String amplitudes in AdS: lessons from the Regge limit

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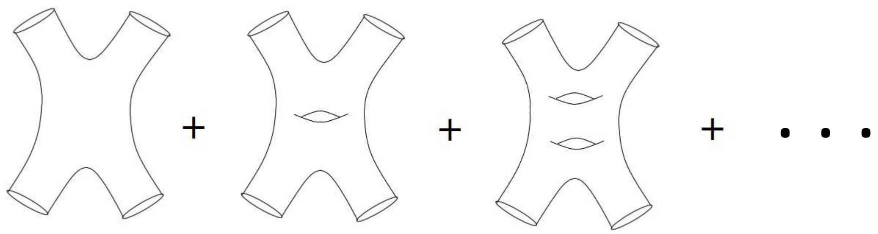
Outline

- Introduction
- Special functions in String Theory
- The *AdS* Virasoro-Shapiro amplitude
- The Regge limit
- Summary & Conclusions

1 Introduction

Why string amplitudes in AdS?

- Scattering amplitudes
  - probability for a certain process to happen
  - predictive tool
  - uncover structure and reveal symmetries
  - Particle Physics, Mathematics, String Theory,...



- String scattering in flat space
  → perturbative String Theory
  → worldsheet methods

$$A^{(n)}(\Lambda_i, p_i) = \sum_{\text{topologies}} g_s^{-\chi} \frac{1}{\text{Vol}} \int DX Dg \; e^{-S_{\text{Poly}}} \prod_{i=1}^{n} V_{\Lambda_i}(p_i)$$

- When considering scattering processes, we sum over all possible configurations of worldsheets.

- In some regimes, the sum is dominated by a **saddle point**.

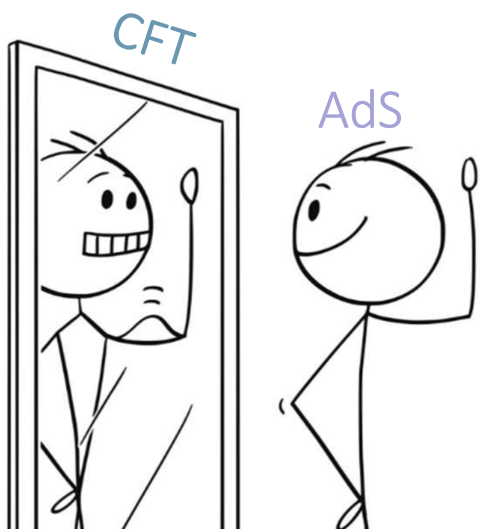
Parameters:
g_s (string coupling constant)
α' (size of the string)

Why string amplitudes in AdS?

- An outstanding question: what about curved spacetimes?
  → difficulties with standard formulations
  → **perturbative genus expansion** but no direct worldsheet approach, even at tree-level

We need additional tools!

- In our favor, AdS/CFT: string scattering amplitudes on AdS from CFT correlators of the dual boundary theory.
- *<u>Strategy</u>: combine worldsheet intuition with alternative powerful tools.*



Why high-energy limits?

Low Energy ($\alpha' \to 0$, field theory limit)

High Energy ($\alpha' \to \infty$)

HIGH ENERGY LIMIT

- Large *S*, *T*, with *S*/*T* fixed.
- QFT: the short-distance behavior of the theory plays a crucial role (OPE, RG flow…) What about String Theory?
- Universality.
- Direct contact with the string worldsheet, at least classically!

REGGE LIMIT

- Large *T*, finite *S*.
- Richer limit.
- Full information on intermediate operators.
- No need to know the full amplitude.

Main object of study

- Scattering of four graviton states at tree level.

- Flat space : Virasoro Shapiro amplitude

Prefactor: polarisation vectors

$$A_4(\varepsilon_i, p_i) = K(\varepsilon_i, p_i) \int d^2z |z|^{-2S-2} |1-z|^{-2T-2}$$

The integrand is a single-valued function of z.

- $\boldsymbol{AdS_5 \times S^5}$: correlator of 4 stress-tensor multiplets, to leading non-trivial order in a *1/c* expansion. Paradigmatic example. Stringy corrections challenging to compute!

- «Right language»: Mellin space. [Penedones]

- Then, Borel transform: AdS analog of the Virasoro Shapiro amplitude.

The AdS Virasoro-Shapiro program

Analyze and include curvature corrections systematically and efficiently.

Exploit and emphasize the interplay between String Theory and Number Theory.

2

Special functions in String Theory

Tree-level perturbative string amplitudes

- We consider the family of integrals:

$$M_{N+3}(\mathbf{s},\mathbf{n},\tilde{\mathbf{n}}) = \left(\frac{i}{2\pi}\right)^N \int_{\mathbb{C}^N} \prod_{0<i<j<N+1} |z_i - z_j|^{2s_{ij}} (z_i - z_j)^{n_{ij}} (\bar{z}_i - \bar{z}_j)^{\tilde{n}_{ij}} \prod_{i=1}^{N} dz_i d\bar{z}_i$$

s : collection of Mandelstam kinematic invariants: $s_{ij} = \alpha' p_i \cdot p_j$

$$z_0 = 0\,, \qquad z_{N+1} = 1, \qquad N \in \mathbb{N}, \qquad n_{ij}, \tilde{n}_{ij} \in \mathbb{Z}$$

$N = 1,\ n_{12} = \tilde{n}_{12} = -1$: Virasoro-Shapiro amplitude

- The global (any s) and local properties are related to the theory of single-valued periods, such as **zeta values** and **polylogs**. [Vanhove, Zerbini]

- The Low Energy expansion $(\alpha' \to 0)$ contains only SVMZVs.

Polylogarithms

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad \rightarrow \quad \int_0^z dz' \, \frac{\mathrm{Li}_{n-1}(z')}{z'}$$

- Classical polylogs:
  - convergent series on the unit disk $|z| < 1$
  - can be continued to the cut plane $\mathbb{C} \setminus [1, \infty)$ by an iterated integral representation
  - reduce to zeta-values at *z=1* $\mathrm{Li}_n(1) = \zeta_n \, , n > 1$
  - reduce to the standard logarithm at *n=1*
- Multiple polylogs (MPLs):
  - *w* = word with letters in the alphabet {0,1}
  - function of a single variable labelled by the word *w*, whose length we call weight:

$$\frac{d}{dz} L_{0w}(z) = \frac{1}{z} L_w(z) \, , \qquad \frac{d}{dz} L_{1w}(z) = \frac{1}{z-1} L_w(z)$$

$$L_{0^p}(z) = \frac{\log^p z}{p!} \qquad L_{0^{n-1}1} = -\mathrm{Li}_n(z)$$

multiple zeta values (MZVs)
= MPLs at unity

$\log|z|^2 = \log z + \log \bar{z}$

Single-valued polylogs

- For the classical polylog, the discontinuity across the cut is

$$\mathrm{Disc}(\mathrm{Li}_n(z)) = 2\pi i \frac{\log^{n-1} z}{(n-1)!}$$

Polylogs = special analytic functions of a single complex variable, with branch points (multi-valued functions on the complex plane).

- Can consider combinations of polylogs such that all the branch cuts cancel, and define single-valued functions in the $(z, \bar{z})$ plane!
- Can do the same for multiple polylogs: SVMPLs.
- $L_w(1) = \zeta(w)$

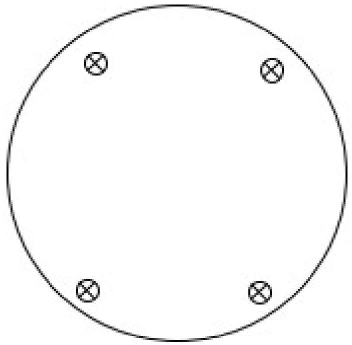
Single-valued polylogs

- At any given weight, there is a finite-dimensional vector space of available functions!
- SVMZVs = single-valued projection of MZVs = SVMPLs at unity
- AdS amplitude:
  curvature expansion around flat space
  building blocks = SVMPLs

The Virasoro-Shapiro amplitude

NOTE: *reduced* amplitude with no overall factor (graviton polarizations)

$$A^{(0)}(S,T) = -\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$



- Crossing symmetry in the 3 Mandelstam variables $S+T+U=0$
- Fix *T* and vary *S*.
  Poles at mass of the tachyon + higher states of the closed string

⇒ STRING AMPLITUDE AS AN INFINITE NUMBER OF (s-channel) TREE-LEVEL QFT DIAGs

- Regge behaviour
- Low/High Energy

$$S = -\frac{\alpha'}{4}(p_1+p_2)^2\,, \qquad T = -\frac{\alpha'}{4}(p_1+p_3)^2\,, \qquad U = -\frac{\alpha'}{4}(p_1+p_4)^2$$

Virasoro-Shapiro amplitude and single-valued periods

- Low Energy expansion of VS

$$A^{(0)}(S,T) = \frac{1}{STU} + 2\sum_{a,b=0}^{\infty} \sigma_2^a \sigma_3^b \alpha_{a,b}^{(0)} \qquad \sigma_2 = \frac{1}{2}(S^2+T^2+U^2)\ , \sigma_3 = STU$$

SUGRA + TOWER OF STRINGY CORRECTIONS

$$\alpha_{a,0} = \zeta(3+2a)$$

- Only odd ζ values appear!
  The Wilson coefficients live in the ring of SVMZVs.

$$A^{(0)}(S,T) = \frac{\exp\left(\sum_{n=1}^{\infty} \frac{\zeta^{\text{sv}}(2n+1)(S^{2n+1}+T^{2n+1}+U^{2n+1})}{2n+1}\right)}{STU}$$

- This reflects the single-valued nature of the integral representation.

$$A^{(0)}(S,T) = \frac{1}{U^2}\int d^2z |z|^{-2S-2}|1-z|^{-2T-2}$$

Key Takeaways

In flat space:

- we can use the **worldsheet theory** to compute string amplitudes
- the Low Energy expansion of closed string amplitudes contains only single-valued MZVs.

Let's use what we learnt to compute **strings amplitudes on curved backgrounds**, where we lack a worldsheet technology.

3

The AdS Virasoro-Shapiro amplitude

The AdS Virasoro-Shapiro amplitude

Scattering of massless strings: 4 **gravitons** on $\boldsymbol{AdS_5 \times S_5}$ in Type IIB superstring theory

$\frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}}$

AdS/CFT

$g_s \sim \frac{1}{N}$

Correlator of four **stress-tensor** multiplets in **Mellin** space, at large central charge

Borel

$$M(s,t) = \frac{1}{2R^6} \int_0^\infty d\beta e^{-\beta} \beta^5 A\left(\frac{s\beta}{2R^2}, \frac{t\beta}{2R^2}\right)$$

$$A(S,T) = A^{(0)}(S,T) + \frac{\alpha'}{R^2} A^{(1)}(S,T) + \dots$$

VS in flat space

Curvature corrections

The CFT point of view

- Four-point correlator in N=4 SYM at leading non-trivial order in the large central charge expansion:

$$\langle \mathcal{O}^{I_1 J_1}(x_1)\mathcal{O}^{I_2 J_2}(x_2)\mathcal{O}^{I_3 J_3}(x_3)\mathcal{O}^{I_4 J_4}(x_4)\rangle\Big|_{\frac{1}{c}} = G^{I_i J_i}(x_i)_{tree}$$

- Superconformal primary operator of the stress tensor multiplet:
  - -scalar operator
  - -protected dimension $\Delta = 2$
  - -symmetric traceless representation of rank 2 of the SO(6) R-symmetry group

- Superconformal Ward identities: extract a R-symmetry prefactor

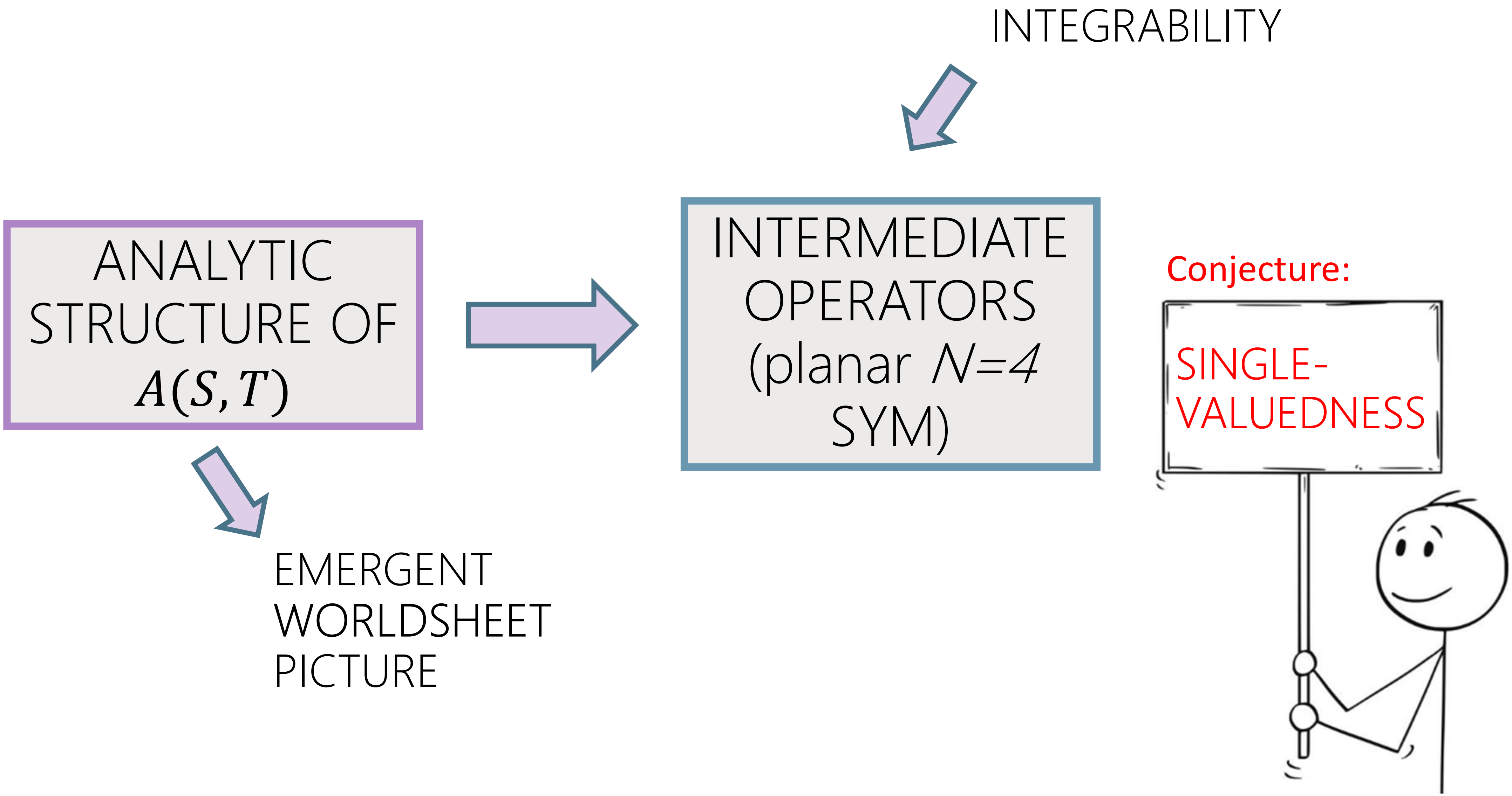
$$G^{I_i J_i}(x_i)_{tree} = factor(I_i, J_i, x_i) \times \mathcal{T}(U,V)$$

REDUCED CORRELATOR → AdS VS

Why Borel space?

$$M(s,t) = \frac{1}{2R^6}\int_0^\infty d\beta e^{-\beta}\beta^5 A\left(\frac{s\beta}{2R^2}, \frac{t\beta}{2R^2}\right)$$

- Technical reason: cannot resum the Low Energy expansion in Mellin space.
- To make contact with the string worldsheet we need to understand how to resum the Low Energy expansion.
- Contact with flat space: at leading order in 1/R it implements the flat space limit [Penedones] with S and T interpreted as Mandelstam variables, but we keep the corrections.
- Our working definition of AdS corrections!


INTEGRABILITY
ANALYTIC STRUCTURE OF $A(S,T)$
INTERMEDIATE OPERATORS (planar $\mathcal{N}=4$ SYM)
Conjecture:
SINGLE-VALUEDNESS
EMERGENT WORLDSHEET PICTURE

The AdS Virasoro-Shapiro amplitude

Three key points:

AdS CORRECTIONS: HIGHER ORDER POLES (jump by 3)

- Structure of poles

$$A^{(k)}(S,T) = \frac{R^{(k)}_{3k+1}(T,\delta)}{(S-\delta)^{3k+1}} + \frac{R^{(k)}_{3k}(T,\delta)}{(S-\delta)^{3k}} + \ldots + \frac{R^{(k)}_{1}(T,\delta)}{S-\delta} + \text{regular}\,, \quad \delta = 1,2,\ldots$$

- **Low Energy expansion**: assume the unknown coefficients to be **single-valued** zetas as in flat space

- Intuition from the **worldsheet**:

$$\int d^2z |z|^{-2S-2}|1-z|^{-2T-2}G(z,\bar{z})$$

The AdS Virasoro-Shapiro amplitude

- *What is the relevant space of functions?* Linear combination of single-valued functions such that the Low Energy expansion contains only SVMZVs.

- The *k*-th order answer takes the form of a genus 0 worldsheet integral involving weight *3k* SVMPLs, and rational in *S,T*. [Alday, Hansen]

$$A(S,T) = \int d^2z|z|^{-2S}|1-z|^{-2T}W_0(z,\bar{z})\left(1+\frac{S^2}{R^2}W_3(z,\bar{z})+\frac{S^4}{R^4}W_6(z,\bar{z})+\dots\right)$$

$$W_0(z,\bar{z}) = \frac{1}{2\pi U^2|z|^2|1-z|^2}$$

- Precise proposal for the structure of the tree-level amplitude on $AdS_5 \times S^5$!

NOTE: This is NOT the result of a direct worldsheet computation!

Key Takeaways

- Single-valuedness plays a fundamental role in the construction of AdS scattering amplitudes, as in flat space!
- Can extract the CFT-data and **compare with integrability and localization** results for planar $N = 4$ SYM at strong coupling!

Big goal

Determine *A(S,T)* to all orders in *1/R*.

- In the meanwhile… focus on more accessible limits to:
  - make connections to more direct worldsheet computations
  - explore the structure of the proposal by probing the theory in different regimes
- High Energy ✓

Next step towards the worldsheet theory: investigate the Regge limit!

High Energy limit

The amplitude can be computed to **all orders** in S^2/R^2 and **curvature corrections exponentiate**!

Spacetime point of view:

Flat space: path integral dominated by a (single-valued) classical solution [Gross, Mende]

$$X_0^\mu(\zeta) = -i \sum_k p_{k,0}^\mu \log \left| 1 - \frac{\zeta}{z_k} \right|$$

AdS: bosonic model for scattering of classical strings

$$X^\mu = \mathcal{L}_1(\zeta) + \frac{1}{R^2}\mathcal{L}_3(\zeta) + \frac{1}{R^4}\mathcal{L}_5(\zeta) + \dots$$

Worldsheet point of view:

Flat space: saddle point approximation

$$z = \bar{z} = \frac{S}{S+T} \equiv z_0$$

$$A^{(0)}(S,T)_{HE} \sim e^{-2S\log|S|-2T\log|T|-2U\log|U|}$$

AdS: same saddle point

Now, can we explore a richer limit?

$$A_4^{AdS}(S,T)_{HE} \sim A_4^{flat}(S,T)_{HE} \times e^{\frac{S^2}{R^2}W_3(z_0)}$$

4 The Regge limit

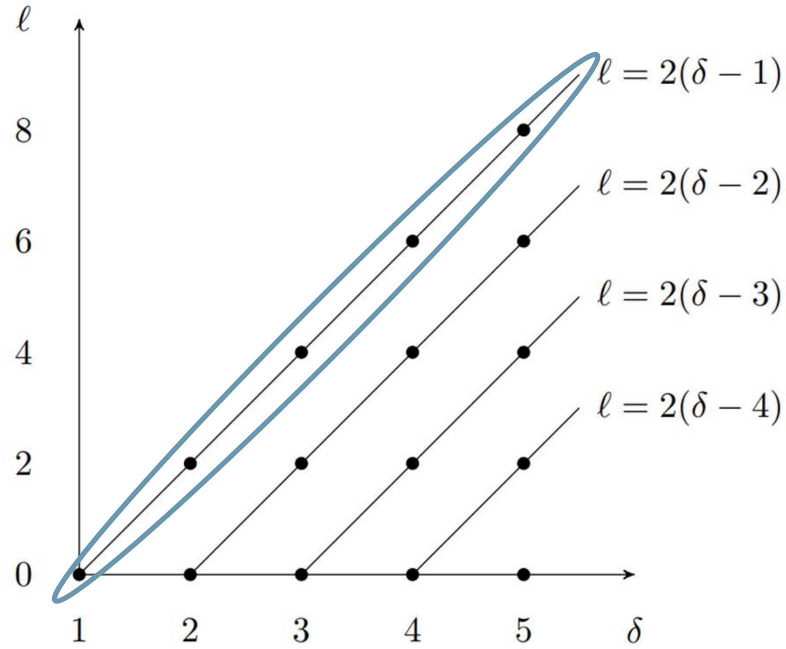
Why Regge?

- Storically: very useful in the presence of infinitely many resonances in scattering amplitudes.

- AdS/CFT: look for a generalization to scattering in AdS [Costa, Gonçalves, Penedones].

- Virasoro-Shapiro amplitude in flat space: infinitely many poles for the exchanged particles, which organize in Regge trajectories.

- <u>Big achievement</u>: amplitude in the Regge limit without knowing the full result! Need only spectrum of the particles in the leading trajectory and cubic couplings.
  Here: Konishi-like operators (target of integrability)



$\delta = 1,2, \ldots$
flat space string mass level

What do we expect for AdS?

- **Flat space:** large *T*, *S* finite
  The exchange of a spin *J* state makes the amplitude scaling as T^J.

$$A^{(0)}_{\text{Regge}}(S,T) = e^{i\pi S}\frac{\Gamma(-S)}{\Gamma(S+1)}T^{2S-2}$$

- **AdS corrections:** higher and higher order poles (quartic, seventh order,...)

$$A^{(k)}(S,T) = \frac{R^{(k)}_{3k+1}(T,\delta)}{(S-\delta)^{3k+1}} + \frac{R^{(k)}_{3k}(T,\delta)}{(S-\delta)^{3k}} + \ldots + \frac{R^{(k)}_{1}(T,\delta)}{S-\delta} + \text{regular}\,, \quad \delta = 1,2,\ldots$$

⇒ we expect logarithmic corrections of the form $(log\, T)^{\#}$, where the power of the log is fixed by the order of the poles.

The Regge limit in Mellin space

- Mellin amplitude for the exchange of an operator of spin J and twist τ [m = 0,1,2,...] :

$$M_{\tau,J}(s,t) = \sum_{m=0}^{\infty} \frac{Q_{J,m}^{\tau+4,d=4}(t)}{s+\frac{4}{3}-\tau-2m} + P_{J-1}(s,t)$$

Regular terms

$$\left(\tau - \frac{4}{3}\right)\left(2J+\tau+\frac{16}{3}\right) = \tilde{m}^2 R^2$$

- Mack polynomials are particularly simple in the leading large t behavior:

$$Q_{J,m}^{\tau,d}(t) = t^J + \cdots \qquad M_{\tau,J}(s,t) = M_{\tau,J}(s)t^J + \cdots$$

- We have two options:
  - compute the exchange amplitude explicitly in the limit
  - look for a difference equation it satisfies

Let's apply the ideas of Regge theory to *A(S,T)*!

The Regge limit in Borel space

- We start from

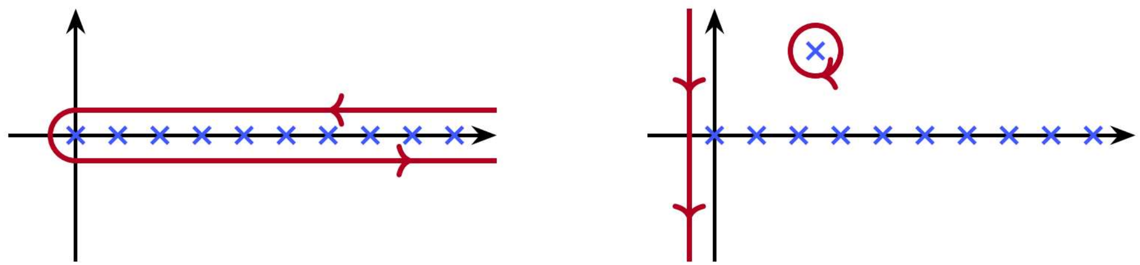
$$A(S,T) \simeq \sum_{J=0} C^2(J) A_{\tau,J}(S) T^J$$

First step in Regge theory: analytically continue the **partial waves** as a function of the spin *J*.

- We write the sum over the spin as a contour integral, which picks the poles at *J=0,2,...*

$$A(S,T) = \int \frac{dJ}{2i} \frac{1+(-1)^J}{2\sin(\pi J)} C^2(J) A_{\tau,J}(S) T^J$$

- We deform the contour, picking up the Regge poles:



- Large *T* limit: the pole with the largest real value of *J* dominates (leading Regge trajectory: operators of lowest dimension for each even spin).

The Regge limit in Borel space

- In a *1/R* expansion, $A_{\tau,J}(S)$ contains poles of higher and higher order at $J=J^*(S)$.

- The locations and residues of these poles are determined by the difference equation satisfied by the Borel amplitude (solved in a 1/R expansion) in terms of the mass squared:

$$m(J^*)^2 = 4S \qquad -4S + m(J)^2 \equiv (J - J^*)\beta(J)$$

- Using Cauchy's theorem:

$$\int \frac{dJ}{2i} \frac{1+(-1)^J}{2\sin(\pi J)} C^2(J) \frac{r_k(m^2)}{(J-J^*)^k \beta(J)^k} T^J = -\pi \frac{\partial_J^{k-1}}{\Gamma(k)} \frac{1+(-1)^J}{2\sin(\pi J)} C^2(J) \frac{r_k(J)}{\beta(J)^k} T^J \Big|_{J=J^*}$$

- We define the operator $y = \partial_J \frac{1}{\beta(J)}$.

For every pole of order *k*, we write its contribution as the operator $\frac{\partial_J^{k-1}}{\Gamma(k)} \frac{r_k(J)}{\beta(J)^k}$

The AdS Virasoro-Shapiro in the Regge limit

- The AdS Virasoro-Shapiro amplitude in the Regge limit is given by the CFT data of the exchanged operators (*N=4* SYM at strong coupling):

$$A_{Regge}(S,T) =: \hat{\mathcal{R}}(y) : \frac{1+(-1)^J}{2\sin(\pi J)} \mathcal{C}(J) \frac{1}{\beta(J)} T^J \Big|_{J=J^*(S)}$$

OPE

Effect of the curvature of the background

Dimension of the operators in the leading Regge trajectory

- At **all orders** in *1/R*, we have explicit solutions as derivatives on the flat space result in the Regge limit!

- The derivatives produce powers of $\log T$.

Results to all orders

$$A^{LL}_{Regge}(S,T) = A^{(0)}_{Regge}(S,T) \times e^{-\frac{4}{3}\frac{S^2}{R^2}\log^3 T}$$

$$\times \log^2 T\left(\frac{16S^3\log^3 T}{5R^2} + 2\pi S^2\cot(\pi S) + 4S^2\psi^{(0)}(S) - \frac{4S}{3} - 2i\pi S^2\right)$$

- Can resum all leading logs: leading logs exponentiate!
- Can also resum subleading logs.

➡ All order results in the limit of large *R*, *T* with $\frac{log^3\,T}{R^2}$ fixed!

Worldsheet: from the integrand to the right Regge limit

$$A(S,T) = \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} G(S,T,z)$$

$$\hat{G}(S,T,z) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} G\left(S, \frac{T}{\epsilon}, \epsilon z\right)$$

- At each order in *1/R*, $\hat{G}$ is a polynomial in $z, \bar{z}, \log|z|$(asymptotics of SVMPLs).
- The Regge limit of the Virasoro-Shapiro to order $1/\mathrm{R}^4$ agrees with our prediction!

- Ambiguities when working with integrands:

  - We assume the integrand remains finite as we take $\epsilon \to 0$.
  - This is true for G_1, G_2 provided the ambiguities are chosen appropriately.
  - For this choice, we can take the Regge limit at the level of the integrand.

Worldsheet: from the Regge limit to the right integrand

- Now let's revers our logic: look for integrands that lead to the correct result in the Regge limit, to all orders in *1/R* (derivatives of flat-space Regge result).

$$\frac{1}{T^2}\int d^2z|z|^{-J-4}|1-z|^{-2T-2}F\left(S,-\log|z|\right)\simeq -F(S,\partial_J)e^{i\pi\frac{J}{2}}\frac{\Gamma(-1-J/2)}{\Gamma(2+J/2)}T^J$$

- We get the functional equation

known to **any** order

$$-F(S,\partial_J)e^{i\pi\frac{J}{2}}\frac{\Gamma(-1-J/2)}{\Gamma(2+J/2)}T^J\Big|_{J=J^*(S)} =: \hat{\mathcal{R}}\left(\partial_J\frac{1}{\beta(J)}\right): \frac{1+(-1)^J}{2\sin(\pi J)}\mathcal{C}(J)\frac{1}{\beta(J)}T^J\Big|_{J=J^*(S)}$$

- Families of solutions, for the world-sheet integral, that lead to the correct Regge behaviour **to all orders** in the curvature expansion (up to ambiguities that integrate to zero in this limit).

Key Takeaways

The AdS Virasoro-Shapiro amplitude in the Regge limit

- can be written as derivatives of the flat space result;
- admits an integral representation with single-valued logarithms! ☺

Proposal: curvature corrections as genus zero integrals with insertions of transcendental single-valued functions of weight $3k$.

Summary and conclusions

- Compute string amplitudes on AdS from
  - *AdS/CFT*
  - *Number Theory*
  - *Integrability*
  - *Worldsheet intuition*

- Single valuedness to understand/construct scattering amplitudes in AdS (as in flat space).

- AdS Virasoro-Shapiro amplitude as a «worldsheet» integral.

- Further step towards the worldsheet theory: High Energy and Regge limit.

- More direct comparison to a putative worldsheet description of strings on $AdS_5 \times S^5$.

Summary and conclusions

- High Energy limit:
  -Curvature corrections **exponentiate**.
  -The leading behavior is captured by a **bosonic model** for classical strings on *AdS*.

- Regge limit:
  -Leading logs **exponentiate**.
  -Contact with integrability.
  -Manifest single-valuedness of the integrand around $z=0$.

→ Single-valuedness at 1?

→ Ambiguities?

→ Other backgrounds?

→ Properties of curvature corrections?

Thanks for your attention!