Spinning and spinless two-particle dynamics from gravitational scattering amplitudes

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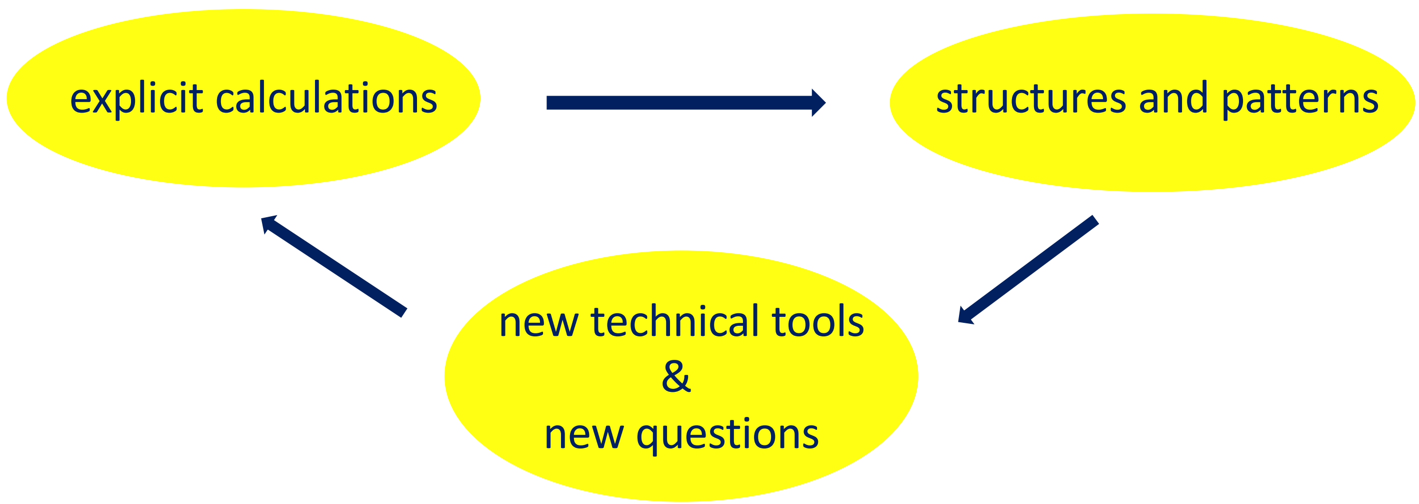
Gravitational interactions – still understanding them, despite long history

Different approaches -- field theory, string theory, world line, twistors, numerical, etc – bring different perspectives, emphasize different aspects

Apart from direct applications, explicit calculations can expose structures and patterns



For classical gravitational interactions --

Not structure (xPM spinless Hamiltonian) --

$$\mathbf{n} = \frac{\boldsymbol{r}}{r} \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$\widehat{H}_{\mathrm{N}}(\mathbf{r},\mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{G_N}{r},$$

$$c^2\,\widehat{H}_{\mathrm{1PN}}(\mathbf{r},\mathbf{p}) = \frac{1}{8}(3\nu-1)(\mathbf{p}^2)^2 - \frac{1}{2}\Big\{(3+\nu)\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{G_N}{r} + \frac{G_N^2}{2r^2},$$

$$\begin{aligned} c^4\,\widehat{H}_{\mathrm{2PN}}(\mathbf{r},\mathbf{p}) &= \frac{1}{16}\left(1-5\nu+5\nu^2\right)(\mathbf{p}^2)^3 + \frac{1}{8}\Big\{\left(5-20\nu-3\nu^2\right)(\mathbf{p}^2)^2 - 2\nu^2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - 3\nu^2(\mathbf{n}\cdot\mathbf{p})^4\Big\}\frac{G_N}{r} \\ &\quad + \frac{1}{2}\Big\{(5+8\nu)\mathbf{p}^2 + 3\nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{G_N^2}{r^2} - \frac{1}{4}(1+3\nu)\frac{G_N^3}{r^3}, \end{aligned}$$

$$\begin{aligned} c^6\,\widehat{H}_{\mathrm{3PN}}(\mathbf{r},\mathbf{p}) &= \frac{1}{128}\left(-5+35\nu-70\nu^2+35\nu^3\right)(\mathbf{p}^2)^4 + \frac{1}{16}\Big\{\left(-7+42\nu-53\nu^2-5\nu^3\right)(\mathbf{p}^2)^3 \\ &\quad + (2-3\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + 3(1-\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - 5\nu^3(\mathbf{n}\cdot\mathbf{p})^6\Big\}\frac{G_N}{r} \\ &\quad + \left\{\frac{1}{16}\left(-27+136\nu+109\nu^2\right)(\mathbf{p}^2)^2 + \frac{1}{16}(17+30\nu)\nu(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \frac{1}{12}(5+43\nu)\nu(\mathbf{n}\cdot\mathbf{p})^4\right\}\frac{G_N^2}{r^2} \\ &\quad + \left\{\left(-\frac{25}{8} + \left(\frac{\pi^2}{64} - \frac{335}{48}\right)\nu - \frac{23\nu^2}{8}\right)\mathbf{p}^2 + \left(-\frac{85}{16} - \frac{3\pi^2}{64} - \frac{7\nu}{4}\right)\nu(\mathbf{n}\cdot\mathbf{p})^2\right\}\frac{G_N^3}{r^3} + \left\{\frac{1}{8} + \left(\frac{109}{12} - \frac{21}{32}\pi^2\right)\nu\right\}\frac{G_N^4}{r^4}. \end{aligned}$$

$$\begin{aligned} c^8\,\widehat{H}_{\mathrm{4PN}}^{\mathrm{local}}(\mathbf{r},\mathbf{p}) &= \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4\right)(\mathbf{p}^2)^5 \\ &\quad + \left\{\frac{45}{128}(\mathbf{p}^2)^4 - \frac{45}{16}(\mathbf{p}^2)^4\,\nu + \left(\frac{423}{64}(\mathbf{p}^2)^4 - \frac{3}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2\right)\nu^2\right. \\ &\quad + \left(-\frac{1013}{256}(\mathbf{p}^2)^4 + \frac{23}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 + \frac{69}{128}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{64}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 + \frac{35}{256}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^3 \\ &\quad \left. + \left(-\frac{35}{128}(\mathbf{p}^2)^4 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 - \frac{35}{128}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^4\right\}\frac{G_N}{r} \\ &\quad + \left\{\frac{13}{8}(\mathbf{p}^2)^3 + \left(-\frac{791}{64}(\mathbf{p}^2)^3 + \frac{49}{16}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{889}{192}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{369}{160}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu\right. \\ &\quad + \left(\frac{4857}{256}(\mathbf{p}^2)^3 - \frac{545}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + \frac{9475}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - \frac{1151}{128}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^2 \\ &\quad \left. + \left(\frac{2335}{256}(\mathbf{p}^2)^3 + \frac{1135}{256}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{1649}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{10353}{1280}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^3\right\}\frac{G_N^2}{r^2} \\ &\quad + \left\{\frac{105}{32}(\mathbf{p}^2)^2 + \left(\left(\frac{2749\pi^2}{8192} - \frac{589189}{19200}\right)(\mathbf{p}^2)^2 + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)(\mathbf{n}\cdot\mathbf{p})^4\right)\nu\right. \\ &\quad + \left(\left(\frac{18491\pi^2}{16384} - \frac{1189789}{28800}\right)(\mathbf{p}^2)^2 + \left(-\frac{127}{3} - \frac{4035\pi^2}{2048}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{57563}{1920} - \frac{38655\pi^2}{16384}\right)(\mathbf{n}\cdot\mathbf{p})^4\right)\nu^2 \\ &\quad \left. + \left(-\frac{553}{128}(\mathbf{p}^2)^2 - \frac{225}{64}(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - \frac{381}{128}(\mathbf{n}\cdot\mathbf{p})^4\right)\nu^3\right\}\frac{G_N^3}{r^3} \\ &\quad + \left\{\frac{105}{32}\mathbf{p}^2 + \left(\left(\frac{185761}{19200} - \frac{21837\pi^2}{8192}\right)\mathbf{p}^2 + \left(\frac{3401779}{57600} - \frac{28691\pi^2}{24576}\right)(\mathbf{n}\cdot\mathbf{p})^2\right)\nu\right. \\ &\quad \left. + \left(\left(\frac{672811}{19200} - \frac{158177\pi^2}{49152}\right)\mathbf{p}^2 + \left(\frac{110099\pi^2}{49152} - \frac{21827}{3840}\right)(\mathbf{n}\cdot\mathbf{p})^2\right)\nu^2\right\}\frac{G_N^4}{r^4} \\ &\quad + \left\{-\frac{1}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169199}{2400}\right)\nu + \left(\frac{7403\pi^2}{3072} - \frac{1256}{45}\right)\nu^2\right\}\frac{1}{r^5}. \end{aligned}$$

Not structure (“too explicit” expressions for observables) --

$$H = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2} + \sum_A H^A(p)\mathbb{O}^A \qquad H^A(\boldsymbol{r}^2, \boldsymbol{p}^2) = \frac{G}{|\boldsymbol{r}|} c_1^A(\boldsymbol{p})^2 + \frac{G^2}{|\boldsymbol{r}|^2} c_2^A(\boldsymbol{p}^2) + \dots$$

$$\begin{aligned}
\Delta_1^{(0)}\boldsymbol{p} &= \frac{2E\xi\bar{c}_1^{(0)}}{b^2 p_\infty}\boldsymbol{b} \\
\Delta_1^{(1)}\boldsymbol{p} &= -\frac{2E\xi}{b^2 p_\infty}\sum_i \bar{c}^{(1,i)}(\boldsymbol{p}\times\boldsymbol{S}_i) + \frac{4E\xi}{b^4 p_\infty^2}\sum_i \bar{c}_1^{(1,i)}(\boldsymbol{L}\cdot\boldsymbol{S}_i)\,\boldsymbol{b} \\
\Delta_1^{(2,1)}\boldsymbol{p} &= \frac{8E\xi\bar{c}_{1,12}^{(2,1)}}{3b^4 p_\infty}\left(\frac{4}{b^2}(\boldsymbol{b}\cdot\boldsymbol{S}_1)(\boldsymbol{b}\cdot\boldsymbol{S}_1)) + \frac{1}{p_\infty^2}(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_1))\right)\boldsymbol{b} \\
&\quad - \frac{8E\xi\bar{c}_{1,12}^{(2,1)}}{3b^4 p_\infty}(\boldsymbol{b}\cdot\boldsymbol{S}_2)\,\boldsymbol{S}_{1,\perp} - \frac{8E\xi\bar{c}_{1,12}^{(2,1)}}{3b^4 p_\infty}(\boldsymbol{b}\cdot\boldsymbol{S}_1)\,\boldsymbol{S}_{2,\perp} \\
\Delta_1^{(2,2)}\boldsymbol{p} &= \frac{8E\xi\bar{c}_{1,12}^{(2,2)}}{b^4 p_\infty}(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\,\boldsymbol{b} \\
\Delta_1^{(2,3)}\boldsymbol{p} &= \frac{8E\xi\bar{c}_{1,12}^{(2,3)}}{b^4 p_\infty}(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2)\,\boldsymbol{b}
\end{aligned}$$

$$\begin{aligned}
\overline{\Delta\boldsymbol{S}}_{1,1} = \frac{2G\xi E}{L|\boldsymbol{b}|}\Bigg[& c_1^{(1,1)}\,\boldsymbol{L}\times\boldsymbol{S}_1 + \frac{c_1^{(2,1)}}{3}\left(\frac{1}{\boldsymbol{p}^2}\,\boldsymbol{p}\times\boldsymbol{S}_1\,\boldsymbol{p}\cdot\boldsymbol{S}_2 + \frac{2}{\boldsymbol{b}^2}\,\boldsymbol{b}\times\boldsymbol{S}_1\,\boldsymbol{b}\cdot\boldsymbol{S}_2\right) \\
& - c_1^{(2,2)}\,\boldsymbol{S}_1\times\boldsymbol{S}_2 + c_1^{(2,3)}\,\boldsymbol{p}\times\boldsymbol{S}_1\,\boldsymbol{p}\cdot\boldsymbol{S}_2\Bigg]
\end{aligned}$$

$$\begin{aligned}
\Delta_2^{(0)}\boldsymbol{p} =& \frac{\pi}{2b^3 p_\infty}(A[2,1,0,\bar{c}_1^{(0)2}] - 2E\xi\bar{c}_2^{(0)})\,\boldsymbol{b} - \frac{1}{b^2 p_\infty^4}(2E^2\xi^2\bar{c}_1^{(0)2})\,\boldsymbol{p} \\
\Delta_2^{(1)}\boldsymbol{p} =& -\frac{\pi}{2b^5 p_\infty^3}\sum_{i=1}^{2}(A[2,1,2,\bar{c}_1^{(0)}\bar{c}_1^{(1,i)}] - Ep_\infty^2\xi\bar{c}_2^{(1,i)})(2b^2\boldsymbol{p}\times\boldsymbol{S}_i + 3\boldsymbol{L}(\boldsymbol{b}\cdot\boldsymbol{S}_i)) \\
& - \frac{4E^2\xi^2}{b^4 p_\infty^4}\sum_{i=1}^{2}\bar{c}_1^{(0)}\bar{c}_1^{(1,i)}(\boldsymbol{L}\cdot\boldsymbol{S}_i)\,\boldsymbol{p} - \frac{2E^2\xi^2}{b^4 p_\infty^4}\sum_{i=1}^{2}\bar{c}_1^{(1,i)}(2\bar{c}_1^{(0)} + p_\infty^2\bar{c}_1^{(1,i)})(\boldsymbol{p}\cdot\boldsymbol{S}_1)\,\boldsymbol{L} \\
& - \frac{2E^2\xi^2}{b^4 p_\infty^2}\bar{c}_1^{(1,1)}\bar{c}_1^{(1,2)}\,(\boldsymbol{S}_1\times(\boldsymbol{p}\times\boldsymbol{S}_2) + \boldsymbol{S}_2\times(\boldsymbol{p}\times\boldsymbol{S}_1)) \\
& + \frac{3\pi\boldsymbol{b}}{8b^7 p_\infty^3}A[2,1,4,\bar{c}_1^{(1,1)}\bar{c}_1^{(1,2)}]\left(2p_\infty^2(\boldsymbol{b}\cdot\boldsymbol{S}_1)(\boldsymbol{b}\cdot\boldsymbol{S}_2) - 3(\boldsymbol{L}\cdot\boldsymbol{S}_1)(\boldsymbol{L}\cdot\boldsymbol{S}_2)\right) \\
& - \frac{8E^2\xi^2\,\boldsymbol{L}}{b^6 p_\infty^4}\bar{c}_1^{(1,1)}c[1,1,2,p_\infty^2]\,((\boldsymbol{L}\cdot\boldsymbol{S}_2)(\boldsymbol{p}\cdot\boldsymbol{S}_1) + (\boldsymbol{L}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2)) \\
& - \frac{3\pi}{8b^5 p_\infty}A[2,1,4,\bar{c}_1^{(1,1)}\bar{c}_1^{(1,2)}]\,((\boldsymbol{b}\cdot\boldsymbol{S}_2)\boldsymbol{S}_{1,\perp} + (\boldsymbol{b}\cdot\boldsymbol{S}_1)\boldsymbol{S}_{2,\perp}) \\
\Delta_2^{(2,1)}\boldsymbol{p} =& \frac{4E^2\xi^2}{3b^4 p_\infty^4}\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,1)}\,(\boldsymbol{S}_1\times(\boldsymbol{p}\times\boldsymbol{S}_2) + \boldsymbol{S}_2\times(\boldsymbol{p}\times\boldsymbol{S}_1)) \\
& + \frac{16E^2\xi^2\,\boldsymbol{L}}{3b^6 p_\infty^6}\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,1)}((\boldsymbol{L}\cdot\boldsymbol{S}_2)(\boldsymbol{p}\cdot\boldsymbol{S}_1) + (\boldsymbol{L}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2)) \\
& - \frac{3\pi\boldsymbol{b}}{4b^7 p_\infty^5}(A[2,1,4/3,\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,1)}] - Ep_\infty^2\xi\bar{c}_{2,12}^{(2,1)})(5p_\infty^2(\boldsymbol{b}\cdot\boldsymbol{S}_1)(\boldsymbol{b}\cdot\boldsymbol{S}_2) + b^2(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2)) \\
& - \frac{8E^2\xi^2\,\boldsymbol{p}}{3b^6 p_\infty^6}\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,1)}\left(4p_\infty^2(\boldsymbol{b}\cdot\boldsymbol{S}_1)(\boldsymbol{b}\cdot\boldsymbol{S}_2 + b^2(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2) + p_\infty^2 b^2(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right) \\
& + \frac{3\pi}{4b^5 p_\infty^3}(A[2,1,4/3,\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,1)}] - Ep_\infty^2\xi\bar{c}_{2,12}^{(2,1)})\,((\boldsymbol{b}\cdot\boldsymbol{S}_2)\boldsymbol{S}_{1,\perp} + (\boldsymbol{b}\cdot\boldsymbol{S}_1)\boldsymbol{S}_{2,\perp}) \\
\Delta_2^{(2,2)}\boldsymbol{p} =& -\frac{8E^2\xi^2}{b^4 p_\infty^4}\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,2)}\,(\boldsymbol{S}_1\times(\boldsymbol{p}\times\boldsymbol{S}_2) + \boldsymbol{S}_2\times(\boldsymbol{p}\times\boldsymbol{S}_1)) \\
& - \frac{3\pi\,\boldsymbol{b}}{b^5 p_\infty^3}(A[2,1,2,\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,2)}] - Ep_\infty^2\xi\bar{c}_{2,12}^{(2,2)})(\boldsymbol{S}_1\cdot\boldsymbol{S}_2) \\
& - \frac{8E^2\xi^2}{b^4 p_\infty^4}\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,2)}\,((\boldsymbol{p}\cdot\boldsymbol{S}_2)\boldsymbol{S}_1 + (\boldsymbol{p}\cdot\boldsymbol{S}_1)\boldsymbol{S}_2) \\
\Delta_2^{(2,3)}\boldsymbol{p} =& -P((16E^2\xi^2\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,3)}(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2))/(b^4 p_\infty^4)) \\
& - \frac{3\pi\,\boldsymbol{b}}{b^5 p_\infty^3}\left(A[2,1,4,\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,3)}] - Ep_\infty^2\xi\bar{c}_{2,12}^{(2,3)}\right)(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2) \\
& + \frac{4E^2\xi^2}{b^4 p_\infty^2}\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,3)}\,((\boldsymbol{p}\cdot\boldsymbol{S}_2)\boldsymbol{S}_{1,\perp} + (\boldsymbol{p}\cdot\boldsymbol{S}_1)\boldsymbol{S}_{2,\perp})
\end{aligned}$$

$$\begin{aligned}
\overline{\Delta\boldsymbol{S}}_{1,2a} =& \frac{\pi\xi E}{2Lb^2}c_2^{(1,1)}\,\boldsymbol{L}\times\boldsymbol{S}_1 + \frac{1}{2Lb^2}\left[-\pi(1-3\xi) - 2\pi\xi^2E^2\frac{1}{\boldsymbol{p}^2} - 2\pi\xi^2E^2\partial\right]\left(c_1^{(0)}c_1^{(1,1)}\right)\boldsymbol{L}\times\boldsymbol{S}_1 \\
\overline{\Delta\boldsymbol{S}}_{1,2b} =& \frac{\boldsymbol{p}^2}{8Lb^2}\left[3\pi(1-3\xi) + 12\pi\xi^2E^2\frac{1}{\boldsymbol{p}^2} + 6\pi\xi^2E^2\partial\right]\left(c_1^{(1,1)}c_1^{(1,2)}\right)\left(-S_{1,z}S_{2,y}\,\hat{\boldsymbol{x}} + S_{1,x}S_{2,y}\,\hat{\boldsymbol{z}}\right) \\
& + \frac{1}{Lb^2}\left[-2\xi^2E^2\right]\left(c_1^{(1,1)}c_1^{(1,2)}\right)\left(S_{1,y}S_{2,x}\,\hat{\boldsymbol{x}} + S_{1,y}S_{2,z}\,\hat{\boldsymbol{z}} - (S_{1,x}S_{2,x} + S_{1,z}S_{2,z})\,\hat{\boldsymbol{y}}\right) \\
\overline{\Delta\boldsymbol{S}}_{1,2c} =& +\frac{1}{8Lb^2}[\pi\xi E]\left(c_2^{(2,1)}\right)\left(-S_{1,y}S_{2,z}\,\hat{\boldsymbol{x}} + (-3S_{1,z}S_{2,x} + S_{1,x}S_{2,z})\,\hat{\boldsymbol{y}} + 3S_{1,y}S_{2,x}\,\hat{\boldsymbol{z}}\right) \\
& + \frac{1}{3L^3}\left[2\xi^2E^2\right]\left(c_1^{(0)}c_1^{(2,1)}\right)\left(-S_{1,y}S_{2,x}\,\hat{\boldsymbol{x}} + (S_{1,x}S_{2,x} - S_{1,z}S_{2,z})\,\hat{\boldsymbol{y}} + S_{1,y}S_{2,z}\,\hat{\boldsymbol{z}}\right) \\
& + \frac{1}{8Lb^2}\left[\pi(1-3\xi) + 4\pi\xi^2E^2\frac{1}{\boldsymbol{p}^2} + 2\pi\xi^2E^2\partial\right]\left(c_1^{(0)}c_1^{(2,1)}\right)\left((S_{1,y}S_{2,z}\,\hat{\boldsymbol{x}}) - S_{1,x}S_{2,z}\,\hat{\boldsymbol{y}}\right) \\
& + \frac{1}{8Lb^2}\left[3\pi(1-3\xi) + 4\pi\xi^2E^2\frac{1}{\boldsymbol{p}^2} + 6\pi\xi^2E^2\partial\right]\left(c_1^{(0)}c_1^{(2,1)}\right)\left(S_{1,z}S_{2,x}\,\hat{\boldsymbol{y}} - S_{1,y}S_{2,x}\,\hat{\boldsymbol{z}}\right) \\
\overline{\Delta\boldsymbol{S}}_{1,2d} =& -\frac{1}{2Lb^2}[\pi\xi E]\left(c_2^{(2,2)}\right)\boldsymbol{S}_1\times\boldsymbol{S}_2 \\
& + \frac{1}{2Lb^2}\left[\pi(1-3\xi) + 2\pi\xi^2E^2\frac{1}{\boldsymbol{p}^2} + 2\pi\xi^2E^2\partial\right]\left(c_1^{(0)}c_1^{(2,2)}\right)\boldsymbol{S}_1\times\boldsymbol{S}_2 \\
\overline{\Delta\boldsymbol{S}}_{1,2e} =& -\frac{\boldsymbol{p}^2}{2Lb^2}[\pi\xi E]\left(c_2^{(2,3)}\right)\left(S_{1,y}S_{2,z}\,\hat{\boldsymbol{x}} - S_{1,x}S_{2,z}\,\hat{\boldsymbol{y}}\right) \\
& - \frac{1}{Lb^2}\left[2\xi^2E^2\right]\left(c_1^{(0)}c_1^{(2,3)}\right)\left(-S_{1,y}S_{2,x}\,\hat{\boldsymbol{x}} + (S_{1,x}S_{2,x} - S_{1,z}S_{2,z})\,\hat{\boldsymbol{y}} + S_{1,y}S_{2,z}\,\hat{\boldsymbol{z}}\right) \\
& + \frac{\boldsymbol{p}^2}{2Lb^2}\left[\pi(1-3\xi) + 4\pi\xi^2E^2\frac{1}{\boldsymbol{p}^2} + 2\pi\xi^2E^2\partial\right]\left(c_1^{(0)}c_1^{(2,3)}\right)\left(S_{1,y}S_{2,z}\,\hat{\boldsymbol{x}} - S_{1,x}S_{2,z}\,\hat{\boldsymbol{y}}\right) \\
\overline{\Delta\boldsymbol{S}}_{1,2f} =& +\frac{\pi\xi^2E^2}{8Lb^2}\left(c_1^{(1,1)} + c_1^{(1,2)}\right)c_1^{(2,1)}\left(S_{1y}S_{2,z}\,\hat{\boldsymbol{x}} - (S_{1,z}S_{2,x} + S_{1,x}S_{2,z})\,\hat{\boldsymbol{y}} + S_{1,y}S_{2,x}\,\hat{\boldsymbol{z}}\right) \\
& + \frac{\xi^2E^2}{3Lb^2}\Big[\left(4c_1^{(1,1)} + 2c_1^{(1,2)}\right)c_1^{(2,1)}\,S_{1y}S_{2,x}\,\hat{\boldsymbol{x}} + \left(2c_1^{(1,1)} + 4c_1^{(1,2)}\right)c_1^{(2,1)}\,S_{1y}S_{2,z}\,\hat{\boldsymbol{z}} \\
& \qquad + \left(4c_1^{(1,1)} - 2c_1^{(1,2)}\right)c_1^{(2,1)}\,S_{1x}S_{2,x}\,\hat{\boldsymbol{y}} + \left(2c_1^{(1,1)} - 4c_1^{(1,2)}\right)c_1^{(2,1)}\,S_{1z}S_{2,z}\,\hat{\boldsymbol{y}}\Big] \\
& + \frac{\xi^2E^2}{Lb^2}\Big[2\left(c_1^{(1,1)} + c_1^{(1,2)}\right)c_1^{(2,2)}\,S_{1y}(S_{2,x}\,\hat{\boldsymbol{x}} + S_{2,z}\,\hat{\boldsymbol{z}}) - 4c_1^{(1,1)}\,c_1^{(2,2)}\,S_{2,y}(S_{1,x}\,\hat{\boldsymbol{x}} + S_{1,z}\,\hat{\boldsymbol{z}}) \\
& \qquad + 2\left(c_1^{(1,1)} - c_1^{(1,2)}\right)c_1^{(2,2)}\,(S_{1,x}S_{2,x} - S_{1,z}S_{2,z})\,\hat{\boldsymbol{y}}\Big] \\
& + \frac{2\xi^2E^2}{Lb^2}\boldsymbol{p}^2\left[c_1^{(1,2)}\,c_1^{(2,3)}\,S_{2,x}(S_{1,y}\,\hat{\boldsymbol{x}} - S_{1,x}\,\hat{\boldsymbol{y}}) + c_1^{(1,1)}\,c_1^{(2,3)}\,S_{2,z}(S_{1,z}\,\hat{\boldsymbol{y}} + S_{1,y}\,\hat{\boldsymbol{z}})\right]
\end{aligned}$$

Examples of structure: (1) scattering observables of spinning particles directly from amplitudes

$$\Delta\mathcal{O} = e^{-i\chi\mathcal{D}}[\mathcal{O}, e^{i\chi\mathcal{D}}]$$

$$[f,g] \equiv i\{f,g\} \qquad \chi\,\mathcal{D}\,g \equiv \chi g + iD_{SL}(\chi, g) \qquad \mathcal{D}_{SL}\,(f,g) \equiv -\sum_{a=1,2} \epsilon^{ijk} S_a^k \frac{\partial f}{\partial S_a^i} \frac{\partial g}{\partial L^j}$$

$$\chi = \chi_1 + \chi_2 + \cdots$$ eikonal phase

– points to remarkable simplicity and structure in the solution to spinning Hamilton's eqs.

– tested with complete new state of the art calculations to second order in spin at $\mathcal{O}(G_N^2)$

(2) Analytic structure at fixed order in Newton's constant; closer connections between amplitudes and observables, etc

A plan

- Classical limits of quantum field theories

- From classical amplitudes to observables

- Spinless observables at 3rd and 4th Post-Minkowskian order (i.e. $\mathcal{O}(G_N^3)$ and $\mathcal{O}(G_N^4)$)

- A classical higher-spin formalism

- The spinning eikonal conjecture and its checks

- Conclusions and outlook

Goal: interaction of classical heavy particles while integrating out off-shell gravitons

$$\text{GR + (spinning) matter} \xrightarrow{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{\text{off shell}} + h_{\mu\nu}^{\text{radiation}}} S_{eff} = S_{eff}(\text{matter}, h_{\mu\nu}^{\text{radiation}})$$

Goal: interaction of classical heavy particles while integrating out off-shell gravitons

GR + (spinning) matter $\xrightarrow{g_{\mu\nu} = \eta_{\mu\nu} + h^{\text{off shell}}_{\mu\nu} + h^{\text{radiation}}_{\mu\nu}}$ $S_{eff} = S_{eff}(\text{matter}, h^{\text{radiation}}_{\mu\nu})$

Why quantum $\longrightarrow$ classical?

- Quantum mechanical framework is more constraining (manifest unitarity, Lorentz invariance)

Intuitive analogy: Integrable models --

diagonalize simultaneously all commuting Hamiltonians rather than
only the one giving the desired time evolution

- Lots of advanced efficient technology developed that can be repurposed

Generalized unitarity, double copy, integration methods for Feynman integrals

Goal: interaction of classical heavy particles while integrating out off-shell gravitons

GR + (spinning) matter $\xrightarrow{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{\text{off shell}} + h_{\mu\nu}^{\text{radiation}}}$ $S_{eff} = S_{eff}(\text{matter}, h_{\mu\nu}^{\text{radiation}})$

Why quantum $\longrightarrow$ classical?

- Quantum mechanical framework is more constraining (manifest unitarity, Lorentz invariance)

Intuitive analogy: Integrable models --

diagonalize simultaneously all commuting Hamiltonians rather than only the one giving the desired time evolution

- Lots of advanced efficient technology developed that can be repurposed

Generalized unitarity, double copy, integration methods for Feynman integrals

(Semi) classical limit: "all conserved charges are large"

Analogous in spirit with the semiclassical limit for strings in anti-de-Sitter space

Gubser, Klebanov, Polyakov; Tseytlin et al…

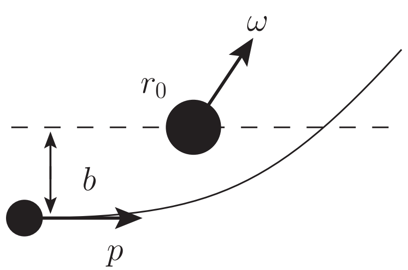
Goal: interaction of classical heavy particles with spin while integrating out short distance gravitons

GR + (spinning) matter $\xrightarrow[\text{Subtlety: radiation can also be off shell}]{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{\text{off shell}} + h_{\mu\nu}^{\text{radiation}}}$ $S_{eff} = S_{eff}(\text{matter}, h_{\mu\nu}^{\text{radiation}})$

(Semi) classical limit: "all conserved charges are large"

- In $\hbar = 1$ units:

"classical": de Broglie wavelength λ of particles is much smaller than their



- separation: $\lambda \sim \dfrac{1}{|\boldsymbol{p}|} \ll |\boldsymbol{b}| \sim \dfrac{1}{|\boldsymbol{q}|} \quad \Longrightarrow \quad |\boldsymbol{L}| = |\boldsymbol{b} \times \boldsymbol{p}| \gg 1 \; ; \; |\boldsymbol{p}| \gg |\boldsymbol{q}|$

- size: $\lambda \sim \dfrac{1}{|m r_0 \boldsymbol{\omega}|} \ll r_0 \quad \Longrightarrow \quad |\boldsymbol{S}| = m r_0^2 |\boldsymbol{\omega}| \gg 1$

Classical limit/expansion: $\mathcal{O}(1/|\boldsymbol{L}|) \sim \mathcal{O}(1/|\boldsymbol{S}|) \sim \mathcal{O}(|\boldsymbol{q}|/m) \sim \mathcal{O}(q/m)$

► A field theory approach to classical dynamics requires higher-spin particles

Structure of two-body potential: $\mathcal{F}_{\boldsymbol{r}}\Big[V(\boldsymbol{p}, \boldsymbol{r}, \boldsymbol{S}), \boldsymbol{q}\Big] \sim \dfrac{c_{ijk}(\boldsymbol{p})}{|\boldsymbol{q}|^3} \, (\, Gm|\boldsymbol{q}|\,)^i \left(\dfrac{\boldsymbol{q} \cdot \boldsymbol{S}}{m}\right)^j (\, R\,|\boldsymbol{q}|\,)^k$

Two related approaches to interactions of matter and gravity

First quantized/worldline

$$S^{wl} = -m\int d\tau \sqrt{u^2} + \mathcal{L}_{\text{int}}^{wl}(S(u))$$

- Matter particles are treated as sources
  - always on shell
  - no antiparticles

- The effective action obtained by integrating out short distance gravitons

$$Z = \int Dh_{\mu\nu}^{\text{off shell}} e^{iS^{wl}} = e^{i\Gamma[S]}$$

is the desired action

Second quantized

$$S = \int d^4x \phi_s(\Box + m^2)\phi_s + \mathcal{L}_{\text{int}}(\phi_s)$$

- Formally geared towards quantum physics
  - includes antiparticles
  - may separate them
    1. matching onto EFT with no antiparticles
    2. Foldy-Wouthuysen transformation(s)
    3. other means?

Cheung, Rothstein, Solon

- The desired effective action obtained by integrating out short distance gravitons is the action of that EFT

- Amplitudes contain superclassical/iteration terms while the action doesn't:

$$e^{i\Gamma} = 1 + i\Gamma + \frac{1}{2}(i\Gamma)^2 + \dots$$

"Interesting" off shell gravitons:

hard: $(\omega, \ell) \sim (\sqrt{s}, \sqrt{s})$

soft: $(\omega, \ell) \sim (|\boldsymbol{q}|, \boldsymbol{q})$

- potential $(\omega, \ell) \sim (v|\boldsymbol{q}|, \boldsymbol{q})$
- radiation $(\omega, \ell) \sim (v|\boldsymbol{q}|, v\boldsymbol{q})$

Beneke, Smirnov

Classical physics from quantum scattering amplitudes

Construct the effective interaction Lagrangian; two main requirements

1. Enforces that there are no anti-matter particles (restriction to classical phys.)
2. Loop-by-loop order, it yields the same physics/amplitudes as the full theory

Goldberger, Rothstein
Neill, Rothstein; Cheung, Rothstein, Solon

- Same approach as in NRQCD/NRGR
  - Matter EFT: $L = L_{\rm kin} + L_{\rm int}$

$$L_{\rm kin} = A^\dagger(-\mathbf{p})(i\partial_t + \sqrt{\mathbf{p}^2 + m_A^2})A(\mathbf{p}) + B^\dagger(-\mathbf{p})(i\partial_t + \sqrt{\mathbf{p}^2 + m_B^2})B(\mathbf{p})$$

$$L_{\rm int} = -V(G_N, \mathbf{k}, \mathbf{k}')A^\dagger(-\mathbf{p}')B^\dagger(\mathbf{p}')A(-\mathbf{p})B(\mathbf{p})$$

$$V(G_N, \mathbf{p}, \mathbf{p}') = \sum_{n\geq 1} G_N^n V_n(\mathbf{p}, \mathbf{p}')$$

- Fix $V(G_N, \mathbf{p}, \mathbf{p}')$ to match the classical limit of amplitudes of the full theory

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$$L_{\text{int}} = -V(G_N, \mathbf{k}, \mathbf{k}')A^\dagger(-\mathbf{p}')B^\dagger(\mathbf{p}')A(-\mathbf{p})B(\mathbf{p})$$

$$V(G_N, \mathbf{p}, \mathbf{p}') = \sum_{n\geq 1} G_N^n V_n(\mathbf{p}, \mathbf{p}')$$

- Fix $V(G_N, \mathbf{p}, \mathbf{p}')$ to match the classical limit of amplitudes of the full theory

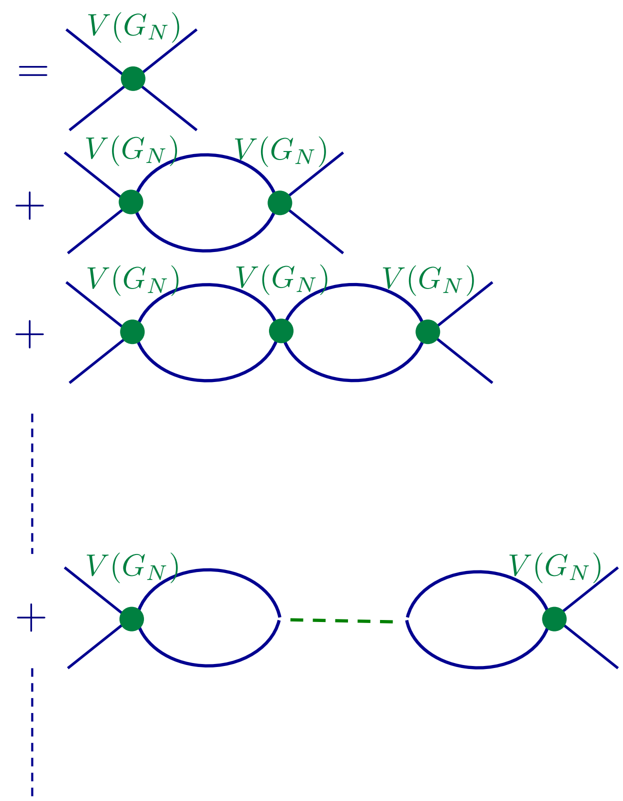
- Relation to classical mechanics Hamiltonian:

$$H = \sqrt{\mathbf{p}^2 + m_A^2} + \sqrt{\mathbf{p}'^2 + m_B^2} + V(G_N, \mathbf{p}, \mathbf{p}')$$

- EFT also covering nonconservative processes, has also graviton-dependent terms; focus on conservative

Amplitudes of the effective field theory

$$A_{\text{EFT}} = \sum_{n\geq 1} G_N^n A_{\text{EFT}}^{(n)}$$

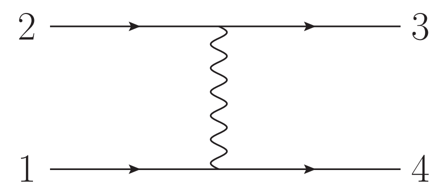


$$V(G_N, \mathbf{p}, \mathbf{p}') = \sum_{n\geq 1} G_N^n V_n(\mathbf{p}, \mathbf{p}')$$

- propagators have only matter poles
- each vertex has (in principle) all powers of G_N
- by construction, integration gives the same IR divergences as the amplitude of the full theory
- No need for fancy technology to construct the integrands
- Equate them with amplitudes of the full theory and extract $V_n(\mathbf{p}, \mathbf{p}')$
- $V_1 = A_{\text{EFT}}^{(1)}$ is the tree-level amplitude of the full theory

Starting point: the tree-level potential

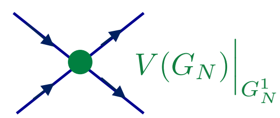
$$\mathcal{M}_{\text{tree}} \propto \frac{G_N}{q^2}\frac{(2(p_1\cdot p_2)^2 - m_1^2 m_2^2)}{m_1^2 m_2^2} = \frac{G_N}{q^2}(2\sigma^2 - 1)$$



$$\sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

$$H(\mathbf{p},\mathbf{r}) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{p},\boldsymbol{r}) \ , \quad V(\mathbf{p},\mathbf{r}) = \sum_{i=1}^{\infty} c_i(\boldsymbol{p}^2)\left(\frac{G_N}{|\boldsymbol{r}|}\right)^i$$

$$c_1(\boldsymbol{p}^2) \propto (1 - 2\sigma^2)$$



However, matching can be tedious at higher orders because of the structure of EFT integrals

Is there a better way?

Conservative amplitudes are (by definition) ellastic

Potential region $\longrightarrow$ no on-shell gravitons

only 2-particle cuts

$$S = 1 + iT \qquad SS^\dagger = 1 \quad \Longrightarrow \quad 2\mathrm{Im}T = TT^\dagger$$

Theory of a single scalar $\longrightarrow$ better solution: $S = e^{i\hat{\Phi}}$ $\longleftarrow$ phase shift operator

S - matrix elements:

2-particle phase space

$$\langle p_{1f}, p_{2f}|S|p_{1i}, p_{2i}\rangle = \langle p_{1f}, p_{2f}|p_{1i}, p_{2i}\rangle + \langle p_{1f}, p_{2f}|i\hat{\Phi}|p_{1i}, p_{2i}\rangle + \frac{1}{2}\langle p_{1f}, p_{2f}|\underbrace{(i\hat{\Phi})(i\hat{\Phi})}_{\text{iteration terms}}|p_{1i}, p_{2i}\rangle + \dots$$

Both the complete theory and the EFT should have this property; testable in EFT

Classical limit: $\langle \boldsymbol{p}+\boldsymbol{q}, E|S|\boldsymbol{p}, E\rangle = e^{i\tilde{I}_r(\boldsymbol{q})}\big(1+\mathcal{O}(\hbar)\big) = \sum_k \mathcal{M}_k(\boldsymbol{q}) \qquad \tilde{I}_r(\boldsymbol{q}) = \sum_n \tilde{I}_{r,n}(\boldsymbol{q})$

$$\mathcal{M}_1(\boldsymbol{q}) = \tilde{I}_{r,1}(\boldsymbol{q}) \qquad \mathcal{M}_2(\boldsymbol{q}) = \tilde{I}_{r,2}(\boldsymbol{q}) + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,1}\tilde{I}_{r,1}}{Z_1}$$

$$\mathcal{M}_3(\boldsymbol{q}) = \tilde{I}_{r,3}(\boldsymbol{q}) + \int_{\boldsymbol{\ell}} \frac{\tilde{I}^3_{r,1}}{Z_1 Z_2} + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,1}\tilde{I}_{r,2}}{Z_1}$$

$$\mathcal{M}_4(\boldsymbol{q}) = \tilde{I}_{r,4}(\boldsymbol{q}) + \int_{\boldsymbol{\ell}} \frac{\tilde{I}^4_{r,1}}{Z_1 Z_2 Z_3} + \int_{\boldsymbol{\ell}} \frac{\tilde{I}^2_{r,1}\tilde{I}_{r,2}}{Z_1 Z_2} + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,1}\tilde{I}_{r,3}}{Z_1} + \int_{\boldsymbol{\ell}} \frac{\tilde{I}^2_{r,2}}{Z_1}$$

$$Z_j = -4E|\boldsymbol{p}|\big((\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2 + \dots + \boldsymbol{\ell}_j)\cdot\hat{\boldsymbol{z}} + i\epsilon\big)$$

# of integrals = # of 2-particle cuts

Conservative amplitudes are (by definition) elastic

S - matrix elements:

2-particle phase space

$$\langle p_{1f},p_{2f}|S|p_{1i},p_{2i}\rangle = \langle p_{1f},p_{2f}|p_{1i},p_{2i}\rangle + \langle p_{1f},p_{2f}|i\hat{\Phi}|p_{1i},p_{2i}\rangle + \frac{1}{2}\langle p_{1f},p_{2f}|\underbrace{(i\hat{\Phi})(i\hat{\Phi})}|p_{1i},p_{2i}\rangle + \dots$$

iteration terms

Classical limit: $\langle \boldsymbol{p}+\boldsymbol{q},E|S|\boldsymbol{p},E\rangle = e^{i\tilde{I}_r(\boldsymbol{q})}\big(1+\mathcal{O}(\hbar)\big)$ $\qquad \tilde{I}_r(\boldsymbol{q}) = \sum_n \tilde{I}_{r,n}(\boldsymbol{q})$

What is $\tilde{I}_r(\boldsymbol{q})$:

- With fixed time and coordinate:

$$\langle t_f,x_f|S|t_i,x_i\rangle = e^{iS_{cl}(\boldsymbol{x}(t),\boldsymbol{p}(t))}\big(1+\mathcal{O}(\hbar)\big) \qquad S_{cl}(\boldsymbol{x}(t),\boldsymbol{p}(t)) = \int_{\text{trajectory}}(\boldsymbol{p}d\boldsymbol{x} - H(\boldsymbol{p},\boldsymbol{x}))$$

- Legendre-transform to fixed energy:

$$I_r = S_{cl} + Et\Big|_{t_i}^{t_f} = \int_{\text{trajectory}}\boldsymbol{p}d\boldsymbol{x} \qquad dI_r = -\chi dJ + \Delta t dE + \dots$$

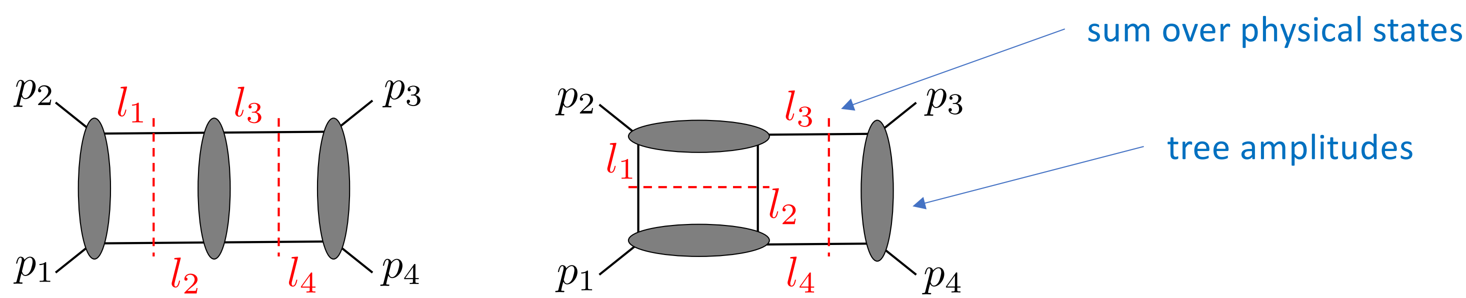
Now that we know what we want to compute…

Amplitudes from generalized unitarity and double copy

Amplitudes' integrands = rational functions with prescribed poles and residues

Poles = graph structure with given number of external lines and loops

Residues = generalized cuts = products of tree amplitudes



Generalized unitarity = systematic reconstruction of integrands from this data
= effectively, a reorganization of Feynman graphs
which makes explicit use if gauge invariance

Bern, Dixon, Kosower
Britto, Cachazo, Feng
Bern, Carrasco, Johansson, Kosower;...

Amplitudes from generalized unitarity and double copy

Gravity trees from gauge theory trees through KLT relations: Kawai, Lewellen, Tye

$$M_4^{\rm tr}(1,2,3,4) = -is_{12}A_4^{\rm tr}(1,2,3,4)A_4^{\rm tr}(1,2,4,3)$$

$$M_5^{\rm tr}(1,2,3,4,5) = \ \ is_{12}s_{34}A_5^{\rm tr}(1,2,3,4,5)A_5^{\rm tr}(2,1,4,3,5) + (2 \leftrightarrow 3)$$

$$M_6^{\rm tr} = 12 \text{ terms of the type } s^3 A_6 A_6$$

- Hold state-by-state for external lines, following addition of helicities

e.g. $\text{scalar} \longleftrightarrow \text{scalar} \times \widetilde{\text{scalar}}$ $\quad h^{++} \longleftrightarrow A^+ \times \tilde{A}^+$ $\quad \varphi^{+-} \longleftrightarrow A^+ \times \tilde{A}^-$

- Hold in any dimension; implements all simplifications required by gauge invariance

Color/kinematics, double-copy and generalized double-copy

Bern, Carrasco, Johansson;
Bern, Carrasco, Chen, Johansson, RR

- Full power in relating loop level gauge and gravity amplitudes
- May be used cut-by-cut; important for obtaining a graph-based organization

Simplify physical state projector with generalized Ward identities:

$$\mathcal{P}_{\mu\nu}(p) = \eta_{\mu\nu} - \frac{p_\mu r_\nu + p_\nu r_\mu}{p\cdot r}$$

$$p_{3_{\gamma_1}}\mathcal{M}^{\gamma_1\gamma_2,\delta_1\delta_2}(1^s,2^s,3^h,4^h) = 0 \qquad p_{3_{\gamma_2}}\mathcal{M}^{\gamma_1\gamma_2,\delta_1\delta_2}(1^s,2^s,3^h,4^h) = 0$$

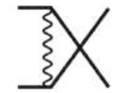
$$p_{4_{\delta_1}}\mathcal{M}^{\gamma_1\gamma_2,\delta_1\delta_2}(1^s,2^s,3^h,4^h) = 0 \qquad p_{4_{\delta_2}}\mathcal{M}^{\gamma_1\gamma_2,\delta_1\delta_2}(1^s,2^s,3^h,4^h) = 0$$

Algorithm to put amplitudes in this form Kosmopoulos

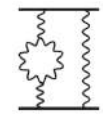
Cheung, Rothestein, Solon
Bern, Cheung, RR, Shen, Solon, Zeng

Cuts should weed out graphs with:

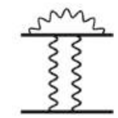
- intersection of matter lines (looking for long distance interactions)



- pure graviton loops (intuitively quantum mechanical)



- gravitons starting and ending on the same matter line (intuitively not potential)



Integral evaluation:

Sudakov
Beneke, Smirnov;
HQET

1. Expand truncated integrand in the soft region; use special choice of momenta

$$\bar{p}_1 = p_1 + q/2 \qquad \bar{p}_2 = p_2 - q/2 \qquad \bar{p}_i^2 = \bar{m}_i^2 \qquad \bar{p}_i \cdot q = 0 \qquad y = \frac{\bar{p}_1 \cdot \bar{p}_2}{\bar{m}_1 \bar{m}_2} = \sigma + \mathcal{O}(q^2)$$

$$\ell_i^2 \to \ell_i^2 \qquad (\ell_i + p_j)^2 - m_j^2 \to 2\bar{p}_j \cdot \ell_i + \mathcal{O}(q^2) \qquad I(p_1, p_2, q) \to \sum_k (q^2)^{a_k} \bar{m}_1^{b_{1k}} \bar{m}_2^{b_{2k}} I_k(y)$$

Single-scale to all orders in G!

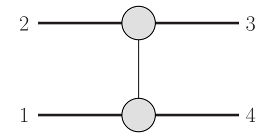
2. Construct differential equations for the independent integrals $\quad \dfrac{dI_i}{dy} = A_{ij}(y, \epsilon) I_i$

Henn;
Parra-Martinez, Ruf, Zeng

3. Solve them using boundary conditions in the potential region (i.e. evaluate integrals to leading order in v)

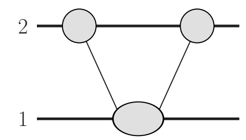
The classical limit of the 4-scalar amplitude through 2 loops:

$$\tilde{I}_{r,1}(\boldsymbol{q}) = \frac{16\pi G}{\boldsymbol{q}^2} m^4 \nu^2 (2\sigma^2 - 1)$$



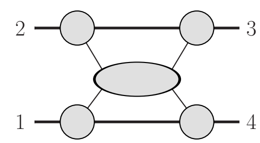
Westphal

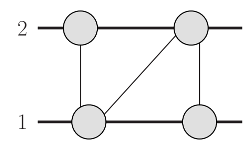
$$\tilde{I}_{r,2}(\boldsymbol{q}) = \frac{6\pi^2 G^2}{|\boldsymbol{q}|} m^5 \nu^2 (5\sigma^2 - 1)$$

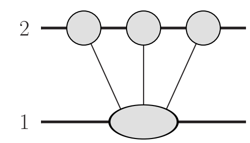


Damour

$$\tilde{I}_{r,3}(\boldsymbol{q}) = 2\pi G^3 m^6 \nu^2 \ln \boldsymbol{q}^2 \left[\frac{2(2\sigma^2-1)^3 h^2}{3(\sigma^2-1)^2} - \frac{3h^2(5\sigma^2-1)(2\sigma^2-1)}{\sigma^2-1} - 18\sigma^2 + 1 \right.$$
$$+ \frac{2\nu}{3}(2\sigma^3 + 54\sigma^2 + 103\sigma - 3)$$
$$\left. - \frac{16\nu(-4\sigma^4 + 12\sigma^2 + 3)}{\sqrt{\sigma^2-1}} \operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}} \right]$$







Bern, Cheung, RR, Solon, Shen, Zeng

(organized in this form by Kälin, Porto)

$$m = m_1 + m_2\ , \quad \nu = \frac{m_1 m_2}{m^2}\ , \quad \sigma = \frac{p_1 \cdot p_2}{m_1 m_2}\ , \quad h = \sqrt{1 + 2\nu(\sigma - 1)}$$

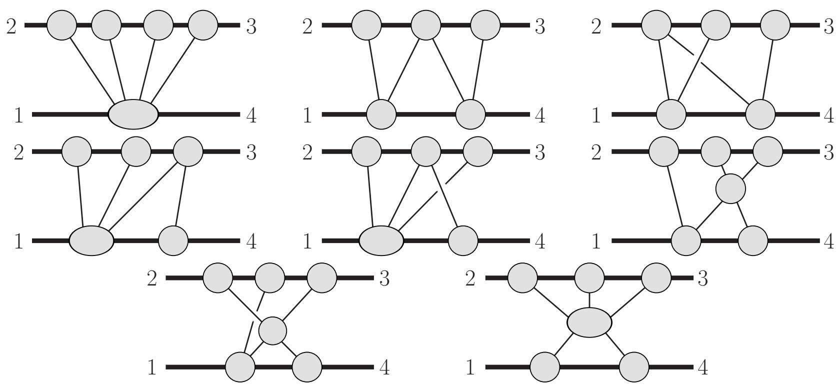
Independent verification of $\mathcal{O}(G_N^3)$ amplitude/Hamiltonian

Cheung, Solon

Kälin, Liu, Porto

The classical limit of the 4-scalar amplitude *in the potential region* at 3 loops:

Bern, Parra-Martinez, RR, Ruf, Solon, Shen, Zeng



$$\tilde{I}_{r,4}(\boldsymbol{q}) = G^4 m^7 \nu^2 |\boldsymbol{q}| \left(\frac{\boldsymbol{q}^2}{4^{\frac{1}{3}}\tilde{\mu}^2}\right)^{-3\epsilon} \pi^2 \left[\mathcal{M}_4^{\mathrm{p}} + \nu\left(\frac{\mathcal{M}_4^{\mathrm{t}}}{\epsilon} + \mathcal{M}_4^{\mathrm{f}}\right)\right]$$

$$\mathcal{M}_4^{\mathrm{p}} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2 - 1\right)} \qquad \mathcal{M}_4^{\mathrm{t}} = h_1 + h_2 \log\left(\tfrac{\sigma+1}{2}\right) + h_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}$$

$$\begin{aligned}
\mathcal{M}_4^{\mathrm{f}} = {} & h_4 + h_5 \log\left(\tfrac{\sigma+1}{2}\right) + h_6 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + h_7 \log(\sigma) - h_2 \frac{2\pi^2}{3} + h_8 \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} + h_9 \left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) + \frac{1}{2}\log^2\left(\tfrac{\sigma+1}{2}\right)\right] \\
& + h_{10}\left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) - \frac{\pi^2}{6}\right] + h_{11}\left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) - \mathrm{Li}_2\left(\tfrac{\sigma-1}{\sigma+1}\right) + \frac{\pi^2}{3}\right] + h_2 \frac{2\sigma(2\sigma^2-3)}{(\sigma^2-1)^{3/2}}\left[\mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right] \\
& + \frac{2h_3}{\sqrt{\sigma^2-1}}\left[\mathrm{Li}_2\left(1-\sigma-\sqrt{\sigma^2-1}\right) - \mathrm{Li}_2\left(1-\sigma+\sqrt{\sigma^2-1}\right) + 5\mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - 5\mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) + 2\log\left(\tfrac{\sigma+1}{2}\right)\operatorname{arccosh}(\sigma)\right] \\
& + h_{12}\mathrm{K}^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{13}\mathrm{K}\left(\tfrac{\sigma-1}{\sigma+1}\right)\mathrm{E}\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{14}\mathrm{E}^2\left(\tfrac{\sigma-1}{\sigma+1}\right)
\end{aligned}$$

$$m = m_1 + m_2\,, \quad \nu = \frac{m_1 m_2}{m^2}\,, \quad \sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

$$\begin{aligned}
h_1 &= \frac{1151 - 3336\sigma + 3148\sigma^2 - 912\sigma^3 + 339\sigma^4 - 552\sigma^5 + 210\sigma^6}{12(\sigma^2-1)} \\
h_2 &= \frac{1}{2}\left(5 - 76\sigma + 150\sigma^2 - 60\sigma^3 - 35\sigma^4\right) \\
h_3 &= \sigma\frac{\left(-3+2\sigma^2\right)}{4(\sigma^2-1)}\left(11 - 30\sigma^2 + 35\sigma^4\right) \\
h_4 &= \frac{1}{144(\sigma^2-1)^2\sigma^7}\left(-45 + 207\sigma^2 - 1471\sigma^4 + 13349\sigma^6\right. \\
&\quad - 37566\sigma^7 + 104753\sigma^8 - 12312\sigma^9 - 102759\sigma^{10} - 105498\sigma^{11} \\
&\quad \left. + 134745\sigma^{12} + 83844\sigma^{13} - 101979\sigma^{14} + 13644\sigma^{15} + 10800\sigma^{16}\right) \\
h_5 &= \frac{1}{4(\sigma^2-1)}\left(1759 - 4768\sigma + 3407\sigma^2 - 1316\sigma^3 + 957\sigma^4\right. \\
&\quad \left. - 672\sigma^5 + 341\sigma^6 + 100\sigma^7\right) \\
h_6 &= \frac{1}{24(\sigma^2-1)^2}\left(1237 + 7959\sigma - 25183\sigma^2 + 12915\sigma^3 + 18102\sigma^4\right. \\
&\quad \left. - 12105\sigma^5 - 9572\sigma^6 + 2973\sigma^7 + 5816\sigma^8 - 2046\sigma^9\right) \\
h_7 &= 2\sigma\frac{\left(-852 - 283\sigma^2 - 140\sigma^4 + 75\sigma^6\right)}{3(\sigma^2-1)} \\
h_8 &= \frac{\sigma}{8(\sigma^2-1)^2}\left(-304 - 99\sigma + 672\sigma^2 + 402\sigma^3 - 192\sigma^4 - 719\sigma^5\right. \\
&\quad \left. - 416\sigma^6 + 540\sigma^7 + 240\sigma^8 - 140\sigma^9\right) \\
h_9 &= \frac{1}{2}\left(52 - 532\sigma + 351\sigma^2 - 420\sigma^3 + 30\sigma^4 - 25\sigma^6\right) \\
h_{10} &= 2\left(27 + 90\sigma^2 + 35\sigma^4\right) \\
h_{11} &= 20 + 111\sigma^2 + 30\sigma^4 - 25\sigma^6 \\
h_{12} &= \frac{834 + 2095\sigma + 1200\sigma^2}{2(\sigma^2-1)} \\
h_{13} &= -\frac{1183 + 2929\sigma + 2660\sigma^2 + 1200\sigma^3}{2(\sigma^2-1)} \\
h_{14} &= \frac{7\left(169 + 380\sigma^2\right)}{4(\sigma-1)}
\end{aligned}$$

Simple dependence on symmetric mass ratio consistent with arguments of Damour & Bini, Damour, Geralico

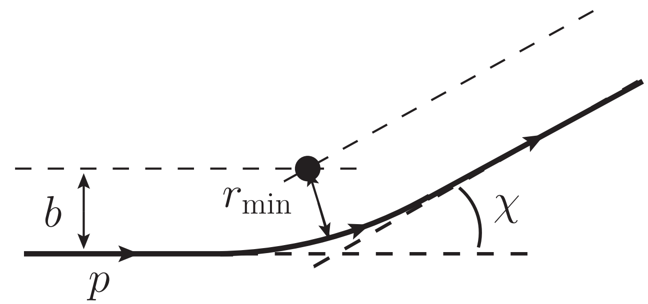
Open-orbit observables:

Bern, Cheung, RR, Solon, Shen, Zeng

Bern, Parra-Martinez, RR, Ruf, Solon, Shen, Zeng

$\propto$ transverse Fourier

$$\chi = \left(\frac{\partial I_r(J,E)}{\partial J}\right)_{E,m} = \frac{1}{|\boldsymbol{p}|}\left(\frac{\partial I_r}{\partial b}\right)_{E,m}$$



All available PN data is reproduced through $\mathcal{O}(G_N^4)$ — Blümlein, Maier, Marquard, Schäfer

Other approaches — Kosower, Maybee, O'Connell

Direct valuation of the impulse $\langle \Delta p^\mu \rangle = \langle \psi | i[\mathbb{P}^\mu, T]\psi\rangle + \langle \psi | T^\dagger [\mathbb{P}^\mu, T]\psi\rangle$

Used to radiated momentum at 2PM — Herrmann, Parra-Martinez, Ruf, Zeng

Closed-orbit observables:

- Quasi-circular orbits: analytic continuation from open-orbit observables — Kälin, Porto

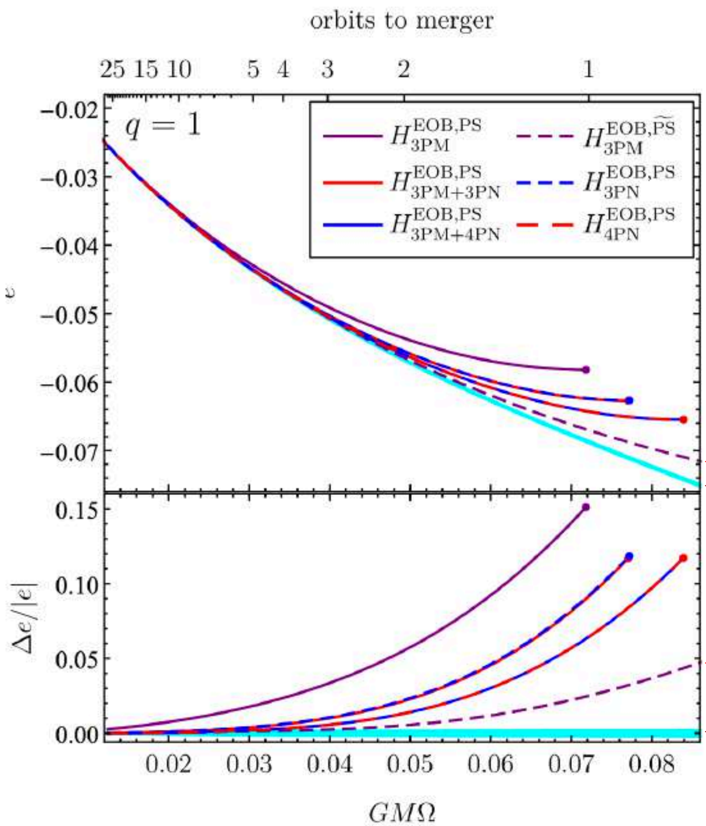
$\chi \longleftrightarrow \Delta\Phi$ period $\longleftrightarrow$ time delay

- Construct Hamiltonian from angle or directly by matching and solve equations of motion

Prospects for LIGO applications at $\mathcal{O}(G_N^3)$

Antonelli, Buonanno, Steinhoff, van de Meent, Vines, 1901.07102

Comparison with numerical GR and other approximation schemes and phenomenological models for a quasi-circular inspiraling binary



based on 3PM

NGR

based on 3PM

NGR

Binding energy vs. orbital frequency

Radiation not included $\longrightarrow$ not completely conclusive

But sufficiently promising to call for a more comprehensive study

"This rather encouraging result motivates a more comprehensive study of EOB resummations of PM results."

Prospects for LIGO applications at $\mathcal{O}(G_N^4)$

Less straightforward because of IR divergence and scale dependence

$$\tilde{I}_{r,4}(\boldsymbol{q}) = G^4 m^7 \nu^2 |\boldsymbol{q}| \left(\frac{\boldsymbol{q}^2}{4^{\frac{1}{3}}\tilde{\mu}^2} \right)^{-3\epsilon} \pi^2 \left[\mathcal{M}_4^{\mathrm{p}} + \nu \left(\frac{\mathcal{M}_4^{\mathrm{t}}}{\epsilon} + \mathcal{M}_4^{\mathrm{f}} \right) \right]$$

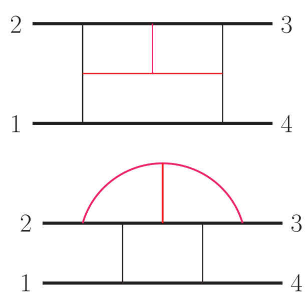
- Originate in the separation of potential and radiation modes

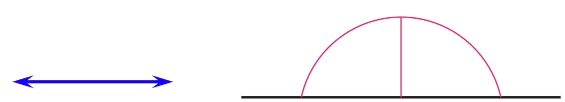
$$(\omega, \boldsymbol{\ell}) \sim (v|\boldsymbol{q}|, \boldsymbol{q}) \qquad\qquad (\omega, \boldsymbol{\ell}) \sim (v|\boldsymbol{q}|, v\boldsymbol{q})$$

- Inclusion of radiation modes *will* remove both and also add finite terms

$$\tilde{\mu} \mapsto \text{physical scale}$$

- New contributions to the amplitude! e.g.





(another) EFT

What about interactions of spinning objects?

- QFT approach requires large/infinite spin

- General parity-invariant effective one-graviton spin-dependent interaction known

Khriplovich, Pomeransky

$$T_{\mu\nu} = \frac{1}{4}\phi_S(p_2)\left[\, p_\mu p_\nu F_1(q^2,(S\cdot q)^2) + p_{(\mu}M_{\nu)\lambda}q^\lambda F_2(q^2,(S\cdot q)^2) + (q_\mu q_\nu - q^2\eta_{\mu\nu})F_3(q^2,(S\cdot q)^2) \right.$$
$$\left. +(S_\mu S_\nu q^2 - 2S_{(\mu}q_{\nu)}\, S\cdot q + \eta_{\mu\nu}(S\cdot q)^2)F_4(q^2,(S\cdot q)^2)\right]\phi_S(p_3)$$

- Three-point interaction of Kerr black hole and one graviton constructed with GR methods

Vines

- World-line action for the covariantization of generic one-graviton interactions

Levi, Steinhoff

earlier work by Perrodin; Porto; Porto and Rothstein

$$S_{\rm pp} = \int d\tau \Big[- m\sqrt{u^2} - \frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu} + L_{\rm SI}[u^\mu, S_{\mu\nu}, g_{\mu\nu}]\Big]$$

$$E_{\mu\nu} \equiv R_{\mu\alpha\nu\beta}u^\alpha u^\beta \qquad B_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\alpha\beta\gamma\mu}R^{\alpha\beta}{}_{\delta\nu}u^\gamma u^\delta$$

$$L_{\rm SI} = \sum_{n=1}^{\infty}\frac{(-1)^{n+m}}{(2n)!}\frac{C_{ES^{2n}}}{m^{2n-1}}D_{\mu_{2n}}\cdots D_{\mu_3}\frac{E_{\mu_1\mu_2}}{\sqrt{u^2}}S^{\mu_1}S^{\mu_2}\cdots S^{\mu_{2n-1}}S^{\mu_{2n}}$$
$$+\sum_{n=1}^{\infty}\frac{(-1)^{n}}{(2n+1)!}\frac{C_{BS^{2n+1}}}{m^{2n}}D_{\mu_{2n+1}}\cdots D_{\mu_3}\frac{B_{\mu_1\mu_2}}{\sqrt{u^2}}S^{\mu_1}S^{\mu_2}\cdots S^{\mu_{2n-1}}S^{\mu_{2n}}S^{\mu_{2n+1}}$$

$$\Omega^{\mu\nu} = e^\mu_A\frac{de^{A\nu}}{d\tau}$$

- Conjectured 3-point HS-HS-graviton amplitude

Arkani-Hamed, Huang, Huang

$$\mathcal{M}_3 \propto \langle \mathbf{12}\rangle^{2S} \longrightarrow e^{-q\cdot S/m}\mathcal{M}_3^{\rm scalar}$$

Higher-spin field theories have a long history

Fierz; Pauli (1939)

Interacting theories of finitely-many higher-spin fields are said not to exist

w/ infinitely-many fields, list is short; w/ propagating DOF is shorter: chiral theories

Metsaev; Ponomarev, Skvortsov; Skvortsov, Tran, Tsulaia;…

However:

- World-line Lagrangian defines a field theory; no consistency problems observed in the classical limit

- Tree amplitudes built on those postulated by Arkani-Hamed, Huang, Huang don’t exhibit unusual problems

Higher-spin field theories have a long history

Fierz; Pauli (1939)

Currently, no fully consistent quantum theory with finitely many higher-spin fields is available

- Recent no-go arguments based on causality

  - Higher-curvature terms require infinitely many massive higher-spin fields (necessary interactions are $h - h - S$)

    Camanho, Edelstein, Maldacena, Zhiboedov

  - Nonminimal interactions of massive higher-spin particles with gravity break causality (interactions considered are $h - S - S$)

    Afkhami-Jeddi, Kundu, Tajdini

The argument: time delay due to propagation & scattering $\propto$ eikonal phase $\delta(s,b,\varepsilon)$, $\mathcal{M} \propto e^{i\delta(s,b,\varepsilon)}$

$$\Longrightarrow \quad \delta(s,b,\varepsilon) > 0 \qquad\qquad \delta(s,b,\varepsilon) = \sum_{k\geq 0} \frac{c_k(s,\mathrm{nonmin},\varepsilon)}{b^{2+k}} \quad D > 4$$

Positivity *order by order in* b and for *all polarization tensors* ε $\Longrightarrow$ all nonminimal couplings vanish

- (Possible) way around: 1. one polarization tensor dominates in the classical limit
  in the classical limit 2. amplitude(s) are valid only above some value of the impact parameter

$\Longrightarrow$ one single loose constraint involving all nonminimal coupling coefficients

An action which covariantizes the most general higher-spin $T_{\mu\nu}$: *modeled* on Levi, Steinhoff's worldline action

$$\mathcal{L} = -R(e,\omega) + \frac{1}{2} g^{\mu\nu} \nabla(\omega)_\mu \phi_s \nabla(\omega)_\nu \phi_s - \frac{1}{2} m^2 \phi_s \phi_s + \mathcal{L}_{\text{non-min}}$$

$$\nabla(\omega)_\mu \phi_s \equiv \partial_\mu \phi_s + \frac{i}{2} \omega_{\mu e f} M^{ef} \phi_s$$

$$\mathcal{L}_{\text{non-min}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{ES^{2n}}}{m^{2n}} \nabla(\omega)_{f_{2n}} \cdots \nabla(\omega)_{f_3} R_{f_1 a f_2 b} \nabla(\omega)^a \phi_s \, \mathbb{S}^{(f_1} \ldots \mathbb{S}^{f_{2n})} \nabla(\omega)^b \phi_s$$
$$- \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{BS^{2n}}}{m^{2n+1}} \nabla(\omega)_{f_{2n+1}} \cdots \nabla(\omega)_{f_3} \frac{1}{2} \epsilon_{ab(c|f_1} R^{ab}{}_{|d)\, f_2} \nabla(\omega)^c \phi_s \mathbb{S}^{(f_1} \ldots \mathbb{S}^{f_{2n+1})} \nabla(\omega)^d \phi_s$$

$$\mathbb{S}^a \equiv \frac{-i}{2m} \epsilon^{abcd} M_{cd} \nabla(\omega)_b$$

Stress tensor – expected general structure; linearized spin operator is $\mathbb{S}_0^a(p) \equiv \frac{1}{2m} \epsilon^{abcd} M_{cd}\, p_b$

With an appropriate definition of asymptotic states:

- three-point amplitudes are the same as those from the world line action
- higher-point amplitudes *may* differ by the contribution of *n* ≥ 2 -graviton gauge-invariant op's

Asymptotic states: from fields to spin vectors

Standard description of integer higher-spin fields: symmetric traceless transverse rank-*s* tensors

$$\varepsilon^{a_1 a_2 \cdots a_i \cdots a_j \cdots a_m} = \varepsilon^{a_1 a_2 \cdots a_j \cdots a_i \cdots a_m}\,, \qquad \eta_{a_1 a_2}\varepsilon^{a_1 a_2 \cdots\cdots a_m} = 0\,, \qquad p_{a_1}\varepsilon^{a_1 a_2 \cdots\cdots a_m} = 0$$

They are needed to project onto spin-*s* representation of $SO(3) \subset SO(3,1)$ Singh, Hagen

More important however: prevent use of representation-specific relations between Lorentz generators

Asymptotic states: from fields to spin vectors

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They are needed to project onto spin-*s* representation of $SO(3) \subset SO(3,1)$ Singh, Hagen

More important however: prevent use of representation-specific relations between Lorentz generators

$\Downarrow$ Relax them *and* keep representation generic

Fields: $$\phi_{s\,\alpha_1\ldots\alpha_s}^{\;\dot\beta_1\ldots\dot\beta_s} = \phi_s{}^{a_1\ldots a_s}(\sigma_{a_1})_{(\alpha_1}{}^{(\dot\beta_1}\ldots(\sigma_{a_s})_{\alpha_s)}{}^{\dot\beta_s)}$$

Polarization tensors: $$\varepsilon(p)^{(\alpha_1\ldots\alpha_s)}_{(\dot\beta_1\ldots\dot\beta_s)} = D(p)\xi(0)^{\alpha_1}\otimes\cdots\otimes\xi(0)^{\alpha_s}\otimes\bar\xi(0)_{\dot\beta_1}\otimes\cdots\otimes\bar\xi(0)_{\dot\beta_s}$$

$\xi(0)^\alpha$: localize the spin in some direction while minimizing its standard deviation

Spin coherent state: $$\xi(0) = \exp(zS_+ - z^* S_-)\xi(0)_+$$
$$n^i = \xi(0)\sigma^i\xi(0) = U^i{}_j(z,z^*)\xi(0)_+\sigma^j\xi(0)_+$$

Klauder, Skagerstam

for general Lie groups see Perelomov

Rest-frame spin vector: $$\boldsymbol{S} = s\,\boldsymbol{n} = |S|\boldsymbol{n}$$

Asymptotic states: from fields to spin vectors

Standard description of integer higher-spin fields: symmetric traceless transverse rank-*s* tensors

$$\varepsilon^{a_1 a_2\cdots a_i\cdots a_j\cdots a_m} = \varepsilon^{a_1 a_2\cdots a_j\cdots a_i\cdots a_m}\,, \qquad \eta_{a_1 a_2}\varepsilon^{a_1 a_2\cdots\cdots a_m} = 0\,, \qquad p_{a_1}\varepsilon^{a_1 a_2\cdots\cdots a_m} = 0$$

They are needed to project onto spin-*s* representation of $SO(3)\subset SO(3,1)$ Singh, Hagen

More important however: prevent use of representation-specific relations between Lorentz generators

Relax them *and* keep representation generic

Fields: $\phi_s{}_{\alpha_1\ldots\alpha_s}^{\dot\beta_1\ldots\dot\beta_s} = \phi_s{}^{a_1\ldots a_s}(\sigma_{a_1})_{(\alpha_1}{}^{(\dot\beta_1}\ldots(\sigma_{a_s})_{\alpha_s)}{}^{\dot\beta_s)}$

Polarization tensors: $\varepsilon(p)^{(\alpha_1\ldots\alpha_s)}_{(\dot\beta_1\ldots\dot\beta_s)} = D(p)\xi(0)^{\alpha_1}\otimes\cdots\otimes\xi(0)^{\alpha_s}\otimes\bar\xi(0)_{\dot\beta_1}\otimes\cdots\otimes\bar\xi(0)_{\dot\beta_s}$

Rest-frame spin vector: $\boldsymbol{S} = |S|\,U\xi(0)\boldsymbol{\sigma}\xi(0) = |S|\boldsymbol{n}$

- Playing with boosts: $S(p,\boldsymbol{S})^\mu = \left(\dfrac{\boldsymbol{p}\cdot\boldsymbol{S}}{m}, \boldsymbol{S} + \dfrac{\boldsymbol{p}\cdot\boldsymbol{S}}{m(E+m)}\boldsymbol{p}\right) \qquad S^{\alpha\beta}(p) = -\dfrac{1}{m}\epsilon^{\alpha\beta\gamma\delta}p_\gamma S_\delta(p)$

$$\varepsilon(\boldsymbol{s},p_1)\cdot\varepsilon(\boldsymbol{s},p_2) = \exp\left(-i\frac{\epsilon_{rsk}p_1^r p_2^s S^k}{m(m+E(\boldsymbol{p}_1))}\right) + \mathcal{O}(q) \qquad \varepsilon(\boldsymbol{s},p_1)M^{ab}\varepsilon(\boldsymbol{s},p_2) = S(p_1,\boldsymbol{S})^{ab}\,\varepsilon(\boldsymbol{s},p_1)\cdot\varepsilon(\boldsymbol{s},p_2) + \mathcal{O}(q^0)$$

Revisit higher-spin field theories in the classical limit:

- Recent no-go arguments based on causality

The argument: time delay due to propagation & scattering $\propto$ eikonal phase $\delta(s,b,\varepsilon)$, $\mathcal{M} \propto e^{i\delta(s,b,\varepsilon)}$

$$\Longrightarrow \quad \delta(s,b,\varepsilon) > 0 \qquad \delta(s,b,\varepsilon) = \sum_{k\geq 0} \frac{c_k(s,\text{nonmin},\varepsilon)}{b^{2+k}} \quad D > 4$$

Positivity *order by order in* b and for *all polarization tensors* ε $\Longrightarrow$ all nonminimal couplings vanish

- In our classical limit: 1. one polarization tensor dominates in the classical limit
  2. amplitude(s) are valid only above some value of the impact parameter

$\Longrightarrow$ one single loose constraint involving all nonminimal coupling coefficients

► Long distance/low energy scattering should be consistent

- Alternative approach using massive spinor helicity, with the same conclusion — Guevara

$$\mathcal{M}_3 \propto \langle \mathbf{12} \rangle^{2S} \longrightarrow e^{-q\cdot S/m} \mathcal{M}_3^{\text{scalar}}$$

An action which covariantizes the most general $T_{\mu\nu}$:

modeled on Levi, Steinhoff's worldline action

$$\mathcal{L} = -R(e,\omega) + \frac{1}{2} g^{\mu\nu} \nabla(\omega)_\mu \phi_s \nabla(\omega)_\nu \phi_s - \frac{1}{2} m^2 \phi_s \phi_s + \mathcal{L}_{\text{non-min}}$$

$$\nabla(\omega)_\mu \phi_s \equiv \partial_\mu \phi_s + \frac{i}{2} \omega_{\mu e f} M^{ef} \phi_s$$

$$\mathcal{L}_{\text{non-min}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{ES^{2n}}}{m^{2n}} \nabla(\omega)_{f_{2n}} \cdots \nabla(\omega)_{f_3} R_{f_1 a f_2 b} \nabla(\omega)^a \phi_s \, \mathbb{S}^{(f_1} \ldots \mathbb{S}^{f_{2n})} \nabla(\omega)^b \phi_s$$
$$- \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{BS^{2n}}}{m^{2n+1}} \nabla(\omega)_{f_{2n+1}} \cdots \nabla(\omega)_{f_3} \frac{1}{2} \epsilon_{ab(c|f_1} R^{ab}{}_{|d)\, f_2} \nabla(\omega)^c \phi_s \mathbb{S}^{(f_1} \ldots \mathbb{S}^{f_{2n+1})} \nabla(\omega)^d \phi_s$$

$$\mathbb{S}^a \equiv \frac{-i}{2m} \epsilon^{abcd} M_{cd} \nabla(\omega)_b$$

Stress tensor – expected general structure; linearized spin operator is $\mathbb{S}_0^a(p) \equiv \frac{1}{2m} \epsilon^{abcd} M_{cd}\, p_b$

The 3-point amplitude for $C_{ES^{2n}} = 1 = C_{BS^{2n}}$

1. Little-group eigenstates $\longrightarrow$ 3-point "minimal" amplitude
   Massive spinor helicity
   (different, but essentially equivalent strategy)

Guevara, Ochirov, Vines
Chung, Huang, Kim, Lee

2. Spin coherent states $\longrightarrow$ "covariant stress tensor of a Kerr black hole" defined by Vines

$$T^{\mu\nu}(p_1,q) = \frac{p_1^\mu p_1^\nu}{m} \sum_{n=0}^{\infty} \frac{C_{ES^{2n}}}{(2n)!} \left(\frac{q\cdot S(p_1)}{m}\right)^{2n} - \frac{i}{m} q_\rho p_1^{(\mu} S(p_1)^{\nu)\rho} \sum_{n=1}^{\infty} \frac{C_{BS^{2n+1}}}{(2n+1)!} \left(\frac{q\cdot S(p_1)}{m}\right)^{2n}$$

Bern, Luna, RR, Shen, Zeng

From Lagrangian to amplitudes

- Tree-level amplitudes from Feynman rules and/or double copy

Bern, Luna, RR, Shen, Zeng
see also Bautista, Guevara for Kerr

The most general *on-shell* arbitrary-spin stress tensor is a double-copy

$$-i\varepsilon(\boldsymbol{s},p_2)V_{3,\mathrm{DC}}^{\mu_0\nu_0}\varepsilon(\boldsymbol{s},p_1) = \left[-i\varepsilon(\boldsymbol{s}_L,p_2)V_{3,\mathrm{L}}^{(\mu_0}\varepsilon(\boldsymbol{s}_L,p_1)\right]\left[-i\varepsilon(\boldsymbol{s}_R,p_2)V_{3,\mathrm{R}}^{\nu_0)}\varepsilon(\boldsymbol{s}_R,p_1)\right]\Big|$$

- $s_L = 0,\ s_R = s$ allowed

- With suitable higher-graviton couplings, higher-point amplitudes may have similar properties

e.g. double-copy structure at 4 pts
w/o nonminimal couplings

$$i\mathcal{M}(1^s,2^s,3^h,4^h) = -4\pi i G\,\frac{p_1\cdot p_3\, p_1\cdot p_4}{p_3\cdot p_4}A(1^0,2^0,3^A,4^A)\,A(1^s,2^s,3^A,4^A)$$

- Tree amplitudes $\xrightarrow{\text{unitarity}}$ Integral representation of loop amplitudes

Lorentz algebra is essential to identify *all* classical contributions:

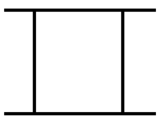
$$M^{\mu\nu}M^{\rho\sigma} = \frac{1}{2}\{M^{\mu\nu},M^{\rho\sigma}\} + \frac{1}{2}[M^{\mu\nu},M^{\rho\sigma}] = \frac{1}{2}\{M^{\mu\nu},M^{\rho\sigma}\} + if^{\mu\nu;\rho\sigma}{}_{\eta\zeta}M^{\eta\zeta}$$

Symmetric product of Lorentz generator $\longleftrightarrow$ product of spin vectors/tensors

The $S_1 S_2 \to S_1 S_2$ one-loop amplitude:

Bern, Luna, RR, Shen, Zeng

$$i\mathcal{M}_4^{1\text{ loop}} = d_{\mathrm{B}}\, I_{\mathrm{B}} + d_{\overline{\mathrm{B}}}\, I_{\overline{\mathrm{B}}} + c_{\triangle}\, I_{\triangle} + c_{\bigtriangledown}\, I_{\bigtriangledown} + \text{non-classical}$$



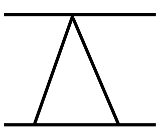
$$\begin{aligned} d_{\mathrm{B}} =& 64G^2 m_1 m_2 \pi^2 \Big\{ 4m_1^3 m_2^3 (2\sigma^2-1)^2 \varepsilon_1\cdot\varepsilon_4 \varepsilon_2\cdot\varepsilon_3 \\ &+ 8i m_1^2 m_2^2 \sigma(2\sigma^2-1)(\varepsilon_1\cdot\varepsilon_4 M_{23}(p_1,q) - M_{14}(p_2,q)\varepsilon_2\cdot\varepsilon_3) \\ &+4m_1 m_2(2\sigma^2-1) M_{14}(p_2,q) M_{23}(p_1,q) \\ &+4m_1^2 m_2^2 \sigma(2\sigma^2-1)\eta_{\mu\nu} M_{14}(e^\mu,q) M_{23}(e^\nu,q) - 2q^2 m_1^2 m_2^2 \sigma(2\sigma^2-1)\eta_{\mu\nu}\eta_{\rho\sigma} M_{14}(e^\mu,e^\rho) M_{23}(e^\nu,e^\sigma) \\ &+\frac{2q^2 m_1 m_2}{\sigma^2-1}(4\sigma^4-2\sigma^2-1)\eta_{\mu\nu} M_{14}(e^\mu,p_2) M_{23}(e^\nu,p_1) \\ &-\frac{q^2 m_2}{\sigma^2-1}\left((4\sigma^4-2\sigma^2-1)m_1 + \sigma(4\sigma^2-3)m_2\right)\eta_{\mu\nu} M_{14}(e^\mu,q) M_{23}(e^\nu,p_1) \\ &+\frac{q^2 m_1}{\sigma^2-1}\left((4\sigma^4-2\sigma^2-1)m_2 + \sigma(4\sigma^2-3)m_1\right)\eta_{\mu\nu} M_{14}(e^\mu,p_2) M_{23}(e^\nu,q)\Big\} + \mathcal{O}(q^3) \end{aligned}$$

$$M_{ij}(a,b) \equiv \varepsilon(p_i) M^{\mu\nu} \varepsilon(p_j) a_\mu b_\nu$$

$$M_{ij}(e^\mu,a) \equiv \varepsilon_i M^{\mu\nu} \varepsilon_j a_\nu$$

All $\sigma^2 = 1$ poles come from reduction of tensor integrals

$$\sigma = \frac{p_1\cdot p_2}{m_1 m_2}$$



$$\begin{aligned} c_{\triangle} =& -32\pi^2 G^2 m_1^2 \varepsilon_1\cdot\varepsilon_4 \varepsilon_2\cdot\varepsilon_3 \Big\{ 6m_1^2 m_2^2 (5\sigma^2-1) + \frac{2m_1 m_2 (5\sigma^2-3)\sigma}{\sigma^2-1}\Big(3iS_2(p_1,q) - 4iS_1(p_2,q)\Big) \\ &+\frac{2}{\sigma^2-1}\Big\{3(5\sigma^2-1)S_1(p_2,q)S_2(p_1,q) + (5\sigma^2-3)\sigma\eta_{\mu\alpha}\Big[m_1 m_2\Big(S_1(e^\mu,q)S_2(e^\alpha,q) \\ &- q^2 \eta_{\nu\beta} S_1(e^\mu,e^\nu) S_2(e^\alpha,e^\beta)\Big) + \frac{q^2}{\sigma^2-1}\Big(2\sigma S_1(e^\mu,p_2)S_2(e^\alpha,p_1) \\ &+\frac{m_1+m_2\sigma}{m_2} S_1(e^\mu,p_2)S_2(e^\alpha,q) - \frac{m_2+m_1\sigma}{m_1} S_2(e^\mu,p_1)S_1(e^\alpha,q)\Big)\Big]\Big\}\Big\} + \mathcal{O}(q^2 \mathrm{S}_i^2) \end{aligned}$$

$$S(a,b) = S^{\mu\nu} a_\mu b_\nu$$

$$S(e^\mu,b) = S^{\mu\nu} b_\nu$$

Recent extension to spin-squared; sensitive to nonminimal couplings

Kosmopoulos, Luna

Comparison with low-spin calculations at one loop:

$$\varepsilon_i \cdot \varepsilon_j \to 1\,, \qquad \varepsilon_1 M^{\mu\nu} \varepsilon_4 = S^{\mu\nu}(p_1)\,, \qquad \varepsilon_2 M^{\mu\nu} \varepsilon_3 = S^{\mu\nu}(p_2)$$

Reproduces spin-1/2 amplitude from "Heavy BH Effective theory" Damgaard, Haddad, Helset

- Capture only leading order in the large spin expansion
- Makes it subtle to extract complete classical physics

Expansion of $i\mathcal{M}^{1\text{ loop}}_{\triangle+\triangledown} = c_\triangle\, I_\triangle + c_\triangledown\, I_\triangledown$ in COM momentum agrees with earlier results Holstein, Ross; Chung, Huang, Kim, Lee

Spin-squared: - this formalism
- worldline formalism

in agreement

Luna, Kosmopoulos
Liu, Porto, Yang

► The formalism captures the desired conservative classical physics and does not exhibit obvious pathologies

From amplitudes to conservative classical observables

	Open orbits	Closed orbits
Conservative classical "observables"	- Two scattering angles / impulse	$E(\omega)$
	- Two (related) spin rotations	periastron advance

- The effective two-body Hamiltonian captures both; difference is in initial conditions for EOM
- More direct connections between classical observables and scattering data
  - Analytic continuations — Kälin, Porto
  - Eikonal
  - Dressed amplitudes, e.g. $\langle \Delta p^\mu \rangle = \langle \psi | i [\mathbb{P}^\mu, T] \psi \rangle + \langle \psi | T^\dagger [\mathbb{P}^\mu, T] \psi \rangle$ — Kosower, Maybee, O'Connell; + Vines

    other observables: spin kick, energy flux, etc
- "When in doubt, go back to the Hamiltonian":
  - It provides a general, all-order connection between amplitudes and all observables
  - Is at the basis of understanding thoroughly more direct relations

Little doubt that the time will come when the Hamiltonian will look like a clumsy device

Spinning Hamiltonians from amplitudes – extension of EFT framework of

Cheung, Rothstein, Solon

- Existence of EFT description is a reflection of universality of long-distance physics
- All terms allowed by long-distance symmetries may appear
- Not unique; may be changed by canonical transformations.

An algorithm: 0. Settle on a description of the spin

e.g. rest frame spin operator vs. covariant spin operator

- we chose the rest-frame spin $[\hat{\mathrm{S}}_a^i, \hat{\mathrm{S}}_b^j] = i\delta_{ab}\epsilon^{ijk}\hat{\mathrm{S}}^k$

Vaidya; Chung, Huang, Kim, Lee

- classical limit in terms of the rest frame coherent states of full theory

Bern, Luna, RR, Shen, Zeng

1. Choose a basis of spin-dependent ops. through the desired order in spin and G_N

e.g. monomials in $\boldsymbol{L}\cdot\hat{\mathrm{S}}_i \equiv (\boldsymbol{q}\times\boldsymbol{p})\cdot\hat{\mathrm{S}}_i,\ \boldsymbol{q}\cdot\hat{\mathrm{S}}_i,\ \boldsymbol{p}\cdot\hat{\mathrm{S}}_i$ consistent with parity, classical limit, modulo possible relations between monomials

$$H = \sqrt{p_1^2+m_1^2} + \sqrt{p_2^2+m_2^2} + \sum_A H^A(p)\mathbb{O}^A$$

2. Determine free coefficients from $\mathcal{M}_{\mathrm{GR}} = \mathcal{M}_{\mathrm{EFT}}$

commutators of spin operators are essential to capture all classical contributions

All EFT operators to quadratic order in spin

$$\mathbb{O}^{(0)} = \mathbb{I}\,, \qquad \mathbb{O}^{(1,1)} = \frac{1}{\boldsymbol{r}^2}\,\boldsymbol{L}\cdot\hat{\boldsymbol{S}}_1\,, \qquad \mathbb{O}^{(1,2)} = \frac{1}{\boldsymbol{r}^2}\,\boldsymbol{L}\cdot\hat{\boldsymbol{S}}_2\,,$$

$$\mathbb{O}^{(2,1)}_{ij} = \frac{1}{\boldsymbol{r}^4}\,\boldsymbol{r}\cdot\hat{\boldsymbol{S}}_i\,\boldsymbol{r}\cdot\hat{\boldsymbol{S}}_j\,, \qquad \mathbb{O}^{(2,2)}_{ij} = \frac{1}{\boldsymbol{r}^2}\,\hat{\boldsymbol{S}}_i\cdot\hat{\boldsymbol{S}}_j\,, \qquad \mathbb{O}^{(2,3)}_{ij} = \frac{1}{\boldsymbol{r}^2}\,\boldsymbol{p}\cdot\hat{\boldsymbol{S}}_i\,\boldsymbol{p}\cdot\hat{\boldsymbol{S}}_j$$

We determined the 2PM contributions to the 2-spin Hamiltonian: Bern, Luna, RR, Shen, Zeng

$$H = H^{(0)}(\boldsymbol{r}^2,\boldsymbol{p}^2)\,\mathbb{I} + H^{(1,i)}(\boldsymbol{r}^2,\boldsymbol{p}^2)\frac{1}{\boldsymbol{r}^2}\boldsymbol{L}\cdot\hat{\boldsymbol{S}}_i$$
$$+ H^{(2,1)}(\boldsymbol{r}^2,\boldsymbol{p}^2)\frac{1}{\boldsymbol{r}^4}\,\boldsymbol{r}\cdot\hat{\boldsymbol{S}}_1\,\boldsymbol{r}\cdot\hat{\boldsymbol{S}}_2 + H^{(2,2)}(\boldsymbol{r}^2,\boldsymbol{p}^2)\frac{1}{\boldsymbol{r}^2}\,\hat{\boldsymbol{S}}_1\cdot\hat{\boldsymbol{S}}_2 + H^{(2,3)}(\boldsymbol{r}^2,\boldsymbol{p}^2)\frac{1}{\boldsymbol{r}^2}\,\boldsymbol{p}\cdot\hat{\boldsymbol{S}}_1\,\boldsymbol{p}\cdot\hat{\boldsymbol{S}}_2$$

$$H^A(\boldsymbol{r}^2,\boldsymbol{p}^2) = \frac{G}{|\boldsymbol{r}|}c_1^A(\boldsymbol{p})^2 + \frac{G^2}{|\boldsymbol{r}|^2}c_2^A(\boldsymbol{p}^2) + \ldots$$

Extended to include spin-squared operators Luna, Kosmopoulos

A number of other terms at higher orders in spin are known in a PN expansion

Barker & R. O'Connell; Porto, Rothstein; Steinhoff, Schäfer, Hergt, Hartung, Levi, Steinhoff; Holstein, Ross, Vaidya; Levi, McLeod, von Hippel; Levi, Teng, Guevara, Ochirov, Vines, Chung, Huang, Kim, Lee;…

$$H = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2} + \sum_A H^A(p)\mathbb{O}^A \qquad H^A(\boldsymbol{r}^2, \boldsymbol{p}^2) = \frac{G}{|\boldsymbol{r}|} c_1^A(\boldsymbol{p})^2 + \frac{G^2}{|\boldsymbol{r}|^2} c_2^A(\boldsymbol{p}^2) + \dots$$

$$\begin{aligned}
\Delta_1^{(0)}\boldsymbol{p} &= \frac{2E\xi\bar{c}_1^{(0)}}{b^2 p_\infty}\boldsymbol{b} \\
\Delta_1^{(1)}\boldsymbol{p} &= -\frac{2E\xi}{b^2 p_\infty}\sum_i \bar{c}^{(1,i)}(\boldsymbol{p}\times\boldsymbol{S}_i) + \frac{4E\xi}{b^4 p_\infty^2}\sum_i \bar{c}_1^{(1,i)}(\boldsymbol{L}\cdot\boldsymbol{S}_i)\,\boldsymbol{b} \\
\Delta_1^{(2,1)}\boldsymbol{p} &= \frac{8E\xi\bar{c}_{1,12}^{(2,1)}}{3b^4 p_\infty}\left(\frac{4}{b^2}(\boldsymbol{b}\cdot\boldsymbol{S}_1)(\boldsymbol{b}\cdot\boldsymbol{S}_1)) + \frac{1}{p_\infty^2}(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_1))\right)\boldsymbol{b} \\
&\quad - \frac{8E\xi\bar{c}_{1,12}^{(2,1)}}{3b^4 p_\infty}(\boldsymbol{b}\cdot\boldsymbol{S}_2)\,\boldsymbol{S}_{1,\perp} - \frac{8E\xi\bar{c}_{1,12}^{(2,1)}}{3b^4 p_\infty}(\boldsymbol{b}\cdot\boldsymbol{S}_1)\,\boldsymbol{S}_{2,\perp} \\
\Delta_1^{(2,2)}\boldsymbol{p} &= \frac{8E\xi\bar{c}_{1,12}^{(2,2)}}{b^4 p_\infty}(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\,\boldsymbol{b} \\
\Delta_1^{(2,3)}\boldsymbol{p} &= \frac{8E\xi\bar{c}_{1,12}^{(2,3)}}{b^4 p_\infty}(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2)\,\boldsymbol{b}
\end{aligned}$$

$$\begin{aligned}
\overline{\Delta\boldsymbol{S}}_{1,1} = \frac{2G\xi E}{L|\boldsymbol{b}|}\Bigg[& c_1^{(1,1)}\,\boldsymbol{L}\times\boldsymbol{S}_1 + \frac{c_1^{(2,1)}}{3}\left(\frac{1}{\boldsymbol{p}^2}\,\boldsymbol{p}\times\boldsymbol{S}_1\,\boldsymbol{p}\cdot\boldsymbol{S}_2 + \frac{2}{\boldsymbol{b}^2}\,\boldsymbol{b}\times\boldsymbol{S}_1\,\boldsymbol{b}\cdot\boldsymbol{S}_2\right) \\
& - c_1^{(2,2)}\,\boldsymbol{S}_1\times\boldsymbol{S}_2 + c_1^{(2,3)}\,\boldsymbol{p}\times\boldsymbol{S}_1\,\boldsymbol{p}\cdot\boldsymbol{S}_2\Bigg]
\end{aligned}$$

$$\begin{aligned}
\Delta_2^{(0)}\boldsymbol{p} =& \frac{\pi}{2b^3 p_\infty}(A[2,1,0,\bar{c}_1^{(0)2}] - 2E\xi\bar{c}_2^{(0)})\,\boldsymbol{b} - \frac{1}{b^2 p_\infty^4}(2E^2\xi^2\bar{c}_1^{(0)2})\,\boldsymbol{p} \\
\Delta_2^{(1)}\boldsymbol{p} =& -\frac{\pi}{2b^5 p_\infty^3}\sum_{i=1}^{2}(A[2,1,2,\bar{c}_1^{(0)}\bar{c}_1^{(1,i)}] - Ep_\infty^2\xi\bar{c}_2^{(1,i)})(2b^2\boldsymbol{p}\times\boldsymbol{S}_i + 3\boldsymbol{L}(\boldsymbol{b}\cdot\boldsymbol{S}_i)) \\
& - \frac{4E^2\xi^2}{b^4 p_\infty^4}\sum_{i=1}^{2}\bar{c}_1^{(0)}\bar{c}_1^{(1,i)}(\boldsymbol{L}\cdot\boldsymbol{S}_i)\,\boldsymbol{p} - \frac{2E^2\xi^2}{b^4 p_\infty^4}\sum_{i=1}^{2}\bar{c}_1^{(1,i)}(2\bar{c}_1^{(0)} + p_\infty^2\bar{c}_1^{(1,i)})(\boldsymbol{p}\cdot\boldsymbol{S}_1)\,\boldsymbol{L} \\
& - \frac{2E^2\xi^2}{b^4 p_\infty^2}\bar{c}_1^{(1,1)}\bar{c}_1^{(1,2)}\,(\boldsymbol{S}_1\times(\boldsymbol{p}\times\boldsymbol{S}_2) + \boldsymbol{S}_2\times(\boldsymbol{p}\times\boldsymbol{S}_1)) \\
& + \frac{3\pi\boldsymbol{b}}{8b^7 p_\infty^3}A[2,1,4,\bar{c}_1^{(1,1)}\bar{c}_1^{(1,2)}]\left(2p_\infty^2(\boldsymbol{b}\cdot\boldsymbol{S}_1)(\boldsymbol{b}\cdot\boldsymbol{S}_2) - 3(\boldsymbol{L}\cdot\boldsymbol{S}_1)(\boldsymbol{L}\cdot\boldsymbol{S}_2)\right) \\
& - \frac{8E^2\xi^2\,\boldsymbol{L}}{b^6 p_\infty^4}\bar{c}_1^{(1,1)}c[1,1,2,p_\infty^2]\,((\boldsymbol{L}\cdot\boldsymbol{S}_2)(\boldsymbol{p}\cdot\boldsymbol{S}_1) + (\boldsymbol{L}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2)) \\
& - \frac{3\pi}{8b^5 p_\infty}A[2,1,4,\bar{c}_1^{(1,1)}\bar{c}_1^{(1,2)}]\,((\boldsymbol{b}\cdot\boldsymbol{S}_2)\boldsymbol{S}_{1,\perp} + (\boldsymbol{b}\cdot\boldsymbol{S}_1)\boldsymbol{S}_{2,\perp}) \\
\Delta_2^{(2,1)}\boldsymbol{p} =& \frac{4E^2\xi^2}{3b^4 p_\infty^4}\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,1)}\,(\boldsymbol{S}_1\times(\boldsymbol{p}\times\boldsymbol{S}_2) + \boldsymbol{S}_2\times(\boldsymbol{p}\times\boldsymbol{S}_1)) \\
& + \frac{16E^2\xi^2\,\boldsymbol{L}}{3b^6 p_\infty^6}\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,1)}((\boldsymbol{L}\cdot\boldsymbol{S}_2)(\boldsymbol{p}\cdot\boldsymbol{S}_1) + (\boldsymbol{L}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2)) \\
& - \frac{3\pi\boldsymbol{b}}{4b^7 p_\infty^5}(A[2,1,4/3,\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,1)}] - Ep_\infty^2\xi\bar{c}_{2,12}^{(2,1)})(5p_\infty^2(\boldsymbol{b}\cdot\boldsymbol{S}_1)(\boldsymbol{b}\cdot\boldsymbol{S}_2) + b^2(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2)) \\
& - \frac{8E^2\xi^2\,\boldsymbol{p}}{3b^6 p_\infty^6}\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,1)}\left(4p_\infty^2(\boldsymbol{b}\cdot\boldsymbol{S}_1)(\boldsymbol{b}\cdot\boldsymbol{S}_2 + b^2(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2) + p_\infty^2 b^2(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right) \\
& + \frac{3\pi}{4b^5 p_\infty^3}(A[2,1,4/3,\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,1)}] - Ep_\infty^2\xi\bar{c}_{2,12}^{(2,1)})\,((\boldsymbol{b}\cdot\boldsymbol{S}_2)\boldsymbol{S}_{1,\perp} + (\boldsymbol{b}\cdot\boldsymbol{S}_1)\boldsymbol{S}_{2,\perp}) \\
\Delta_2^{(2,2)}\boldsymbol{p} =& -\frac{8E^2\xi^2}{b^4 p_\infty^4}\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,2)}\,(\boldsymbol{S}_1\times(\boldsymbol{p}\times\boldsymbol{S}_2) + \boldsymbol{S}_2\times(\boldsymbol{p}\times\boldsymbol{S}_1)) \\
& - \frac{3\pi\,\boldsymbol{b}}{b^5 p_\infty^3}(A[2,1,2,\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,2)}] - Ep_\infty^2\xi\bar{c}_{2,12}^{(2,2)})(\boldsymbol{S}_1\cdot\boldsymbol{S}_2) \\
& - \frac{8E^2\xi^2}{b^4 p_\infty^4}\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,2)}\,((\boldsymbol{p}\cdot\boldsymbol{S}_2)\boldsymbol{S}_1 + (\boldsymbol{p}\cdot\boldsymbol{S}_1)\boldsymbol{S}_2) \\
\Delta_2^{(2,3)}\boldsymbol{p} =& -P((16E^2\xi^2\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,3)}(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2))/(b^4 p_\infty^4)) \\
& - \frac{3\pi\,\boldsymbol{b}}{b^5 p_\infty^3}\left(A[2,1,4,\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,3)}] - Ep_\infty^2\xi\bar{c}_{2,12}^{(2,3)}\right)(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2) \\
& + \frac{4E^2\xi^2}{b^4 p_\infty^2}\bar{c}_1^{(0)}\bar{c}_{1,12}^{(2,3)}\,((\boldsymbol{p}\cdot\boldsymbol{S}_2)\boldsymbol{S}_{1,\perp} + (\boldsymbol{p}\cdot\boldsymbol{S}_1)\boldsymbol{S}_{2,\perp})
\end{aligned}$$

$$\begin{aligned}
\overline{\Delta\boldsymbol{S}}_{1,2a} =& \frac{\pi\xi E}{2Lb^2}c_2^{(1,1)}\,\boldsymbol{L}\times\boldsymbol{S}_1 + \frac{1}{2Lb^2}\left[-\pi(1-3\xi) - 2\pi\xi^2E^2\frac{1}{\boldsymbol{p}^2} - 2\pi\xi^2E^2\partial\right]\left(c_1^{(0)}c_1^{(1,1)}\right)\boldsymbol{L}\times\boldsymbol{S}_1 \\
\overline{\Delta\boldsymbol{S}}_{1,2b} =& \frac{\boldsymbol{p}^2}{8Lb^2}\left[3\pi(1-3\xi) + 12\pi\xi^2E^2\frac{1}{\boldsymbol{p}^2} + 6\pi\xi^2E^2\partial\right]\left(c_1^{(1,1)}c_1^{(1,2)}\right)\,(-S_{1,z}S_{2,y}\,\hat{\boldsymbol{x}} + S_{1,x}S_{2,y}\,\hat{\boldsymbol{z}}) \\
& + \frac{1}{Lb^2}\left[-2\xi^2E^2\right]\left(c_1^{(1,1)}c_1^{(1,2)}\right)\,(S_{1,y}S_{2,x}\,\hat{\boldsymbol{x}} + S_{1,y}S_{2,z}\,\hat{\boldsymbol{z}} - (S_{1,x}S_{2,x} + S_{1,z}S_{2,z})\,\hat{\boldsymbol{y}}) \\
\overline{\Delta\boldsymbol{S}}_{1,2c} =& +\frac{1}{8Lb^2}\left[\pi\xi E\right]\left(c_2^{(2,1)}\right)\,(-S_{1,y}S_{2,z}\,\hat{\boldsymbol{x}} + (-3S_{1,z}S_{2,x} + S_{1,x}S_{2,z})\,\hat{\boldsymbol{y}} + 3S_{1,y}S_{2,x}\,\hat{\boldsymbol{z}}) \\
& + \frac{1}{3L^3}\left[2\xi^2E^2\right]\left(c_1^{(0)}c_1^{(2,1)}\right)(-S_{1,y}S_{2,x}\,\hat{\boldsymbol{x}} + (S_{1,x}S_{2,x} - S_{1,z}S_{2,z})\,\hat{\boldsymbol{y}} + S_{1,y}S_{2,z}\,\hat{\boldsymbol{z}}) \\
& + \frac{1}{8Lb^2}\left[\pi(1-3\xi) + 4\pi\xi^2E^2\frac{1}{\boldsymbol{p}^2} + 2\pi\xi^2E^2\partial\right]\left(c_1^{(0)}c_1^{(2,1)}\right)\,((S_{1,y}S_{2,z}\,\hat{\boldsymbol{x}}) - S_{1,x}S_{2,z}\,\hat{\boldsymbol{y}}) \\
& + \frac{1}{8Lb^2}\left[3\pi(1-3\xi) + 4\pi\xi^2E^2\frac{1}{\boldsymbol{p}^2} + 6\pi\xi^2E^2\partial\right]\left(c_1^{(0)}c_1^{(2,1)}\right)\,(S_{1,z}S_{2,x}\,\hat{\boldsymbol{y}} - S_{1,y}S_{2,x}\,\hat{\boldsymbol{z}}) \\
\overline{\Delta\boldsymbol{S}}_{1,2d} =& -\frac{1}{2Lb^2}\left[\pi\xi E\right]\left(c_2^{(2,2)}\right)\,\boldsymbol{S}_1\times\boldsymbol{S}_2 \\
& + \frac{1}{2Lb^2}\left[\pi(1-3\xi) + 2\pi\xi^2E^2\frac{1}{\boldsymbol{p}^2} + 2\pi\xi^2E^2\partial\right]\left(c_1^{(0)}c_1^{(2,2)}\right)\,\boldsymbol{S}_1\times\boldsymbol{S}_2 \\
\overline{\Delta\boldsymbol{S}}_{1,2e} =& -\frac{\boldsymbol{p}^2}{2Lb^2}\left[\pi\xi E\right]\left(c_2^{(2,3)}\right)\,(S_{1,y}S_{2,z}\,\hat{\boldsymbol{x}} - S_{1,x}S_{2,z}\,\hat{\boldsymbol{y}}) \\
& - \frac{1}{Lb^2}\left[2\xi^2E^2\right]\left(c_1^{(0)}c_1^{(2,3)}\right)\,(-S_{1,y}S_{2,x}\,\hat{\boldsymbol{x}} + (S_{1,x}S_{2,x} - S_{1,z}S_{2,z})\,\hat{\boldsymbol{y}} + S_{1,y}S_{2,z}\,\hat{\boldsymbol{z}}) \\
& + \frac{\boldsymbol{p}^2}{2Lb^2}\left[\pi(1-3\xi) + 4\pi\xi^2E^2\frac{1}{\boldsymbol{p}^2} + 2\pi\xi^2E^2\partial\right]\left(c_1^{(0)}c_1^{(2,3)}\right)\,(S_{1,y}S_{2,z}\,\hat{\boldsymbol{x}} - S_{1,x}S_{2,z}\,\hat{\boldsymbol{y}}) \\
\overline{\Delta\boldsymbol{S}}_{1,2f} =& +\frac{\pi\xi^2E^2}{8Lb^2}\left(c_1^{(1,1)} + c_1^{(1,2)}\right)c_1^{(2,1)}\,(S_{1y}S_{2,z}\,\hat{\boldsymbol{x}} - (S_{1,z}S_{2,x} + S_{1,x}S_{2,z})\,\hat{\boldsymbol{y}} + S_{1,y}S_{2,x}\,\hat{\boldsymbol{z}}) \\
& + \frac{\xi^2E^2}{3Lb^2}\Bigg[\left(4c_1^{(1,1)} + 2c_1^{(1,2)}\right)c_1^{(2,1)}\,S_{1y}S_{2,x}\,\hat{\boldsymbol{x}} + \left(2c_1^{(1,1)} + 4c_1^{(1,2)}\right)c_1^{(2,1)}\,S_{1y}S_{2,z}\,\hat{\boldsymbol{z}} \\
& \qquad + \left(4c_1^{(1,1)} - 2c_1^{(1,2)}\right)c_1^{(2,1)}\,S_{1x}S_{2,x}\,\hat{\boldsymbol{y}} + \left(2c_1^{(1,1)} - 4c_1^{(1,2)}\right)c_1^{(2,1)}\,S_{1z}S_{2,z}\,\hat{\boldsymbol{y}}\Bigg] \\
& + \frac{\xi^2E^2}{Lb^2}\Bigg[2\left(c_1^{(1,1)} + c_1^{(1,2)}\right)c_1^{(2,2)}\,S_{1y}(S_{2,x}\,\hat{\boldsymbol{x}} + S_{2,z}\,\hat{\boldsymbol{z}}) - 4c_1^{(1,1)}\,c_1^{(2,2)}\,S_{2,y}(S_{1,x}\,\hat{\boldsymbol{x}} + S_{1,z}\,\hat{\boldsymbol{z}}) \\
& \qquad + 2\left(c_1^{(1,1)} - c_1^{(1,2)}\right)c_1^{(2,2)}\,(S_{1,x}S_{2,x} - S_{1,z}S_{2,z})\,\hat{\boldsymbol{y}}\Bigg] \\
& + \frac{2\xi^2E^2}{Lb^2}\boldsymbol{p}^2\left[c_1^{(1,2)}\,c_1^{(2,3)}\,S_{2,x}(S_{1,y}\,\hat{\boldsymbol{x}} - S_{1,x}\,\hat{\boldsymbol{y}}) + c_1^{(1,1)}\,c_1^{(2,3)}\,S_{2,z}(S_{1,z}\,\hat{\boldsymbol{y}} + S_{1,y}\,\hat{\boldsymbol{z}})\right]
\end{aligned}$$

Looking for structure:

Bern, Luna, RR, Shen, Zeng

Leading order in G_N: Amplitude $\xleftrightarrow[\text{transform}]{\text{Fourier}}$ Interaction potential

$$\Delta\mathcal{O} = \int_{-\infty}^{+\infty} dt\dot{\mathcal{O}} = \int_{-\infty}^{+\infty} dt\{\mathcal{O}, H_0 + V\} \qquad V(\boldsymbol{r},\boldsymbol{p}) \propto \int d^3q e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\mathcal{M}^{\text{tree}}(\boldsymbol{q},\boldsymbol{p})$$

- Leading order trajectory is a straight line: $\boldsymbol{r} = \boldsymbol{b} + \boldsymbol{p}t$

$$\Delta\mathcal{O} \propto \{\mathcal{O}, \int d^3q\delta(\boldsymbol{q}\cdot\boldsymbol{p})e^{i\boldsymbol{b}_\perp\cdot\boldsymbol{q}}\mathcal{M}^{\text{tree}}(\boldsymbol{q},\boldsymbol{p})\} \propto \{\mathcal{O}, \chi_1(\boldsymbol{b}_\perp,\boldsymbol{p})\}$$

leading order eikonal phase

$$\mathcal{M} \sim e^{i\chi} - 1 = \chi_1 + \dots$$

- What are the operators $\mathcal{O}$ corresponding to desired observables?

- How to go beyond leading order?

Looking for structure:

Bern, Luna, RR, Shen, Zeng

Leading order in G_N: Amplitude $\xleftrightarrow[\text{transform}]{\text{Fourier}}$ Interaction potential

$$\Delta\mathcal{O} \propto \{\mathcal{O}, \int d^3q\,\delta(\boldsymbol{q}\cdot\boldsymbol{p})e^{i\boldsymbol{b}_\perp\cdot\boldsymbol{q}}\mathcal{M}^{\text{tree}}(\boldsymbol{q},\boldsymbol{p})\} \propto \{\mathcal{O}, \chi_1(\boldsymbol{b}_\perp,\boldsymbol{p})\}$$

leading order eikonal phase

$$\mathcal{M}(\boldsymbol{b}_\perp,\boldsymbol{p}) \sim e^{i\chi(\boldsymbol{b}_\perp,\boldsymbol{p})} - 1 = i\chi_1(\boldsymbol{b}_\perp,\boldsymbol{p}) + \ldots$$

- Operators for observables

$$\Delta\mathcal{O} = \int_{-\infty}^{+\infty} dt\{\mathcal{O}, H\} \ :$$

$$\Delta\boldsymbol{p}_\perp : \mathcal{O} \equiv \boldsymbol{P}_\perp \propto \nabla_{\boldsymbol{b}_\perp}$$

$$\Delta\boldsymbol{S}_i : \mathcal{O} \propto \boldsymbol{S}_i$$

$$\Delta\boldsymbol{p}_{||} = -\frac{1}{2|\boldsymbol{p}|}(\Delta\boldsymbol{p}_\perp)^2$$

- Restoring Jacobians and other factors:

$$\Delta\boldsymbol{p}_\perp = [\boldsymbol{P}_\perp, i\chi_1] + \mathcal{O}(G_N^2) = \nabla_{\boldsymbol{b}_\perp}\chi_1 + \mathcal{O}(G_N^2)$$

$$\Delta\boldsymbol{S}_i = [\boldsymbol{S}, i\chi_1] + \mathcal{O}(G_N^2)$$

$$\chi_1 = \frac{1}{4m_1m_2\sqrt{\sigma^2-1}}\int\frac{d^{2-2\epsilon}\boldsymbol{q}_\perp}{(2\pi)^{2-2\epsilon}}e^{-i\boldsymbol{q}_\perp\cdot\boldsymbol{b}_\perp}\mathcal{M}^{\text{tree}}(\boldsymbol{q}_\perp,\boldsymbol{p})$$

Agrees with Hamiltonian EOM

agrees for aligned spin result of Guevara, Ochirov, Vines

Looking for structure:

Bern, Luna, RR, Shen, Zeng

Higher orders in G_N ?

Observations: - For spinless scattering, the impulse is related to the eikonal phase beyond $\mathcal{O}(G_N)$

- χ_1 acts as a Hamiltonian, so one might think higher orders behave similarly

- nonabelian nature of the spin must become important
  - possible nonlinearities
  - lower orders in χ may appear at higher orders in G_N

Looking for structure:

Bern, Luna, RR, Shen, Zeng

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Observations:
- For spinless scattering, the impulse is related to the eikonal phase beyond $\mathcal{O}(G_N)$
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Through second order in spin at $\mathcal{O}(G_N^2)$:

$$\Delta_2\mathcal{O} = [\mathcal{O}, i\chi_2] + \frac{1}{2}[-i\chi_1, [\mathcal{O}, i\chi_1]] + i\mathcal{D}_{SL}(-i\chi_1, [\mathcal{O}, i\chi_1]) + [\mathcal{O}, \frac{i}{2}\mathcal{D}_{SL}(i\chi_1, i\chi_1)]$$

$$\chi_1 = \frac{1}{4m_1 m_2\sqrt{\sigma^2-1}}\int \frac{d^{2-2\epsilon}\boldsymbol{q}_\perp}{(2\pi)^{2-2\epsilon}} e^{-i\boldsymbol{q}_\perp\cdot\boldsymbol{b}_\perp}\mathcal{M}^{\text{tree}}(\boldsymbol{q}_\perp, \boldsymbol{p})$$

$$\chi_2 = \frac{1}{4m_1 m_2\sqrt{\sigma^2-1}}\int \frac{d^{2-2\epsilon}\boldsymbol{q}_\perp}{(2\pi)^{2-2\epsilon}} e^{-i\boldsymbol{q}_\perp\cdot\boldsymbol{b}_\perp}\underline{\mathcal{M}^{\triangle+\triangledown}(\boldsymbol{q}_\perp, \boldsymbol{p})}$$

suitably-defined finite-part of the 1-loop amplitude

$$\mathcal{D}_{SL}(f,g) \equiv -\sum_{a=1,2}\epsilon^{ijk}S_a^k\frac{\partial f}{\partial S_a^i}\frac{\partial g}{\partial L^j}$$

Looking for structure:

Bern, Luna, RR, Shen, Zeng

An all-order conjecture for open-orbit observables:

$$\Delta\mathcal{O} = e^{-i\chi\mathcal{D}}[\mathcal{O}, e^{i\chi\mathcal{D}}]$$

$$\chi\mathcal{D}g \equiv \chi g + \mathcal{D}_{SL}(\chi, g) \qquad \mathcal{D}_{SL}\left(\chi, g\right) \equiv -\sum_{a=1,2} \epsilon^{ijk} S_a^k \frac{\partial \chi}{\partial S_a^i} \frac{\partial g}{\partial L^j}$$

$\chi \equiv$ conservative eikonal phase of the $S_1 S_2 \to S_1 S_2$ amplitude

- General proof at $\mathcal{O}(G_N)$
- Explicit check at $\mathcal{O}(G_N^2)$ through orders ($S_1 S_2$, S_1^2, S_2^2)
- One might expect that a direct relation between χ and observables exists to all orders

An all-order conjecture:

Bern, Luna, RR, Shen, Zeng

$$\Delta\mathcal{O} = e^{-i\chi^{\mathcal{D}}}[\mathcal{O}, e^{i\chi^{\mathcal{D}}}]$$

- Checks can be carried out within effective field theory framework

$$\begin{tikzcd}
\begin{array}{c}\text{(known)}\\ \text{Hamiltonian}\end{array} \arrow[r, leftrightarrow] \arrow[d] & \text{EFT action} \arrow[d] \\
\begin{array}{c}\text{Impulse,}\\ \text{Spin kick}\end{array} & \begin{array}{c}\text{Eikonal of}\\ \text{EFT amplitude}\end{array} \arrow[l, "?"']
\end{tikzcd}$$

- (More) covariant versions of the conjecture?
- Relation to the formalism of Kosower, Maybee, O’Connell/ Maybee, O’Connell, Vines?
- Connection to bound-orbit observables and the formalism of Kälin, Porto?
- Other aspects of gravitational interactions captured by the eikonal phase?

Summary and outlook

Amplitudes – new powerful ways to look at gravitational problems with and without spin

- pushed state of the art for spinless interaction potential calculations to 3PM and 4PM
- tested and illustrated these ideas by deriving a 2PM spin-dependent Hamiltonian
- agreement in the overlap with available state of the art results

  - provide inspiration to find hidden structure and hidden simplicity

- close relation between amplitude and radial action
- analytic dependence of observables on velocity
- spinning eikonal conjecture; explicit tests through quadratic order in spin
- interactions of spinning and spinless particles and tidal deformations (not discussed here)

Summary and outlook

Many immediate and longer term questions

- PM Hamiltonians at higher orders in G_N and spin
- PM for conservative radiation effects; fix the scale dependence of potential
- Test & understand the origin and construct a proof of the spinning eikonal conjecture
- Interface with other formalisms relating directly amplitudes and observables
- Complete classification of $R^{k\geq 2}_{abcd}S^n$
- Resummation of G_N expansion; interface with effective one-body theory
- Closer connection between QFT and worldline parametrizations
- Closer connection between self-force and amplitudes-based PM expansion w/ & w/o spin
- Connection with other aspects of physics of black holes and other extended bodies
- ...

Expect renewed progress in the future