \boldsymbol{T}_{\mathbf{s}}^{\mu\nu}(\mathbf{y})$

    where the components of $\boldsymbol{T}_{\mathbf{s}}^{\mu\nu}$ are related to $\left\{\mathrm{T}_{\mathbf{s}+1}(\mathbf{z}), \Theta_{\mathbf{s}-1}(\mathbf{z}), \overline{\mathrm{T}}_{\mathbf{s}+1}(\mathbf{z}), \overline{\Theta}_{\mathbf{s}-1}(\mathbf{z})\right\}$
  - non-relativistic deformation of the S-matrix: $S^{(\mathbf{s})}(\theta, \theta', \tau) = S(\theta - \theta') \exp\{\mathbb{i}\tau m \gamma_{\mathbf{s}} \sinh(\theta - \mathbf{s}\theta')\}$
  - not able to identify the perturbing operator

Remark: $\mathrm{J}\overline{\mathrm{T}}$ and $\mathrm{T}\overline{\mathrm{J}}$ deformations are found as particular cases.

- **Example 2**:
  - CDD deformation of the S-matrix: $S^{(\mathbf{s})}(\theta-\theta',\tau)=S(\theta-\theta')\exp\{\mathbb{i}\tau m\gamma_{\mathbf{s}}\sinh(\mathbf{s}\theta-\mathbf{s}\theta')\}$
  - apparently not related to a coordinates transformation
  - related to the operator $\lim_{\mathbf{z}'\to\mathbf{z}} \mathrm{T}_{\mathbf{s}+1}(\mathbf{z})\overline{\mathrm{T}}_{\mathbf{s}+1}(\mathbf{z}')-\Theta_{\mathbf{s}-1}(\mathbf{z})\Theta_{\mathbf{s}-1}(\mathbf{z}')=\frac{1}{4\pi^2}\mathrm{X}_{\mathbf{s}}(\mathbf{z})+\text{derivatives}$

    and $\mathrm{X}_1(\mathbf{z})\equiv\mathrm{T}\overline{\mathrm{T}}(\mathbf{z})$

Other generalizations:

- supersymmetric extensions of the $\mathrm{T}\overline{\mathrm{T}}$ deformation; (Sfondrini et al., Sethi et al., Freedman et al.)
- combinations of various deformations $(\mathrm{T}\overline{\mathrm{T}}+\mathrm{J}\overline{\mathrm{T}}+\cdots)$; (Frolov (2019))

Questions:

- what about the UV behaviour of these models?
- is there a gravitational interpretation of these deformations?

Further directions

The $\mathrm{T\overline{T}}$ deformation and its generalizations apply to $d = 2$ QFTs. Are there consistent extensions to $d \neq 2$?

- $d > 2$:
  - proposals inspired by Holography (Taylor (2018), Hartman et al. (2018))
  - a curious fact: $d = 4$ Maxwell Born-Infeld Lagrangian as deformation of $d = 4$ Maxwell Lagrangian induced by $\sqrt{\det \boldsymbol{T}}$

$$\partial_\tau \mathcal{L}_{\mathrm{MBI}} = \sqrt{\det \boldsymbol{T}^{\mu\nu}_{\mathrm{MBI}}} \quad \text{with} \quad \mathcal{L}_{\mathrm{MBI}} = \frac{1}{2\tau}\left(-1 + \sqrt{\det\left(\delta^{\mu\nu} + \sqrt{2\tau}\, F^{\mu\nu}\right)}\right)$$

- $d = 1$:
  - "$\mathrm{T\overline{T}}$" deformation of quantum mechanical systems (Gross et al. (2019))
  - deformations of integrable lattice models, e.g. quantum spin chains (Sfondrini et al. (2019), Pozsgay et al. (2019))

<u>**Question**</u>: does it provide a discretized version of the $\mathrm{T\overline{T}}$ deformation?

Thank you for the attention!