Thermal correlators in CFT and black holes

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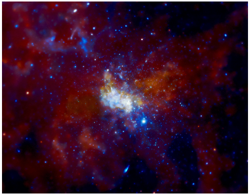
Black hole singularities

A compact and extremely massive object lies at the center of the Milky Way, Sagittarius A *. The object is dark and contains 4 millions solar masses within the size of the Mercury's orbit. According to Penrose's theorems, there must be a singularity in the interior.



- What is inside black holes? What is the nature of singularities?
  Long standing questions since the '60s [Penrose, Hawking, Belinski, Khalatnikov, Lifshitz (random deformations completely change de BH interior, leading to chaotic behavior)]
  Wheeler: "The holy grail of theoretical physics"
- Understanding singularities may shed light on the origin of the universe.
- Obviously it is a problem of quantum gravity in a strongly coupled regime. (Perturbative string theory is not useful)
- Holography: a black hole is described by a CFT at finite temperature. Black hole features are subtly encoded in thermal CFT's correlators.
- Do these correlators only describe physics outside the horizon?
  The issue was investigated since the beginning of AdS/CFT, with many ideas and proposals on how to extract physics behind the horizon from CFT correlators.

Why should we study CFT at finite temperature?

- Many realistic applications in the description of critical points.
- Detailed information of the black hole interior may be encrypted in CFT observables through holography.
- A first challenge is to understand how simple geometric properties are encoded in 1-point, 2-point and higher point correlation functions.
- Recently, Grinberg and Maldacena (2020) studied one-point functions representing decay into gravitons. Remarkably they have information about proper time to the singularity!

For earlier studies, see e.g.

Kraus, Ooguri, Shenker (2002)
Fidkowski, Hubeny, Kleban, Shenker (2003)
Papadodimas and Raju (2013)

Two-point function at $T = 0$

Consider a CFT with a holographic dual. Let ϕ be the bulk scalar field dual to an operator O of dimension Δ. The bulk action is

$$S = \int_{\mathrm{bulk}} dx \left(\partial\bar{\phi}\partial\phi + m^2\bar{\phi}\phi \right)$$

with

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2R^2} \ , \quad \Delta \approx mR \ , \quad mR \gg 1 \ .$$

The propagator can be obtained by solving the KG equation in AdS space.

For operators of large conformal dimension, such propagators can be approximated by $G \sim e^{-m\ell}$, i.e. the exponential of minus the geodesic length between the corresponding points (WKB approximation).

Propagators are thus described by geodesics which start at a boundary point $x_b^{(1)}$, get into the bulk and come back to another boundary point $x_b^{(2)}$.

However, it is possible to regard this single geodesic as the junction of two geodesic arcs, one from $x_b^{(1)}$ to some bulk point x and another from x to $x_b^{(2)}$ upon integration over the junction point x. [see e.g. Minahan, 2012]

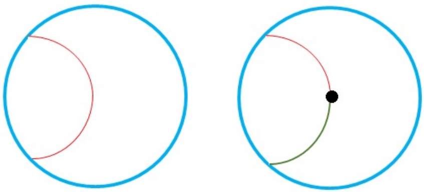
One writes

$$\langle O(x_b^{(1)})\, O(x_b^{(2)})\rangle = \int_{\text{bulk}} dx\, G(x_b^{(1)}, x)\, G(x_b^{(2)}, x)\,,$$

where

$$G(x_b, x) = e^{-iS(x_b,x)}$$

being $S(x_b, x)$ the action of a particle of mass $m \approx \Delta$ which travels between a boundary point x_b and x.



The (euclidean) AdS metric is

$$ds^2 = \frac{R^2}{z^2} \left(d\vec{x}_d^2 + dz^2 \right) .$$

Choosing coordinates where the trajectory is parametrized by z, the action becomes

$$S = -i\,\Delta \int dz\, z^{-1} \sqrt{1 + \dot{\vec{x}}^2} , \qquad \dot{\vec{x}} \equiv \frac{d\vec{x}}{dz}$$

Using conservation of momentum

$$\vec{p}_x = \frac{\Delta \quad \dot{\vec{x}}}{z\sqrt{1 + \dot{\vec{x}}^2}} \quad \longrightarrow \quad \dot{\vec{x}} = \pm \frac{\vec{p}_x z}{\sqrt{\Delta^2 - p_x^2 z}}$$

one finds the solution

$$x = x_1 + \frac{\Delta}{p_x} - \sqrt{\frac{\Delta^2}{p_x^2} - z^2} , \qquad x(z = 0) = x_1 \; .$$

Substituting into the action, we find the on-shell action

$$S_{\rm os} = i\Delta \log\left[\frac{z\,\epsilon}{(\vec{x}-\vec{x}_1)^2+z^2}\right], \qquad G(\vec{x}_1,x) \approx e^{-iS_{\rm os}}$$

Thus

$$\langle O(-\vec{x}_1)\,O(\vec{x}_1)\rangle = \int_{AdS_{d+1}} \frac{dz}{z^d}\, d^d x\; e^{\Delta \log\left[\frac{z\,\epsilon}{(\vec{x}-\vec{x}_1)^2+z^2}\right]+\Delta \log\left[\frac{z\,\epsilon}{(\vec{x}+\vec{x}_1)^2+z^2}\right]}\,.$$

A saddle-point evaluation at fixed x_1 gives the equation $\sqrt{\Delta^2 - p_x^2 z^2} = 0$, which sets $z^* = \Delta/p_x$, leading to

$$\langle O(\vec{x})\,O(\vec{y})\rangle = \frac{1}{|\vec{x}-\vec{y}|^{2\Delta}}\,,$$

just as expected.

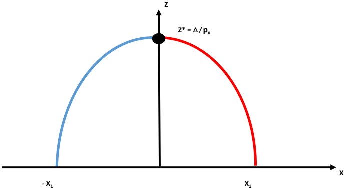


Figure: Note that $z^* = \Delta/p_x$ is precisely the point where $dx/dz = \infty$, i.e. the turning point where $dz/dx = 0$.

FINITE TEMPERATURE

General structure of the OPE

[El-Showk and K. Papadodimas, 2011]; [Iliesiu, Kologlu, Mahajan, Perlmutter, Simmons, (2018)]

On general grounds, the OPE of a scalar operator O of dimension Δ is of the form

$$O(x)O(0) = \sum_{\mathcal{O}} \frac{f_{OO\mathcal{O}}}{c_{\mathcal{O}}} \frac{x_{\nu_1} \cdots x_{\nu_{J_{\mathcal{O}}}}}{|x|^{2\Delta_O - \Delta_{\mathcal{O}} + J_{\mathcal{O}}}} \mathcal{O}^{\nu_1 \cdots \nu_{J_{\mathcal{O}}}}(0) + \text{descendants}\,.$$

Consider the theory on $S^1_\beta \times \mathbb{R}^{d-1}$. Computing the two-point function one obtains

$$\langle O(x)O(0)\rangle = \sum_{\mathcal{O}} \frac{f_{OO\mathcal{O}}}{c_{\mathcal{O}}} \frac{x_{\nu_1} \cdots x_{\nu_{J_{\mathcal{O}}}}}{|x|^{2\Delta_O - \Delta_{\mathcal{O}} + J_{\mathcal{O}}}} \langle \mathcal{O}^{\nu_1 \cdots \nu_{J_{\mathcal{O}}}}(0)\rangle.$$

where we used the fact that descendants do not get a VEV (this is implied by translation invariance).

At finite temperature, the theory on $S^1_\beta \times \mathbb{R}^{d-1}$ has a special direction and a scale β. Because of this, VEV's of operators can be non-zero. By dimensional analysis and symmetry, they are of the form

$$\langle \mathcal{O}_\Delta^{\nu_1\cdots\nu_J}(0)\rangle = \frac{b_\mathcal{O}}{\beta^\Delta}\left(e^{\nu_1}\cdots e^{\nu_J} - \text{traces}\right).$$

where e^μ is the unit vector in the τ- direction.
Next, we substituting this into the 2-point function and do the corresponding tensor contractions

$$\mathcal{C}_J \equiv x_{\nu_1}\cdots x_{\nu_J}\left(e^{\nu_1}\cdots e^{\nu_J} - \text{traces}\right).$$

This leads to

$$\mathcal{C}_J = |x|^J \frac{J!}{2^J (\nu)_J} C_J^{(\frac{d}{2}-1)}(\eta) \qquad \eta \equiv \frac{\tau}{|x|}, \quad |x| = \sqrt{\vec{x}^2+\tau^2},$$

$C_J^{(\frac{d}{2}-1)}(\eta)$ are Gegenbauer polynomials. One finds the general formula

$$\langle O(x)O(0)\rangle = \sum_\mathcal{O} \frac{a_\mathcal{O}}{\beta^{\Delta_\mathcal{O}}} \frac{1}{|x|^{2\Delta_O - \Delta_\mathcal{O}}} C_{J_\mathcal{O}}^{(\frac{d}{2}-1)}(\eta) \qquad a_\mathcal{O} = \frac{f_{OO\mathcal{O}}\, b_\mathcal{O}}{c_\mathcal{O}} \frac{J_\mathcal{O}!}{2^{J_\mathcal{O}}(\nu)_{J_\mathcal{O}}}.$$

[Iliesiu, Kologlu, Mahajan, Perlmutter, Simmons-Duffin, (2018)]

Two-point function at finite T: General properties

In summary, the scale and the special direction provided by the circle now leads to a highly non-trivial structure for the two-point function, with an OPE of the general form

$$\langle O(0)\, O(\tau,\vec{x})\rangle = \sum_{\mathcal{O}\in \mathrm{OPE}[O\times O]} \frac{a_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}}\, C_{J_{\mathcal{O}}}^{(\frac{d}{2}-1)}(\eta)\, |x|^{\Delta_{\mathcal{O}}-2\Delta},$$

where

$$\eta \equiv \frac{\tau}{|x|}\,, \quad |x| = \sqrt{\vec{x}^2+\tau^2}\,.$$

The first few Gegenbauer polynomials are given by

$$d=4: \qquad C_0^{(1)}(\eta)=1\,, \quad C_1^{(1)}(\eta)=2\eta\,, \quad C_2^{(1)}(\eta)=4\eta^2-1\,, \quad \text{etc.}$$

The $J=2$ term multiplying $C_2^{(1)}(\eta)$ is associated with the energy-momentum tensor $\mathcal{T}$. One has

$$a_{\mathcal{T}} = -\frac{2\Delta}{9\pi^2}\frac{\langle T^{00}\rangle_\beta}{c_{\mathcal{T}}\, T^4}$$

where $c_{\mathcal{T}}$ is the normalization of the two-point function of $\mathcal{T}$ (given by the c-anomaly coefficient). Thus from the short-distance expansion we can read the VEV of T^{00}.

Example: Free scalar in $d = 4$

In $d = 4$ the thermal 2-point function in position space is

$$\langle \phi(0)\phi(\tau,\vec{x})\rangle = \frac{\pi}{2\beta}\frac{1}{|\vec{x}|}\left[\coth\left(\frac{\pi}{\beta}(|\vec{x}|-i\tau)\right)+\coth\left(\frac{\pi}{\beta}(|\vec{x}|+i\tau)\right)\right]$$

$$T \to 0: \qquad \langle \phi(0)\phi(\tau,\vec{x})\rangle \to \frac{1}{|x|^{2\Delta}}\ , \qquad \Delta = 1\ .$$

It satisfies the KMS condition of invariance under $\tau \to \tau + \beta$.

For $|x| \ll \beta$ it admits the following low-temperature expansion

$$\langle \phi(0)\phi(\tau,\vec{x})\rangle = \sum_{n=0} 2\zeta(2n)x^{2n-2}\, T^{2n}\, C^{(1)}_{2n-2}(\eta)$$

i.e. the expansion get organized in terms of Gegenbauer polynomials, in agreement with the general form expected from the OPE. The contribution of the stress tensor is represented by the coefficient $2\zeta(4) = \frac{\pi^4}{45}$ multiplying the $C^{(1)}_2(\eta)$. This reproduces the known formula for free fields:

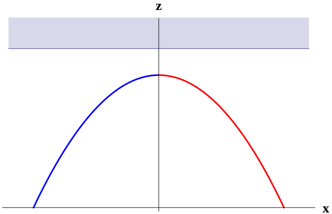
$$\langle T^{00}\rangle_\beta = -\frac{2(d-1)\zeta(d)}{\mathrm{vol}(S^{d-1})}\, T^d\ , \qquad c_T = \frac{d}{d-1}\frac{1}{\left(\mathrm{vol}(S^{d-1})\right)^2}\ .$$

Thermal two-point function from Holography

We now apply the geodesic approximation used earlier. The calculation is similar, but now the geodesics are on the black brane in AdS:

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)\,dt^2 + \frac{dz^2}{f(z)} + d\vec{x}_{d-1}^2\right), \qquad f = 1 - \frac{z^d}{z_0^d}, \qquad z_0 = \frac{d}{4\pi}\beta\,.$$

Consider the same two geodesics as in the $T = 0$ case, meeting at some point in the bulk. We assume that one geodesic arc departs from the $z = 0$ boundary point (t, x) and the other arc departs from $(-t, -x)$.



The action for a single arc is given by

$$S = -\Delta \int dz\, z^{-1} \sqrt{f\,\dot{t}^2 - \frac{1}{f} - \dot{x}^2}\,.$$

The equations are solved by using conservation of momentum $p_t \equiv \mu$ and $p_x \equiv \nu$.

A simple calculation leads to

$$\dot{t} = \mp \frac{\mu\, z}{f\,\sqrt{-f + (\mu^2 - f\,\nu^2)\, z^2}}\,, \qquad \dot{x} = \pm \frac{\nu\, z}{\sqrt{-f + (\mu^2 - f\,\nu^2)\, z^2}}\,.$$

Then, the on-shell action takes the form

$$S = -i\Delta \int dz\, \frac{1}{z\,\sqrt{f\ - \mu^2 z^2 + f \nu^2 z^2}}\,.$$

These equations can be integrated explicitly in terms of elliptic integrals.
We will begin with the case of $\nu = 0$, where the integrals simplify. This implies $\dot{x} = 0$, that is, it corresponds to insertions of operators at equal points on the $\mathbb{R}^3$ but at different times, that is, $\langle O(t_1) O(-t_1) \rangle$.

The geodesic is then obtained by a simple integration leading to elementary functions,

$$\dot{t} = \mp \frac{i}{2}\, \frac{\mu}{(1 - z^4)\, \sqrt{1 - z^4 - \mu^2 z^2}}\,.$$

Parenthesis: Geodesics from WKB

Consider a metric

$$ds^2 = \frac{R^2}{z^2}\Big(- f(z)\, dt^2 + \frac{dz^2}{f(z)} + d\vec{x}^2_{d-1}\Big)$$

To construct the propagator, we solve the equation

$$\left(\frac{1}{\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu - m^2\right)\phi = 0$$

For large m, one can use the WKB approximation $\phi = e^S$ and write $S = mF_0 + F_1 + ...$ To leading order

$$-z^2 f^{-1}(\partial_t F_0)^2 + z^2(\vec{\partial}F_0.\vec{\partial}F_0) + z^2 f(\partial_z F_0)^2 = 1$$

This can be solved by setting

$$F_0 = ip_t t + i\vec{p}.\vec{x} - iH(z) \quad \longrightarrow \quad z^2 f^{-1} p_t^2 - z^2\vec{p}^2 - z^2 f(\partial_z H)^2 = 1$$

which leads to

$$H = \int \frac{dz}{zf}\sqrt{-f + p_t^2 z^2 - p_z^2 f\ z^2}$$

This gives $F_0 = S$, just the action obtained by the geodesic method.

Returning to the calculation...

It is convenient to use as integration variable $r = z^2$. The relevant equations then become

$$S = -i\Delta \int dr \, \frac{1}{2r\sqrt{1-r^2 - \mu^2 r}} \,, \qquad \dot{t} = \mp \frac{i}{2} \frac{\mu}{(1-r^2)\sqrt{1-r^2-\mu^2 r}} \,,$$

Note that close to the boundary $\dot{t} \to \mp i \, \frac{\mu}{2}$. Thus, for real μ, t is imaginary, and the euclidean time is real.

Integrating the action from the boundary at $z = \epsilon$ up to a generic bulk point z, we find

$$S = \frac{1}{2} i\Delta \, \log \left(\frac{\epsilon^2 \left(2 - \mu^2 z^2 + 2\sqrt{1-\mu^2 z^2 - z^4}\right)}{4z^2} \right) .$$

The geodesic with boundary conditions $t(0) = t_1$ is described by

$$\begin{aligned} t - t_1 \quad &= \quad -\frac{1}{4} \log \left(\frac{\mu^2 + \left(\mu^2+2\right) z^2 + 2i\mu\sqrt{-z^4-\mu^2 z^2+1} - 2}{\left(\mu^2+2i\mu-2\right)\left(1-z^2\right)} \right) \\ &- \quad \frac{1}{4} i \log \left(\frac{\mu^2 - \left(\mu^2-2\right) z^2 + 2\mu\sqrt{-z^4-\mu^2 z^2+1} + 2}{\left(\mu^2+2\mu+2\right)\left(z^2+1\right)} \right) . \end{aligned}$$

The bulk joining point is determined by the saddle point equation for z,

$$0 = \frac{dS}{dz} = \frac{\partial S}{\partial z} + \frac{\partial S}{\partial \mu}\frac{d\mu}{dz} \ ,$$

where $\frac{d\mu}{dz}$ is obtained by differentiating $t = t(z, \mu)$ at fixed t. This leads to

$$0 = \sqrt{1 - z^4 - \mu^2 z^2} \ .$$

This is expected, since it ensures a smooth (U-shaped) geodesic. It is the turning point of the geodesic where $dz/dt = 0$.
Substituting into the action, we find the correlation function

$$\langle O(t_1) O(-t_1) \rangle = e^{-2iS}\Big|_{\rm os} = 4^{-\Delta} \left(\mu^4 + 4\right)^{\frac{\Delta}{2}} \ ,$$

where μ is an implicit function of τ_1 defined by

$$\tau_1 \equiv i t_1 = -\frac{1}{4} \log\left(\frac{(\mu-2)\mu+2}{\sqrt{\mu^4+4}}\right) + \frac{i}{4}\log\left(\frac{\sqrt{\mu^4+4}}{\mu(\mu+2i)-2}\right) \ ,$$

We cannot invert this equation to find $\mu = \mu(\tau_1)$. The correlation function is determined in closed form, but implicitly.

To have an explicit formula, we can expand in powers of T at any desired order.

Expanding at small temperature (i.e. short distances), we get ($\tau = 2\tau_1$)

$$\langle O(0)O(\tau)\rangle = \frac{1}{\tau^{2\Delta}}\left(1 + \frac{\Delta\,\pi^4\,T^4\,\tau^4}{40} + \frac{\Delta\,\pi^8\,T^8\tau^8}{28800}(9\,\Delta + 22) + \dots\right)$$

Double scaling limit and exponentiation

A particular large Δ limit exists, with $T\tau \to 0$ and fixed $\Delta(T\tau)^4$, where the correlation function exponentiates:

$$\begin{aligned}\langle O(0)O(\tau)\rangle &= \frac{1}{\tau^{2\Delta}}\left(1 + \frac{\pi^4\,(\Delta\,T^4\,\tau^4)}{40} + \frac{\pi^8\,(\Delta\,T^4\tau^4)^2}{3200} + \frac{\pi^{12}\,(\Delta\,T^4\,\tau^4)^3}{384000} + \cdots\right) \\ &= \frac{1}{\tau^{2\Delta}}\,\exp\left[\frac{\pi^4\,\Delta\,T^4\,\tau^4}{40}\right]\;.\end{aligned}$$

Let us now restore the full spacetime dependence. We find

$$\langle O(0)O(\tau,\vec{x})\rangle = \frac{1}{|x|^{2\Delta}}\,e^{\frac{\Delta\,\pi^4\,T^4}{120}\,C_2^{(1)}(\eta)\,|x|^4}\;,\qquad C_2^{(1)}(\eta) = -1 + 4\eta^2 = \frac{3\tau^2 - \vec{x}^2}{\tau^2 + \vec{x}^2}$$

It exhibits an exponentiation of the energy-momentum tensor block. Expanding the exponential leads to an infinite sequence of predictions for the coefficients of blocks corresponding to contractions of powers of the stress tensor.

In particular, from this formula we can read the VEV of the energy-momentum tensor,

$$\langle T^{00} \rangle_\beta = -\frac{3\pi^6}{80}\, c_{\mathcal{T}}\ T^4$$

We claim that this is a universal relation which should be valid for any four-dimensional CFT with a gravity dual.

We can check it explicitly in the case of $\mathcal{N} = 4$ super Yang-Mills theory.

Field-theory content:

- N^2 gauge fields
- $6N^2$ real scalars
- "$\frac{4}{2}\, N^2$ Dirac fermions"

c-anomaly: $c_{\mathcal{T}} = \frac{1}{4\pi^4}\left(16 n_1 + 8 n_{1/2} + \frac{4}{3} n_0\right) = \frac{10\,N^2}{\pi^4}$. Hence

$$\langle T^{00} \rangle_\beta = -\frac{3\,\pi^2}{8} N^2 T^4$$

Compare with free energy computed in '98 [Gubser, Klebanov, Tseytlin]: $F = E + T\,\frac{dF}{dT}$

If $E = \langle T^{00} \rangle_\beta = -(d-1)\, f\, T^d$, then $F = f\, T^d$. Hence

$$F = \frac{\pi^2}{8} N^2 T^4$$

in precise agreement with the known result.

Thermal two-point function in $d = 2$

By the same method based on studying geodesics in the black brane background, we can find the thermal two-point function in $d = 2$:

$$\langle O(0)O(\tau,x)\rangle = \frac{\pi^{2\Delta}\, T^{2\Delta}}{\Big(\sinh(\pi T(x-i\tau))\,\sinh(\pi T(x+i\tau))\Big)^{\Delta}}\,.$$

In $d = 2$ the geodesic approach gives the *exact* correlation function for any Δ.

Exponentiation:

Now the appropriate limit requires $T|x| \to 0$, $\Delta \to \infty$, with fixed $\Delta T^2\,|x|^2$. We find

$$\langle O(0)O(\tau,x)\rangle = \frac{e^{\frac{\pi^2\,T^2}{3}\,\Delta\,(2\eta^2-1)\,|x|^2}}{|x|^{2\Delta}}\,.$$

Just as we did in the $d = 4$ case, we can derive a formula for the thermal expectation value of the energy-momentum tensor. From the leading correction we find

$$\langle T^{00}\rangle_\beta = -\,\frac{\pi^2}{6\,\beta^2}\,(2\pi c_T)\,.$$

Check: free scalar field.

$$c_T = \frac{1}{2\pi^2} \quad\longrightarrow\quad \langle T^{00}\rangle_\beta = -\frac{\pi}{6}\,T^2$$

just as expected.

Thermal two-point function in d dimensions

Using the geodesic method in the black brane, in d dimensions we obtain the general formula

$$\langle O(\tau)\, O(0)\rangle = \frac{1}{\tau^{2\Delta}}\ \exp\left[\alpha_d\, \Delta\, T^d\, \tau^d\right]\,, \qquad \alpha_d \equiv \frac{\pi^{d+1}\, \Gamma(d)}{d^d\, \Gamma(\frac{d+3}{2})\, \Gamma(\frac{d-1}{2})}$$

For $d = 2, 4$ this formula recovers the previous values.
It then follows that

$$\frac{\langle \mathcal{T}^{00}\rangle_\beta}{c_{\mathcal{T}}} = -\frac{2^{2d-1}\,(d-1)^2\,\pi^{\frac{3d}{2}}\,\Gamma(\frac{d}{2})}{d^d\,\Gamma(d+2)}\,T^d\,.$$

Using that $F = E + T\,\frac{dF}{dT}$, we find

$$F = \frac{2^{2d-1}\,(d-1)\,\pi^{\frac{3d}{2}}\,\Gamma(\frac{d}{2})}{d^d\,\Gamma(d+2)}\,c_{\mathcal{T}}\,T^d\,.$$

Combining this formula with the free energy computed in [Gubser, Klebanov, Tseytlin], one can compute the central charge for $ABJM$ and the 6d $(2,0)$ theory for $d = 3, 6$

$$c_{\mathcal{T}}^{ABJM} = \frac{2\sqrt{2}}{\pi^3}\,N^{\frac{3}{2}}\,, \qquad c_{\mathcal{T}}^{(2,0)} = 4N^3\,\frac{84}{\pi^6}\,.$$

in agreement with the direct calculation. [Bastianelli, Frolov, Tseytlin]; [Chester et al]

Probing the black hole interior

In a recent paper Grinberg and Maldacena, argued that one can measure the proper time to the black hole singularity by examining the asymptotic behavior of VEV of operators with large scaling dimensions.

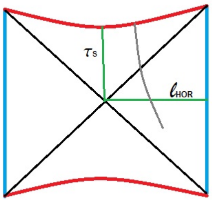
They found

$$\langle O \rangle \propto e^{-\Delta \ell} , \qquad \ell = \frac{R}{d}(\pm i\pi + \log 4) .$$

The parameter ℓ represents the renormalized proper lenght from the boundary to the singularity:

$$\ell = \ell_{\rm hor} \pm i\tau_s = R \lim_{\epsilon \to 0} \left(\int_\epsilon^\infty \frac{dz}{z\sqrt{1 - z^d}} + \log \epsilon \right) .$$

Thus the dependence of $\langle O \rangle$ on the conformal dimension $\Delta \approx m$ provides a measurement of ℓ.



The bulk scalar field which is dual to the scalar operator O has a Lagrangian

$$I = \frac{1}{16\pi G_N} \int d^5x\sqrt{g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi\phi + \alpha\phi W^2\right] \ , \qquad \Delta \approx mR \gg 1 \ .$$

The VEV is induced by a gravitational coupling ϕW^2, where $W^2 = W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$ is the square of the Weyl tensor.
This represents the first possible coupling in a derivative expansion.

In AdS, $W^2 = 0$ and the one-point function vanishes. However, on the black hole background, one has a one-point function

$$\langle O(0)\rangle = \alpha \int d^5x\sqrt{g}\ G(0;x)\, W^2 \ , \qquad W^2 = \mathrm{const.}\, z^8$$

In the geodesic approximation, $G \sim e^{-m\ell}$. For $mR \sim \Delta \gg 1$, the integral is dominated by a saddle point. It must be located at a large $|z|$ so that the log W term in the action balances the term $m\ell$.

This means that the saddle-point sits near the singularity at $z = \infty$.

As a result, ℓ becomes the proper distance from the boundary to the singularity.

It is natural to expect that higher-point thermal correlation functions may codify more information on the black hole geometry. Let us now investigate this.

The model

We will consider scalar operators of large dimensions carrying a $U(1)$ global charge. The relevant part of the action for the dual bulk scalar field is

$$I = \frac{1}{16\pi G_N}\int d^5x\sqrt{g}\left[g^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\phi + m^2\bar{\phi}\phi + \alpha\bar{\phi}\phi W^2\right] \ , \qquad \Delta \approx mR \gg 1 \ .$$

The coupling $\bar{\phi}\phi W^2$ leads to a new interaction vertex, which implies the following correction to the two-point function $\langle O(0)\bar{O}(x)\rangle$:

$$I = \alpha\int_{\mathrm{bulk}} G(x_b^{(1)}, x)\, G(x_b^{(2)}, x)\, W^2(x) \ .$$

In terms of $r = z^2$, in the black brane background, $W^2 = \frac{72}{R^4}\frac{r^4}{r_0^4}$.
In the geodesic approximation, the integral to compute is

$$I = c\int \frac{dr}{r^3}\ G(x_1 - x'(z))G(x_2 - x'(z))\ r^4 \ , \qquad c = 72R\frac{\alpha}{r_0^4} \ .$$

Due to the presence of the extra W^2 factor, the two geodesic arcs will now meet at a different $r_\star$, determined by saddle point.

The integrand can be written as e^{-iS_T}, where

$$S_T = S(z, \mu_1) + S(z, \mu_2) + i \log r \ .$$

S_T depends on the joining point r both explicitly and implicitly through $\mu_{1,2}$.
The calculation now leads to saddles at

$$\begin{aligned} r_\pm &= \pm i\Delta + \frac{\mu^2}{2} + \mathcal{O}(\frac{1}{\Delta}) \ , \\ r_3 &= \frac{1}{2}\left(\sqrt{\mu^4 + 4} - \mu^2\right) + \mathcal{O}(\frac{1}{\Delta^2}) \ . \end{aligned}$$

Having two contributions, $e^{-iS_T(r_3)}$ and $e^{-iS_T(r_+)} + e^{-iS_T(r_-)}$, the dominant contribution for a given τ_1 is the one with largest modulus.

- At small τ, the dominant contribution is given by the saddle point sitting at r_3, since it will have large μ and therefore larger weight factor $|e^{-iS_T}|$. This gives the expected $1/|\tau|^{2\Delta}$ behavior for nearly coincident points.
- However, there is a critical value of τ_1 (namely $\tau_c \sim 1.16$) beyond which the dominant contribution is given by the saddle points at $r_\pm$, where the correction to the 2-point function is given by

$$e^{-iS_T(r_+)} + e^{-iS_T(r_-)} \ , \qquad \text{with} \quad e^{-iS_T(r_\pm)} = \left(-\frac{\mu^2}{4} \mp \frac{i}{2}\right)^\Delta$$

Substituting $r_\pm$ in the geodesic $t = t(\mu, z)$, $z^2 = r_\pm$, and expanding for large Δ, one finds

$$t_1 = -i\tau_1 = \left(\mp \frac{1}{4} + \frac{i}{4}\right) \log\left(\frac{-\mu + (1 \mp i)}{\mu + (1 \mp i)}\right) + \mathcal{O}\left(\frac{1}{\Delta^2}\right) \ . \tag{1}$$

This equation can be inverted to give

$$\mu = -(1 \pm i) \tan\left((1 \pm i)\, \tau_1\right) . \tag{2}$$

This determines μ in terms of τ_1 for the saddle-point contribution at $r_\pm$.

Thus we obtain

$$e^{-iS_T(r_\pm)} = \mathrm{const.}\, 2^{-\Delta}\, e^{\mp \frac{i\pi\Delta}{2}} \frac{1}{\left(\cos\left((1 \pm i)\tau_1\right)\right)^{2\Delta}} \ .$$

The factor $e^{-2\Delta \ell}$, with $\ell = \pm \frac{i\pi\Delta}{2} + \frac{1}{2}\log 2$, is nothing but renormalized length to the black hole singularity discussed earlier. This is the squared of the similar factor appearing in the one-point function in Grinberg-Maldacena, since this factor now accounts for two geodesics.

Quasinormal frequencies

The W^2 correction to the correlation function has a non-trivial time dependence. Indeed, one can get a deeper insight by expanding

$$e^{-iS_T(r_\pm)} \sim 2^\Delta\, e^{\mp i\frac{\Delta\pi}{2}}\, e^{-(1\mp i)\,\Delta\tau_1} \sum_{n=0} \frac{(-1)^n}{n!} \frac{(2\Delta+n-1)!}{(2\Delta-1)!}\, e^{-i\omega_n\,\tau_1}\,,$$

with

$$\omega_n = (\Delta + 2n)\,(\mp 1 - i)\,.$$

The complex frequencies ω_n coincide with the quasinormal frequencies of a scalar field in the black brane background.
[Nunez-Starinet, 2003; Fidkowski et al 2003nf].

In general, quasinormal modes of a black hole describe the exponential decrease of perturbations. The appearance of quasinormal frequencies is due to multi-graviton emission due to the coupling $\bar{\phi}\phi W^2$.

The quasinormal frequencies can also be found directly from the location of the poles in the thermal correlator.

Thermal three-point functions

[D. Rodriguez-Gomez, J.R., work in progress]

Similarly, one can compute thermal higher-point functions. The calculation is simpler in a symmetric case:

$$\langle O(\tau_1)\, O'(0)\, O(-\tau_1)\rangle \approx \frac{e^{\frac{\left(32\Delta^2-18\Delta\Delta'+3\Delta'^2\right)}{640(2\Delta+\Delta')}\pi^4 T^4 \tau^4}}{\tau^{2\Delta+\Delta'}}$$

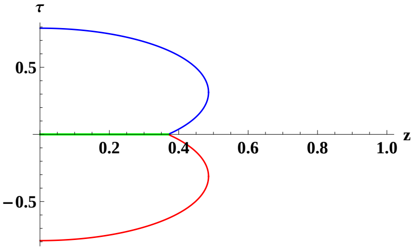


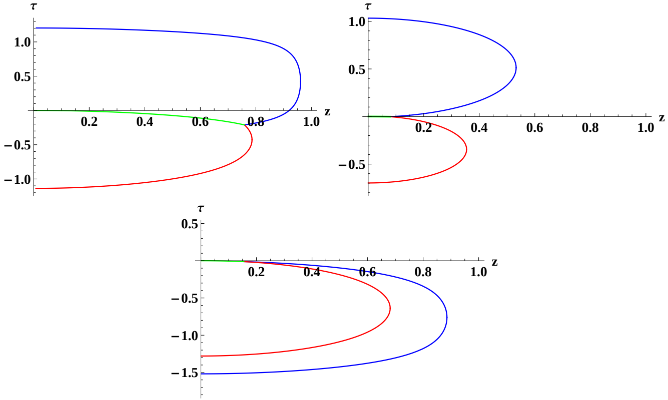
Figure: Symmetric three-point function with two operators of dimension $\Delta = 3$ and one operator in the middle of dimension $\Delta' = 4$.

Non-symmetric 3-point functions and factorization limit

Although analytic formulas for general 3-point functions are very long and complicated, there are some remarkable properties that can be demonstrated explicitly.

When $\Delta_3 \to \Delta_1 + \Delta_2$, the length of the ingoing geodesic shrinks to zero and the 3-point correlation function exhibits the remarkable **factorization property**:

$$\langle O_1(-\tau_1)\, O_3(0)\, O_2(\tau_2)\rangle \to \langle O_1(-\tau_1)\, O_1(0)\rangle\, \langle O_2(0)\, O_2(\tau_2)\rangle$$



Concluding remarks

- To summarize, we have computed the thermal 2-point function for CFT's for operators of large dimension using holography.
  In the OPE regime the expected structure is recovered with the correct VEV of the energy-momentum tensor, and it gives predictions for higher powers.
- Interestingly, there is a double scaling limit where the 2-point exponentiates the energy-momentum tensor block.
- We studied the effect of the first higher-derivative correction $\bar{\phi}\phi W^2$ to the 2-point function. At large charges, the correlator probes the interior of the black hole.
- The correlation function also exhibits the quasinormal frequencies.
- General properties of higher-point functions can be demonstrated by the geodesic approach, including factorization and transitions occurring at critical separations of operators.