Hamiltonian structure of spin Ruijsenaars-Schneider models

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Calogero-Moser-Sutherland models

Rational model

Invert oscillator potential

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \;\to\; H = \frac{1}{2}p^2 + \frac{\varkappa^2}{q^2}$$

Integrable generalisation to many particles $\{q\}_{i=1,\dots,N}$

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \varkappa^2 \sum_{i<j}^{N} \frac{1}{q_{ij}^2}\,, \qquad q_{ij} = q_i - q_j$$

Hyperbolic model

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \varkappa^2 \sum_{i<j}^{N} \frac{1}{\sinh q_{ij}^2}$$

Elliptic model

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \varkappa^2 \sum_{i<j}^{N} \wp(q_{ij})$$

The models were discovered in 1970's.
Wide applications

- soliton theory
- quantum field theory
- solvable models of stat. mechanics
- black hole physics
- condensed matter
- quantum chaos
- representation theory
- harmonic analysis
- random matrix theory
- complex geometry

Ruijsenaars-Schneider models

Rational model

$$H = c^2 \sum_{i=1}^{N} \cosh \frac{p_i}{c} \prod_{i \neq j}^{N} \sqrt{1 + \frac{\varkappa^2}{c^2 q_{ij}^2}}$$

Hyperbolic model

$$H = c^2 \sum_{i=1}^{N} \cosh \frac{p_i}{c} \prod_{i \neq j}^{N} \sqrt{1 + \frac{\varkappa^2}{c^2 \sinh^2 q_{ij}^2}}$$

Elliptic model

$$H = c^2 \sum_{i=1}^{N} \cosh \frac{p_i}{c} \prod_{i \neq j}^{N} \sqrt{\lambda + \mu \wp(q_{ij})}$$

$$\{p_i, q_j\} = \delta_{ij}$$

Expanding in the limit $c \to \infty$ the corresponding Hamiltonians of the CMS models are recovered

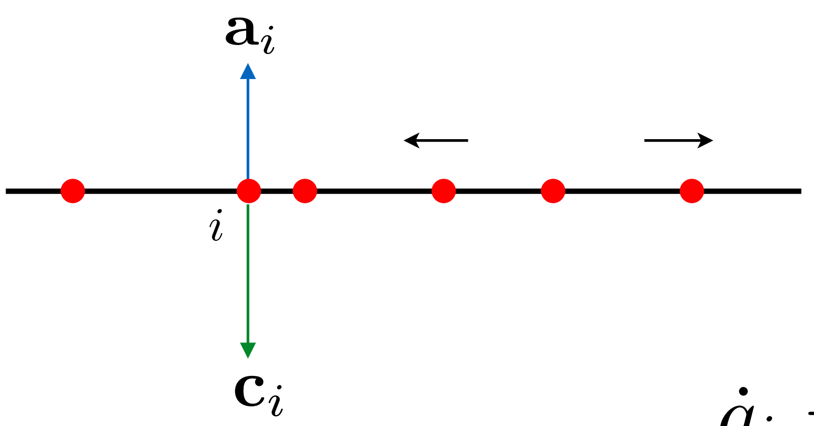
Inclusion of spin degrees of freedom

Outline

- *Spin RS models: equations of motion*
- *Heisenberg double*
- *Oscillator manifold*
- *Poisson-Lie group action on a product manifold*
- *Reduction*
- *Superintegrability*
- *Conclusions and future directions*

with Enrico Olivucci, arXiv:190602619

Equations of motion of the spin RS model



$$V(z) = \zeta(z) - \zeta(z+\gamma)$$

$$\dot{q}_i = f_{ii}$$

$$\dot{\mathbf{a}}_{i\alpha} = \sum_{j \neq i}^{N} V(q_{ij})(\mathbf{a}_{j\alpha} - \mathbf{a}_{i\alpha})$$

$$\dot{\mathbf{c}}_{i\alpha} = \sum_{j \neq i}^{N} \Big(V(q_{ij}) f_{ij} \mathbf{c}_{i\alpha} - V(q_{ji}) f_{ji} \mathbf{c}_{j\alpha} \Big)$$

Krichever & Zabrodin, 1995

$$f_{ij} = \sum_{\alpha=1}^{\ell} \mathbf{a}_{i\alpha} \mathbf{c}_{\alpha j}$$

$$\sum_{\alpha=1}^{\ell} \mathbf{a}_{i\alpha} = 1 \quad \forall \alpha$$

collective spin variables

Hamiltonian structure in the rational case

$$L_{ij} = \frac{f_{ij}}{q_{ij} + \gamma}$$

$$\{q_i, q_j\} = 0\,, \quad \{q_i, \mathbf{a}_{i\alpha}\} = 0\,, \quad \{q_i, \mathbf{c}_{j\alpha}\} = \delta_{ij}\mathbf{c}_{j\alpha}\,,$$

$$\{\mathbf{a}_{i\alpha}, \mathbf{a}_{j\beta}\} = \frac{\delta_{i\neq j}}{q_{ij}}(\mathbf{a}_{i\alpha}\mathbf{a}_{j\beta} + \mathbf{a}_{i\beta}\mathbf{a}_{j\alpha} - \mathbf{a}_{i\alpha}\mathbf{a}_{i\beta} - \mathbf{a}_{j\alpha}\mathbf{a}_{j\beta})$$

$$\{\mathbf{a}_{i\alpha}, \mathbf{c}_{\beta j}\} = \mathbf{a}_{i\alpha}L_{ij} - \delta_{\alpha\beta}L_{ij} - \frac{\delta_{i\neq j}}{q_{ij}}(\mathbf{a}_{i\alpha} - \mathbf{a}_{j\alpha})\mathbf{c}_{\beta j}\,,$$

$$\{\mathbf{c}_{\alpha i}, \mathbf{c}_{\beta j}\} = \frac{\delta_{i\neq j}}{q_{ij}}(\mathbf{c}_{i\alpha}\mathbf{c}_{\beta j} + \mathbf{c}_{\beta i}\mathbf{c}_{\alpha j}) - \mathbf{c}_{\alpha i}L_{ij} + \mathbf{c}_{\beta j}L_{ji}$$

G.A. & Frolov, 1997

Hamiltonian reduction

$$\mathscr{M} = T^*G \times \Sigma\,, \qquad \Sigma = \underbrace{\mathscr{O} \times \mathscr{O} \times \ldots \times \mathscr{O}}_{\ell}$$

$\mathscr{O}$ - coadjoint orbit of minimal dimension

$$G : \mathscr{M} \to \mathscr{M} \quad \Longrightarrow \quad \mu : \mathscr{M} \to \mathfrak{g}^*$$

$$\mathscr{P} = \mu^{-1}(\gamma\mathbb{1})/G$$

More on Hamiltonian structure in the rational case $G = \mathrm{GL}_N(\mathbb{C})$

$$\mathscr{M} = T^*G \times \mathfrak{g}^* = G \times \mathfrak{g}^* \times \mathfrak{g}^* \simeq G \times \mathfrak{g} \times \mathfrak{g} \quad \text{if} \quad \mathfrak{g}^* \simeq \mathfrak{g} \quad \to \quad (g, A, S)$$

Poisson structure

$$\begin{aligned} \{A_1, A_2\} &= \frac{1}{2}[C, A_1 - A_2] \\ \{A_1, g_2\} &= g_2 C, \qquad \{g_1, g_2\} = 0 \end{aligned}$$

$$C = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}$$

$$\{S_1, S_2\} = -\frac{1}{2}[C, S_1 - S_2]$$

Hamiltonian group action

$$A \to hAh^{-1}, \quad g \to hgh^{-1}, \quad S \to hSh^{-1}$$

$$\mu = gAg^{-1} - A + S \qquad \leftarrow \quad \text{moment map}$$

Two simple Hamiltonians

$$H_C = \mathrm{Tr}\, A^2 \quad \text{and} \quad H_R = \mathrm{Tr}\, g$$

Superintegrability

The Poisson bracket of S_{ij} can be realized by means of $2N$ ℓ-dimensional vectors a_i, b_i which form $N\ell$-pairs of canonically conjugated variables:

$$\{a_{i\alpha}, b_{\beta j}\} = -\delta_{ij}\delta_{\alpha\beta}$$

$i, j = 1, \cdots, N$ and $\alpha, \beta = 1, \cdots, \ell$.

$$S_{ij} = \sum_{\alpha=1}^{\ell} a_{i\alpha} b_{\alpha j}$$

Transformations under the group action

$$a_{i\alpha} \quad \rightarrow \quad h_{ij} a_{j\alpha}\,, \qquad b_{\alpha i} \quad \rightarrow \quad b_{\beta j}(h^{-1})_{ji}\,, \qquad h \in G\,.$$

Many $G-$invariants commuting with H_C or H_R exist!

Many G – invariants commuting with H_C or H_R exist!

For H_C the family $I_n^{\alpha\beta} = \mathrm{Tr}\, A^n \mathrm{S}^{\alpha\beta}$

$(\mathrm{S}^{\alpha\beta})_{ij} = a_{i\alpha} b_{\beta j}$

Generating function $T^{\alpha\beta}(\lambda)$ of $I_n^{\alpha\beta}$:

$$T^{\alpha\beta}(\lambda) = \delta^{\alpha\beta} + \mathrm{Tr}\frac{1}{\lambda - A}\mathrm{S}^{\alpha\beta}$$

Yangian algebra

$$\{T_1(\lambda), T_2(\mu)\} = [r(\lambda-\mu), T_1(\lambda)T_2(\mu)]$$

$\mathrm{Tr}\, T(\lambda)^n \leftarrow$ center

split Casimir of $\mathrm{GL}_\ell(\mathbb{C})$

$$r(\lambda-\mu) = \frac{C^{\mathrm{s}}}{\lambda-\mu}$$

$\leftarrow$ rational solution of CYBE

Spin Calogero – Moser model has Yangian as symmetry

Many G – invariants commuting with H_C or H_R exist!

For H_R the family $J_n^{\alpha\beta} = \mathrm{Tr}\, g^n \mathrm{S}^{\alpha\beta}$

$(\mathrm{S}^{\alpha\beta})_{ij} = a_{i\alpha} b_{\beta j}$

Generating function $J^{\alpha\beta}(\lambda)$ of $J_n^{\alpha\beta}$:

$$J^{\alpha\beta}(\lambda) = \sum_{n=-\infty}^{\infty} J_n^{\alpha\beta} \lambda^{-n-1}$$

Current algebra

$$\{J_1(\lambda), J_2(\mu)\} = [C^{\mathrm{s}}, J_2(\mu)]\, \delta\left(\frac{\lambda}{\mu}\right)$$

$\delta(\frac{\lambda}{\mu}) = \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} (\frac{\lambda}{\mu})^n$

$\mathrm{Tr}\, J(\lambda)^n \leftarrow$ center

Spin RS model has the current algebra as symmetry

Superintegrability

For the RS model there exist an algebra of polynomial invariants

$$J^+(\lambda) = \sum_{n=0}^{\infty} J_n \lambda^{-n-1}$$

with algebra

$$\{J_1^+(\lambda), J_2^+(\mu)\} = [r(\lambda - \mu), J_1^+(\lambda) + J_2^+(\mu)]$$

$J_n^+(\lambda) = \mathrm{Tr} J^+(\lambda)^n \quad \leftarrow \quad$ center

Quadratic algebra of the Calogero model is the deformation of this linear one

Quasi – Hamiltonian reduction

Chalykh & Fairon, 2018

quasi-Poisson (Van der Bergh's bracket)

Jordan quiver/G

On the other hand, there is a deformation hierarchy of initial phase spaces

$$T^*G \longrightarrow D_+(G)$$

Heisenberg double

Fock & Rosly, 1993, 1998
Gorsky & Nekrasov, 1994
G.A. & Frolov, 1996
....
Feher & Klimcik, 2009
...

$$\mathscr{M} = D_+(G) \times ???$$

What should be there in the spin case?

Heisenberg double

$(\mathfrak{g}, \mathfrak{g}^*)$ - factorisable Lie bialgebra, $\mathfrak{g}^* \simeq \mathfrak{g}$

$$\mathscr{D} = \mathfrak{g} \oplus \mathfrak{g} \longleftarrow \text{double}$$

$(X, X) \subset \mathscr{D}, \quad \forall X \in \mathfrak{g}$

$(X_+, X_-) = (\hat{r}_+ X, \hat{r}_- X) \subset \mathscr{D}, \quad \forall X \in \mathfrak{g}^* \simeq \mathfrak{g}$

$\hat{r}_\pm = \hat{r} \pm \frac{1}{2}\mathbb{1}$ are two linear operators, $\hat{r}_\pm : \mathfrak{g}^* \to \mathfrak{g}_\pm \subset \mathfrak{g}$

$r \in \mathfrak{g} \wedge \mathfrak{g}$ split solution of mCYBE

$$D = G \times G \longleftarrow \text{double Lie group}$$

$G^* \simeq (u_+, u_-) \subset D$

diffeomorphism $\sigma : G^* \simeq G$ $\qquad \sigma(u_+, u_-) = u_+ u_-^{-1} = u$

Heisenberg double

$D_+(G)$

$A, B \in G = \mathrm{GL}_N(\mathbb{C})$

$$
\begin{aligned}
\frac{1}{\varkappa}\{A_1, A_2\} &= -r_- A_1 A_2 - A_1 A_2 r_+ + A_1 r_- A_2 + A_2 r_+ A_1 , \\
\frac{1}{\varkappa}\{B_1, B_2\} &= -r_- B_1 B_2 - B_1 B_2 r_+ + B_1 r_- B_2 + B_2 r_+ B_1 , \\
\frac{1}{\varkappa}\{A_1, B_2\} &= -r_- A_1 B_2 - A_1 B_2 r_- + A_1 r_- B_2 + B_2 r_+ A_1 , \\
\frac{1}{\varkappa}\{B_1, A_2\} &= -r_+ B_1 A_2 - B_1 A_2 r_+ + B_1 r_- A_2 + A_2 r_+ B_1 .
\end{aligned}
$$

$$
r_\pm = \pm \frac{1}{2} \sum_{i=1}^{N} E_{ii} \otimes E_{ii} \pm \sum_{i \leqslant j}^{N} E_{ij} \otimes E_{ji}
$$

$$
r_+ - r_- = C_{12} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}
$$

$$
r = \frac{1}{2}(r_+ + r_-)
$$

Poisson action of a Poisson-Lie group G

$$A \to hAh^{-1}\,, \quad B \to hBh^{-1}\,, \quad h \in G$$

The Poisson-Lie structure of G is given in terms of the Sklyanin bracket

$$\{h_1, h_2\} = -\varkappa\,[r_\pm, h_1 h_2]\,, \quad h \in G\,.$$

The non-abelian moment map for this action (m_+, m_-)

$$m = m_+ m_-^{-1} \in G \quad \longrightarrow \quad m = BA^{-1}B^{-1}A$$

$$\frac{1}{\varkappa}\{m_1, m_2\} = -r_+ m_1 m_2 - m_1 m_2 r_- + m_1 r_- m_2 + m_2 r_+ m_1$$

Semenov-Tian-Shansky bracket

Involutive family $\{H_k, H_m\} = 0$

$$H_k = \mathrm{Tr}(BA^{-1})^k = \mathrm{Tr}(A^{-1}B)^k\,, \quad k \in \mathbb{Z}$$

$\Sigma_{N,\ell}$: $a_{i\alpha} \equiv (a)_{i\alpha}\,,\ \ b_{\alpha j} \equiv (b)_{\alpha j} \qquad i = 1,\ldots,N\,,\quad \alpha = 1,\ldots,\ell$

$$
\begin{aligned}
\{a_1, a_2\}_\pm &= \varkappa\,(\,r\, a_1 a_2 \mp a_1 a_2\, \rho\,)\;, \\
\{b_1, b_2\}_\pm &= \varkappa\,(\,b_1 b_2\, r \mp \rho\, b_1 b_2\,)\;, \\
\{a_1, b_2\}_\pm &= \varkappa\,(-b_2\, r_+\, a_1 \pm a_1\, \rho_\mp\, b_2\,) - C_{12}^{\mathrm{rec}}\;, \\
\{b_1, a_2\}_\pm &= \varkappa\,(-b_1\, r_-\, a_2 \pm a_2\, \rho_\pm\, b_1\,) + C_{21}^{\mathrm{rec}}\;.
\end{aligned}
$$

$$C_{12}^{\mathrm{rec}} = \sum_{i=1}^{N}\sum_{\alpha=1}^{\ell} E_{i\alpha}\otimes E_{\alpha i}$$

$$\rho_\pm = \pm\frac{1}{2}\sum_{\alpha=1}^{\ell} E_{\alpha\alpha}\otimes E_{\alpha\alpha} \pm \sum_{\alpha \lessgtr \beta}^{\ell} E_{\alpha\beta}\otimes E_{\beta\alpha}$$

$$\rho_+ - \rho_- = C_{12}^{\mathrm{s}} = \sum_{\alpha,\beta=1}^{\ell} E_{\alpha\beta}\otimes E_{\beta\alpha}$$

$$\rho = \tfrac{1}{2}(\rho_+ + \rho_-)$$

$\varkappa = 0$ ⟶

$$\{a_{i\alpha}, b_{\beta j}\} = -\delta_{ij}\delta_{\alpha\beta}$$

$N\ell$ pairs of canonically conjugate variables

Oscillator manifold

$$\omega = \mathbb{1} + \varkappa ab$$

Define the following action of the Poisson-Lie group G on oscillators

$$\delta_X a_{i\alpha} = (\mathrm{Ad}^*_\omega X\, a)_{i\alpha} \qquad \delta_X b_{\alpha i} = -(b\, \mathrm{Ad}^*_\omega X\,)_{\alpha i}\,, \qquad X \in \mathfrak{g}$$

$\mathrm{Ad}^*_g X$ for $g \equiv (g_+, g_-) \in G^*$ is the dressing transformation

★ This action is Poisson

★ If $\omega = \omega_+ \omega_-^{-1}$ then $(\omega_+^{-1}, \omega_-^{-1}) \in G^*$ is the moment map

$$n = \omega_+^{-1}\omega_- \in G$$

$$\frac{1}{\varkappa}\{n_1, n_2\} = -r_+ n_1 n_2 - n_1 n_2 r_- + n_1 r_- n_2 + n_2 r_+ n_1$$

Semenov-Tian-Shansky bracket

Poisson-Lie action on a product manifold

Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two Poisson manifolds with brackets $\{\cdot\,,\cdot\}_{\mathscr{M}_1}$ and $\{\cdot\,,\cdot\}_{\mathscr{M}_2}$

$$m_i \,:\, \mathscr{M}_i \to G^*$$

$$\mathscr{M} = \mathscr{M}_1 \times \mathscr{M}_2$$

$$m = m_1 m_2 \qquad \longrightarrow \qquad G : \mathscr{M} \to \mathscr{M}$$

$$\xi_X f = \langle X, \{m, f\}_{\mathscr{M}} m^{-1} \rangle\,, \quad f \in \mathrm{Fun}(\mathscr{M})$$

$$X \to \xi_X \qquad \text{Lie algebra homomorphism}$$

$\mathscr{M}_1$ $\qquad\qquad$ $\mathscr{M}_2$

$$\mathscr{M} = D_+(G) \times \Sigma^{\pm}_{N,\ell}$$

Moment map equation

$$m, n \in G$$

Product in G^*

$$m \star n = q\,\mathbb{1}$$

Moment map for
the action of G
on the double

Moment map for
the action of G
on the oscillator manifold

$$m = q\,\omega_+\omega_-^{-1} = q\,\omega$$

$$BA^{-1}B^{-1}A = q(\mathbb{1} + \varkappa ab)$$

Reduction

$$\mathscr{P} = \{\text{Solutions of} \;\; BA^{-1}B^{-1}A = q(\mathbb{1} + \varkappa ab)\}/G$$

$$\delta_X a_{i\alpha} \;=\; (\mathrm{Ad}^*_{\omega\star m^{-1}} X\, a)_{i\alpha} \qquad \delta_X b_{\alpha i} \;=\; -(b\,\mathrm{Ad}^*_{\omega\star m^{-1}} X\,)_{\alpha i}\,, \qquad X \in \mathfrak{g}\,,$$

$$\omega\star m^{-1} = \omega_+ m_+^{-1} m_- \omega_-^{-1} \equiv q^{-1}\mathbb{1}$$

$$a_{i\alpha} \longrightarrow (h\,a)_{i\alpha} \qquad b_{\alpha i} \longrightarrow (b\,h^{-1})_{\alpha i}\,, \qquad h = e^X \in G$$

Construction of G-invariants becomes elementary !

Reduction

$$A = TQT^{-1}\,, \qquad B = UP^{-1}T^{-1}$$

G.A. & Frolov, 1996

diagonal

$$\sum_{j=1}^{N} T_{ij} = \sum_{j=1}^{N} U_{ij} = 1\,, \qquad \forall\, i = 1, \ldots, N$$

Frobenius

$$t \text{ diagonal} \rightarrow \; t_{ij} = \delta_{ij} \sum_{\alpha=1}^{\ell} (T^{-1}a)_{i\alpha}$$

$$L = t^{-1}T^{-1}UP^{-1}t\,Q^{-1}\,, \qquad \mathbf{a} = t^{-1}T^{-1}a\,, \qquad \mathbf{c} = bA^{-1}BTt$$

Lax matrix

Invariant spins

$$L - qQ^{-1}LQ = q\varkappa\,\mathbf{ac}\,. \qquad \Longrightarrow \qquad L = q\varkappa \sum_{i,j=1}^{N} \frac{Q_i}{Q_i - qQ_j} (\mathbf{ac})_{ij}\, E_{ij}$$

Poisson algebra

$$Z = Q^{-1} L Q$$

$$\{Q_i, \mathbf{a}_{j\alpha}\} = 0\,, \qquad \{Q_i, \mathbf{c}_{\alpha j}\} = \delta_{ij}\, \mathbf{c}_{\alpha j}\, Q_j$$

$$\begin{aligned}
\{\mathbf{a}_1, \mathbf{a}_2\}_{\pm} &= \varkappa\big[(r^{\bullet} \mp Y)\, \mathbf{a}_1\mathbf{a}_2 \mp \mathbf{a}_1\mathbf{a}_2\, \rho \mp \mathbf{a}_1\, X_{21}\, \mathbf{a}_2 \pm \mathbf{a}_2\, X_{12}\, \mathbf{a}_1\big]\,, \\
\{\mathbf{a}_1, \mathbf{c}_2\}_{\pm} &= \varkappa\big[\mathbf{c}_2(r^{*}_{12} \pm Y)\, \mathbf{a}_1 \pm \mathbf{a}_1\rho_{\mp}\mathbf{c}_2 \pm \mathbf{a}_1\mathbf{c}_2\, X_{21} \mp X^{\mp}_{12}\, \mathbf{a}_1\mathbf{c}_2\big] + K_{21}\, \mathbf{a}_1 Z_2 - C^{\mathrm{rec}}_{12} Z_2\,, \\
\{\mathbf{c}_1, \mathbf{a}_2\}_{\pm} &= \varkappa\big[\mathbf{c}_1(-r^{*}_{21} \pm Y)\, \mathbf{a}_2 \pm \mathbf{a}_2\rho_{\pm}\mathbf{c}_1 \mp \mathbf{a}_2\mathbf{c}_1\, X_{12} \pm X^{\mp}_{21}\, \mathbf{a}_2\mathbf{c}_1\big] - K_{12}\, \mathbf{a}_2 Z_1 + C^{\mathrm{rec}}_{21} Z_1\,, \\
\{\mathbf{c}_1, \mathbf{c}_2\}_{\pm} &= \varkappa\big[\mathbf{c}_1\mathbf{c}_2\, (r^{\circ} \mp Y) \mp \rho\, \mathbf{c}_1\mathbf{c}_2 \pm \mathbf{c}_1\, X^{\mp}_{12}\, \mathbf{c}_2 \mp \mathbf{c}_2\, X^{\mp}_{21}\, \mathbf{c}_1\big] + \mathbf{c}_2 K_{12}\, Z_1 - \mathbf{c}_1 K_{21}\, Z_2\,,
\end{aligned}$$

$$X_{12} = \sum_{i\beta\sigma\delta} (\mathbf{a}_1\rho)_{i\beta\sigma\delta}\, E_{ii} \otimes E_{\sigma\delta}\,, \qquad X^{\pm}_{12} = \sum_{i\beta\sigma\delta} (\mathbf{a}_1\rho^{\pm})_{i\beta\sigma\delta}\, E_{ii} \otimes E_{\sigma\delta}\,,$$

$$K_{12} = \sum_{i\sigma} E_{\sigma i} \otimes E_{ii}\,, \qquad Y_{12} = \sum_{i\beta k\delta} (\mathbf{a}_1\mathbf{a}_2\rho)_{i\beta k\delta}\, E_{ii} \otimes E_{kk}\,.$$

$$r^{\bullet} = \frac{1}{2} \sum_{i,j=1}^{N} \frac{Q_i + Q_j}{Q_i - Q_j}\, (E_{ii} - E_{ij}) \otimes (E_{jj} - E_{ji})\,,$$

$$r^{*} = \frac{1}{2} \sum_{i,j=1}^{N} \frac{Q_i + Q_j}{Q_i - Q_j}\, (E_{ij} - E_{ii}) \otimes E_{jj}\,, \qquad r^{\circ} = \frac{1}{2} \sum_{i,j=1}^{N} \frac{Q_i + Q_j}{Q_i - Q_j}\, (E_{ii} \otimes E_{jj} - E_{ij} \otimes E_{ji})$$

Poisson algebra

$$\frac{1}{\varkappa}\{L_1, L_2\}_\pm = (r_{12} \mp Y)L_1 L_2 - L_1 L_2(\underline{r}_{\,12} \pm Y) + L_1(\bar{r}_{21} \pm Y)L_2 - L_2(\bar{r}_{12} \mp Y)L_1$$

$$r = \sum_{i \neq j}^{N} \left(\frac{Q_j}{Q_{ij}} E_{ii} - \frac{Q_i}{Q_{ij}} E_{ij}\right) \otimes (E_{jj} - E_{ji}),$$

$$\bar{r} = \sum_{i \neq j}^{N} \frac{Q_i}{Q_{ij}} (E_{ii} - E_{ij}) \otimes E_{jj}, \qquad \underline{r} = \sum_{i \neq j}^{N} \frac{Q_i}{Q_{ij}} (E_{ij} \otimes E_{ji} - E_{ii} \otimes E_{jj}).$$

$$Y = \sum_{i\beta j\delta} (\mathbf{a}_1 \mathbf{a}_2 \rho)_{i\beta j\delta} \, E_{ii} \otimes E_{jj}$$

The L-algebra is not the same as in the spin case!

$$\frac{1}{\varkappa}\{L_1, L_2\} = r_{12} L_1 L_2 - L_1 L_2 \underline{r}_{\,12} + L_1 \bar{r}_{21} L_2 - L_2 \bar{r}_{12} L_1$$

G.A. , Klabbers & Olivucci, 2018

Poisson algebra

$$H_m = \mathrm{Tr}(BA^{-1})^m \qquad \longleftarrow \text{ commutative family}$$

$$J_n^+ = \mathrm{Tr}\big[\mathrm{S}(BA^{-1})^n\big]\,, \qquad J_n^- = \mathrm{Tr}\big[\mathrm{S}(A^{-1}B)^n\big]$$

$$\{H_m, I_n\} = \{H_m, J_n\} = 0$$

$$J_n^{+\alpha\beta} = \mathrm{Tr}\big[\mathrm{S}^{\alpha\beta}(BA^{-1})^n\big]\,, \quad J_n^{-\alpha\beta} = \mathrm{Tr}\big[\mathrm{S}^{\alpha\beta}(A^{-1}B)^n\big]$$

$$(\mathrm{S}^{\alpha\beta})_{ij} = a_{i\alpha}b_{\beta j}$$

$$J_n^{+\alpha\beta} = \mathrm{Tr}\big[\mathbf{S}^{\alpha\beta}Q^{-1}L^{-1}QL^n\big]\,, \qquad J_n^{-\alpha\beta} = \mathrm{Tr}\big[\mathbf{S}^{\alpha\beta}Q^{-1}L^{n-1}Q\big]$$

$$(\mathbf{S}^{\alpha\beta})_{ij} = \mathbf{a}_{i\alpha}\mathbf{c}_{\beta j}$$

$$J_0^{+\alpha\beta} = J_0^{-\alpha\beta} = \mathrm{Tr}\,\mathrm{S}^{\alpha\beta}$$

Poisson algebra

$$
\begin{aligned}
\frac{1}{\varkappa}\{J_n^{\alpha\beta}, J_m^{\gamma\delta}\} = {} & \frac{1}{\varkappa}\left(\delta^{\beta\gamma}J^{\alpha\delta}_{n+m} - \delta^{\alpha\delta}J^{\gamma\beta}_{n+m}\right) \\
& \pm \left[\rho_{\alpha\mu,\gamma\nu}J_n^{\mu\beta}J_m^{\nu\delta} + J_n^{\alpha\mu}J_m^{\gamma\nu}\rho_{\mu\beta,\nu\delta} - J_m^{\gamma\nu}\rho_{\pm\alpha\mu,\nu\delta}J_n^{\mu\beta} - J_n^{\alpha\mu}\rho_{\mp\mu\beta,\gamma\nu}J_m^{\nu\delta}\right] \\
& \pm \left[-\frac{1}{2}\left(J_n^{\alpha\delta}J_m^{\gamma\beta} - J_m^{\alpha\delta}J_n^{\gamma\beta}\right) + \sum_{p=0}^{m}\left(J^{\alpha\delta}_{n+m-p}J_p^{\gamma\beta} - J^{\alpha\delta}_{m-p}J^{\gamma\beta}_{n+p}\right)\right] \\
& + \frac{1\mp 1}{2}\left(J^{\alpha\delta}_{n+m}J_0^{\gamma\beta} - J_0^{\alpha\delta}J^{\gamma\beta}_{n+m}\right).
\end{aligned}
$$

$J_n^{\pm\alpha\beta}$

$\Sigma^{\pm}_{N,\ell}$

Poisson algebra of the spin group

$$\{J_0^{\alpha\beta}, J_0^{\gamma\delta}\} = \delta^{\beta\gamma} J_0^{\alpha\delta} - \delta^{\alpha\delta} J_0^{\gamma\beta}$$
$$\pm \varkappa \Big[\rho_{\alpha\mu,\nu\rho} J_0^{\mu\beta} J_0^{\nu\delta} + J_0^{\alpha\mu} J_0^{\gamma\nu} \rho_{\mu\beta,\nu\delta} - J_0^{\gamma\nu} \rho_{\pm\alpha\mu,\nu\delta} J_0^{\mu\beta} - J_0^{\alpha\mu} \rho_{\mp\mu\beta,\gamma\nu} J_0^{\nu\delta} \Big]$$

$$\varpi^{\mu\nu} = \delta^{\mu\nu} + \varkappa J_0^{\alpha\beta}$$

$$\{\varpi_1, \varpi_2\}_\pm = \pm(\rho\, \varpi_1 \varpi_2 + \varpi_1 \varpi_2 \rho - \varpi_2 \rho_\pm \varpi_1 - \varpi_1 \rho_\mp \varpi_2)$$

Semenov-Tian-Shansky bracket

ϖ $\longleftarrow$ moment map for the Poisson action of the spin Poisson-Lie group $S = \mathrm{GL}_\ell(\mathbb{C})$

$$a_{i\alpha} \longrightarrow (ag)_{i\alpha}\,, \qquad b_{\alpha i} \longrightarrow (g^{-1}b)_{\alpha i}\,, \quad g \in S$$

$$\{g_1, g_2\} = \pm\varkappa[\rho, g_1 g_2]$$

Poisson algebra

$$J(\lambda) = \sum_{n=0}^{\infty} J_n^+ \lambda^{-n-1}$$

$$\begin{aligned}\{J_1(\lambda), J_2(\mu)\}_\pm &= \frac{1}{\lambda-\mu}[C_{12}^{\mathrm{s}}, J_1(\lambda)+J_2(\mu)] \\ &\pm \varkappa\Big[\rho_\pm(\lambda,\mu)J_1(\lambda)J_2(\mu) + J_1(\lambda)J_2(\mu)\rho_\mp(\lambda,\mu) - J_2(\mu)\rho_\pm J_1(\lambda) - J_1(\lambda)\rho_\mp J_2(\mu)\Big]\end{aligned}$$

$$\rho_\pm(\lambda,\mu) = \rho \pm \frac{1}{2}\frac{\lambda+\mu}{\lambda-\mu}C_{12}^{\mathrm{s}} = \frac{\lambda\rho_\pm \mp \mu\rho_\mp}{\lambda-\mu}$$

Solving equation of motion

$$H_1 \longrightarrow \qquad \dot{A} = -B\,, \quad \dot{B} = -BA^{-1}B\,, \quad \dot{a} = 0 = \dot{b}$$

$BA^{-1} = I$ is an integral of motion and also $a = \text{const}$, $b = \text{const}$

$$A(\tau) = e^{-I\tau}A(0)\,, \quad B(\tau) = Ie^{-I\tau}A(0)$$

Initial data $\quad A(0) \equiv Q\,, \qquad a_{i\alpha}(0) \equiv a_{i\alpha}, \qquad \sum a_{i\alpha} = 1 \quad \forall i$

$$I = L(0)$$

$$e^{-L(0)\tau}Q = T(\tau)Q(\tau)T(\tau)^{-1}$$

Frobenius, $\quad T(\tau) = \mathbb{1}$

$$\mathbf{a}_{i\alpha}(\tau) = \frac{T(\tau)^{-1}_{ij}a_{j\alpha}}{\sum_{\beta} T(\tau)^{-1}_{ij}a_{j\beta}} = T(\tau)^{-1}_{ij}a_{j\alpha}$$

Conclusions

The Hamiltonian structure of the hyperbolic spin RS model is found from the Poisson reduction of $D_+(G) \times \Sigma^{\pm}_{N,\ell}$

★ Elliptic version?

★ Quantum model?