RG Limit Cycles and “Spooky” Fixed Points in Perturbative QFT

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Introduction

- One of the main tools of modern theoretical physics is RG, that could hint on the behavior of the system in the regime of high or small energies.
- It reduces the study of the complicated theory to some simple system of differential equations

$$\mu\frac{dg^i}{d\mu} = \beta^i(g^j)$$

- These beta-functions usually are calculated perturbatively, but for some models exact beta-functions are known.

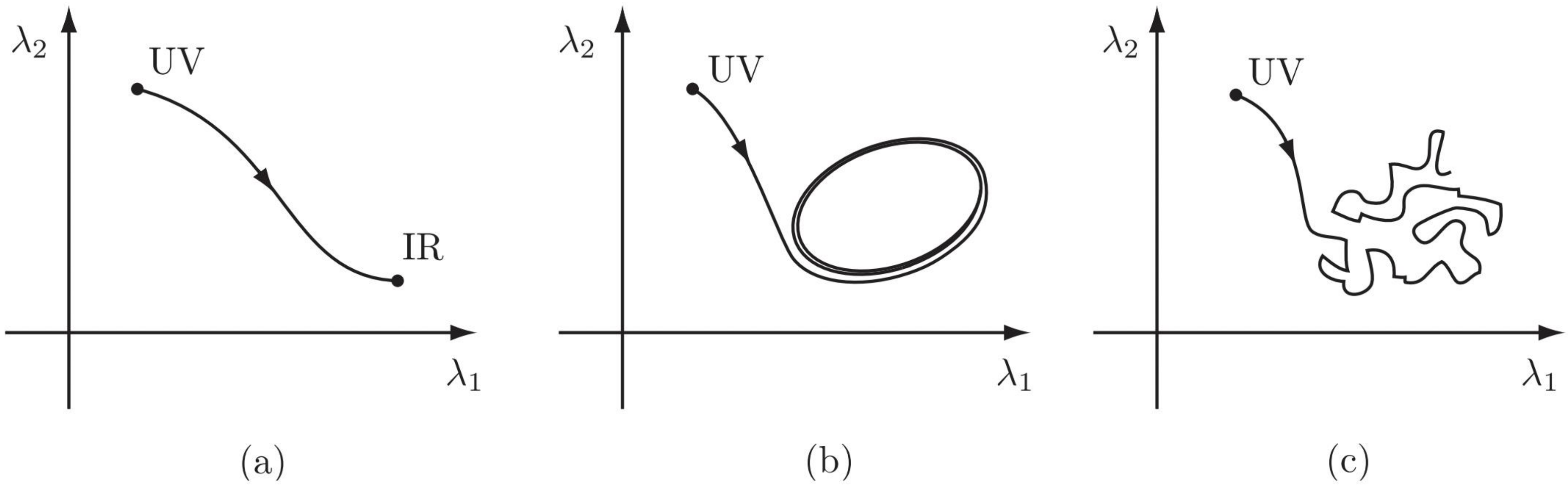
Fixed points

- Since 1971, in hep-th people usually consider fixed points of this equations

$$\beta^i = 0$$

- At these points the theory becomes conformal and some powerful tools could be used to investigate further the theory.
- But is it the only possible behavior?
- It is known that in the system of autonomous differential equations rather complicated phenomena could arise

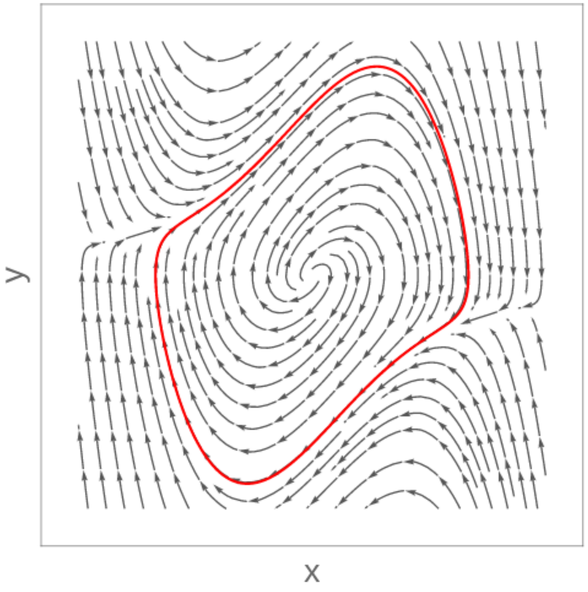
Other types of RG flow behavior



Autonomous differential equation

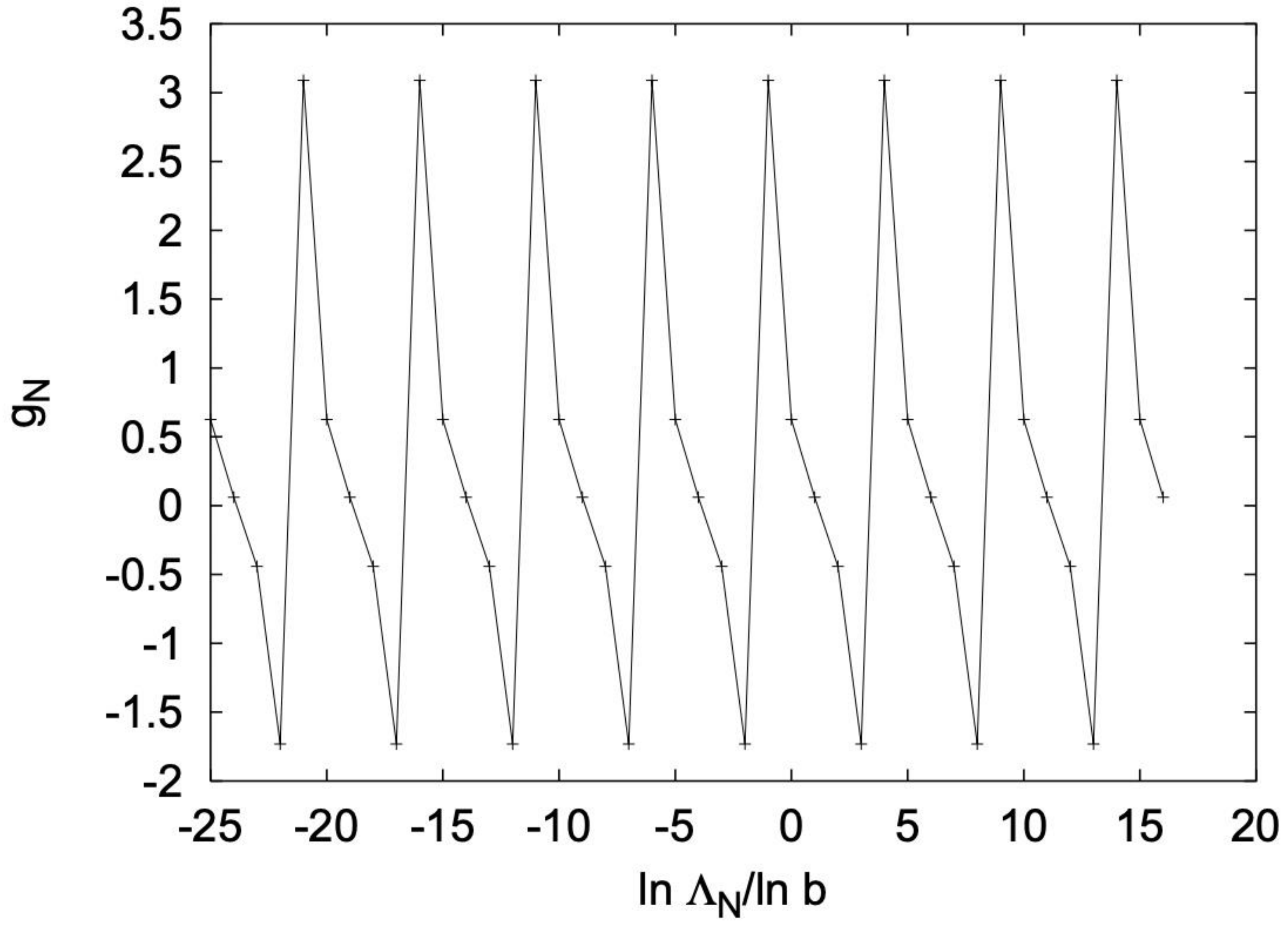
- The limit cycles arise in a different areas of modern science:
- The Hodgkin–Huxley model for action potentials in neurons.
- The Sel'kov model of glycolysis.
- The migration of cancer cells.
- The most famous one is Van der Pol oscillator

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$



The possibility of Limit Cycles

- In 1993 K.Wilson proposed a model of QCD, where the coupling experience limit cycles [Phys. Rev. D 47, 4657]
- The model is singular and the coupling constant jumps from $-\infty$ to $+\infty$
- In 1999, Bedaque et al. proposed a model of 3-body interaction, where such a limit cycle emerge [Phys.Rev.Lett.82.463]
- In 2002, K. Wilson proposed a 0-dim model where the limit cycles also emerge [hep-th/0203088]



3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-25
-20
-15
-10
-5
0
5
10
15
20
g_N
ln Λ_N/ln b

The other known models

- In the previous models coupling constant goes through infinity - can not be trusted.
- In 2004, Le Clair et al. considered a deformed WZW model, where exact beta-function could be found and confirmed the emergence of limit cycles
- All above mentioned limit cycles exists due to the existence of shallow states with Efimov scaling

$$E_n = E_0 e^{-\lambda n}$$

Limit Cycles in Graphene

- Consider an electric charge in graphene [1402.2431]

$$H = v_F \sigma^\alpha \partial_\alpha - \frac{e^2}{r}$$

- The spectrum of such a system is conformal – possibility of the atomic collapse

- The wave function is singular at the origin

$$\psi \sim \sqrt{r} \sin\left(\nu \log \frac{r}{r_0}\right), \quad \nu^2 + \frac{1}{4} = \frac{e^4}{v_F^2}$$

- Therefore we have to regularize the potential at the origin
- We can assume that there is some cut-off at which the potential becomes constant

$$V(r) = \begin{cases} -\frac{e^2}{r}, r > R \\ -\frac{\lambda}{R}, r < R \end{cases}$$

- It will resolve the problem of atomic collapse and introduce the cut-off R.

The discussion

- In all above mentioned models, there is no “true” limit cycles – the coupling constant goes through infinite.
- Could we get a model where we have a true limit cycle like in the Van der Pol oscillator?
- Could we have a limit cycle in a perturbative quantum field theory? - From the first glance it is not possible due to the Zamolodchikov c-theorem and its generalization.

The c-,a- and other theorems

- People tried to investigate the general properties of the perturbative RG flow
- In 2d, Zamolodchikov proposed a c-theorem that suggests that central charge could only decrease along the RG flow [Pisma v Zhetf 43-12-515-517]

$$\beta^i \partial_i c = -12 \beta^i \beta^j G_{ij}, \quad G_{ij} = \langle \Phi_i \Phi_j \rangle > 0$$

- In 4d, Komargoski and Polchinski et al. Managed to get a similar statement that some function is smaller at IR than at UV

$$a_{IR} < a_{UV}$$

- Osborn et al. managed to show the existence of the gradient flow

$$\beta^i = G^{ij} \partial_j A$$

The F-theorem in 3d

- In 3d, a similar theorem could be proved (even though the most weakest one) – the F-theorem, that is a free energy of a theory on a sphere (Klebanov, Jafferies, Pufu)
- There are also some hints that the theory posses a gradient flow
- $$\beta^i = G^{ij}\partial_j A$$
- These 3 theorems seemingly rule out the possibility of the limit cycles.

Loophole

- Let us assume that we have some parameter in a theory (say, N – the number of components in a vector model).
- The metric should be positive for “physical” N. But for non-physical it could be negative
- We “analytically continue” theory from integer N to non-integer N
- It could be done (Binder, Ryckov)
- The theory is non-unitary

Loophole

- For example, in a matrix model the operators

$$(\phi_{ab}\phi_{ab})^2\,, \quad \phi_{ab}\phi_{bc}\phi_{cd}\phi_{da}$$

- are independent, but for N=1 they dependent, therefore

$$\det G_{ij} = 0, \quad N = 1$$

- And around this point we could have violation of a,c,f-theorems

The Tensor Model

[Gurau, Rivasseau, Benedetti, Carrozzo, Tanaza, Witten, Klebanov-Tarnopolsky]

We mostly will study the tensor models where the fields ϕ_{abc} transform under the action of the O(N)^3 group

- The theory with one or two indices are well studied and give an interesting examples of the large N limits (where only diagrams of particular type contribute in the large N limit)
- We pick the interaction for the field ϕ_{abc} to be tetrahedral

$$V\left(\phi_{abc}\right) = \frac{g_t}{4}\phi_{abc}\phi_{ab'c'}\phi_{a'bc'}\phi_{a'b'c} \qquad H = \frac{g_t}{4}\psi_{abc}\psi_{ab'c'}\psi_{a'bc'}\psi_{a'b'c}$$

Renormalization group approach

- So we study the scalar field theory at $d = 4 - \epsilon$ with the interaction

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_{abc}\right)^2 - g_t \mathcal{O}_t - g_p \mathcal{O}_p - g_{ds} \mathcal{O}_{ds}$$

$$\mathcal{O}_t = \phi_{abc}\phi_{ab'c'}\phi_{a'bc'}\phi_{a'b'c}$$

$$\mathcal{O}_p = \left(\phi_{abc}\phi_{a'bc}\right)^2 + \left(\phi_{abc}\phi_{ab'c}\right)^2 + \left(\phi_{abc}\phi_{abc'}\right)^2$$

$$\mathcal{O}_{ds} = \left(\phi_{abc}^2\right)^2$$

- These operators are marginal and may appear during the RG flow, therefore to get the correct understanding they must be included.
- We will not consider the limit of the large N, in order to get the information about the 1/N corrections

The Beta Functions

$$\beta_t = -\epsilon g_1 + \frac{4}{3(4\pi)^2}\Big(3g_1g_2(N+1) + 18g_1g_3 + 2g_2^2\Big)$$
$$+\frac{2}{9(4\pi)^4}\Big(9(N^3-15N-10)g_1^3 - 36g_1^2\big((N^2+4N+13)g_2 + 15Ng_3\big)$$
$$-3g_1\big((N^3+15N^2+93N+101)g_2^2 + 12(5N^2+17N+17)g_2g_3 + 6(5N^3+82)g_3^2\big)$$
$$-4g_2^2\big((2N^2+13N+24)g_2 + 72g_3\big)\Big)\,,$$

$$\gamma_\phi = \frac{1}{6(4\pi)^4}\Big(3g_1^2(N^3+3N+2) + 6g_3^2(N^3+2) + 12g_1\big(g_2(N^2+N+1) + 3g_3N\big)$$
$$+12g_2g_3(N^2+N+1) + g_2^2(N^3+3N^2+9N+5)\Big)\,.$$

$$\beta_p = -\epsilon g_2 + \frac{2}{3(4\pi)^2}\Big(9g_1^2(N+2) + 12g_2g_1(N+2) + g_2^2(N^2+5N+12) + 36g_2g_3\Big)$$
$$-\frac{2}{9(4\pi)^4}\Big(108(N^2+N+4)g_1^3 + 9g_1^2\big((N^3+12N^2+99N+98)g_2 + 72(N+2)g_3\big)$$
$$+36g_1g_2\big((4N^2+18N+29)g_2 + 3(13N+16)g_3\big) + g_2\big((5N^3+45N^2+243N+343)g_2^2$$
$$+36(7N^2+15N+29)g_2g_3 + 18(5N^3+82)g_3^2\big)\Big)\,,$$

These beta-functions are calculated at finite N

$$\beta_{ds} = -\epsilon g_3 + \frac{2}{3(4\pi)^2}\Big(3g_3^2\,(N^3+8) + 6g_3g_2\,(N^2+N+1) + g_2^2(2N+3) + 18g_1g_3N + 6g_1g_2\Big)$$
$$-\frac{2}{9(4\pi)^4}\Big(54Ng_1^3 + 9g_1^2\big(4(N^2+N+4)g_2 + 5(N^3+3N+2)g_3\big)$$
$$+36g_1\big(4(N+1)g_2^2 + (5N^2+5N+17)g_2g_3 + 33Ng_3^2\big) + 14(N^2+3N+5)g_2^3$$
$$+3(5N^3+15N^2+93N+97)g_2^2g_3 + 396(N^2+N+1)g_2g_3^2 + 54(3N^3+14)g_3^3\Big)\,.$$

The fixed point

- As one can see there is a complex fixed point

$$\tilde{g}_1^* = (\epsilon/2)^{1/2}, \quad \tilde{g}_2^* = \pm 3i(\epsilon/2)^{1/2}, \quad \tilde{g}_3^* = \mp i(3 \pm \sqrt{3})(\epsilon/2)^{1/2}.$$

- But it reproduces the right anomalous dimensions for the operators. For example for double trace we have

$$\Delta_O = d - 2 + 2(\tilde{g}_2^* + \tilde{g}_3^*) = 2 \pm i\sqrt{6\epsilon} + \mathcal{O}(\epsilon).$$

- The dimension of the tetrahedral operator also nicely fits with the exact solution.

$$\Delta_{\text{tetra}} = d + \beta_t'(g_1^*) = 4 + \epsilon + \mathcal{O}(\epsilon^2),$$

The matrix model

- Maybe studying eps-expasnsion of matrix models we could get more information about these theories
- We consider the theory of traceless-symmetric matrices

$$O_1 = \operatorname{tr}\phi^6, \quad O_2 = \operatorname{tr}\phi^2 \operatorname{tr}\phi^4, \quad O_3 = (\operatorname{tr}\phi^2)^3, \quad O_4 = (\operatorname{tr}\phi^3)^2$$

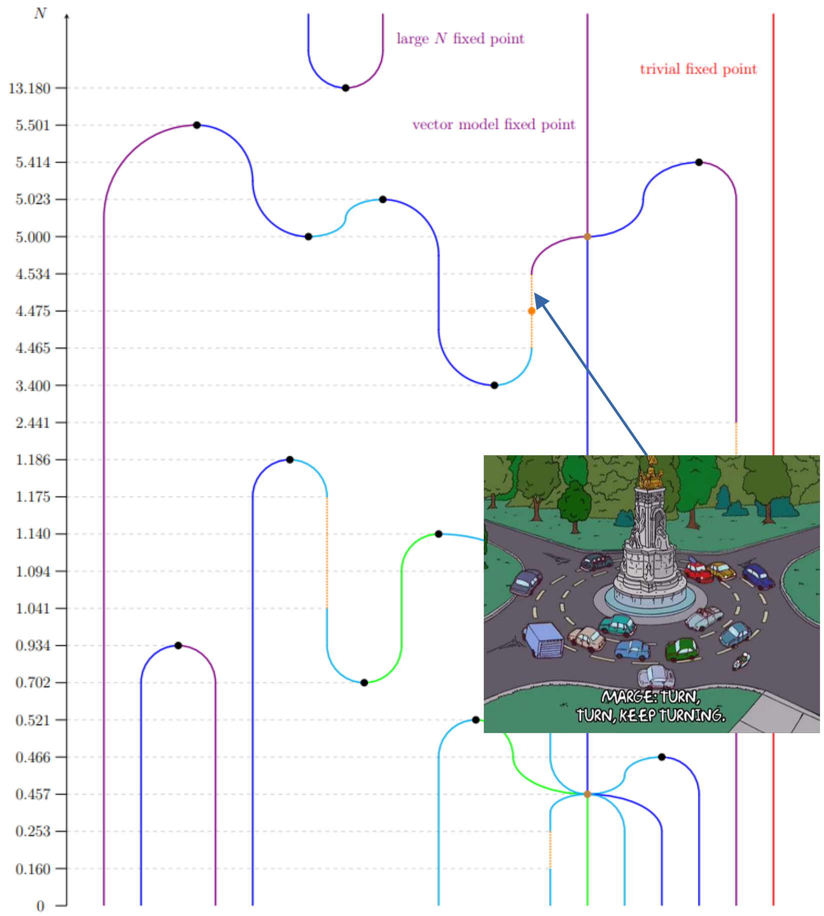
The metric

- The theory is defined for integer N
- But we can study it at fractional N
- The operators are overcounted for N=2,3,4,5

$$N = 2 : O_4 = 0, O_3 = 2O_2 = 4O_1$$

$$N = 3 : O_3 = 2O_2, 2O_4 = 3O_3 + 6O_1$$

$$N = 4, 5 : 18O_2 + 8O_4 = 24O_1 + 3O_3$$



The trivial fixed point correspond to a free theory

The vector model fixed point correspond that only
O4 operator has non-zero coupling

Large N fixed point correspond to solution that at
Large N converges to large N solution

At this point the coupling g_4 is complex the theory
Is described by compelx CFT[Gorbenko, Rychkov]

It is a breaking of orbifold equivalence

There is a region where the theory is real but
The eigenvalues are complex.

We shall call them ***spooky*** fixed points

Stability of the fixed point

As it was mentioned before RG flow is just a system of autonomous differential equations

One can ask the question whether this point is stable
We just linearize system

$$\beta_i = \frac{dg_j}{d\log\mu} \approx \left(\frac{\partial\beta_i}{\partial g_j}\right)(g_j - g_j^*)$$

If all eigenvalues have non-zero real part we can change variables(=redefine operators) such that

$$\operatorname{Re}\lambda_i \neq 0$$

then linearization is exact

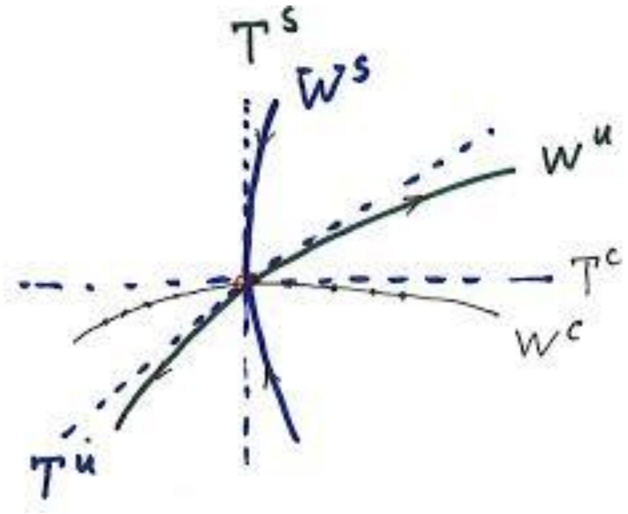
From that we can easily understand the stability conditions
At N=4.475 there is a pair of purely imaginary eigenvalues – it is not clear what is happening

$$\forall i : \operatorname{Re}\lambda_i < 0$$

Central manifold theorem

Actually we can say even more – we can always split our consider submanifold s.t. beta-function preserves it and stable

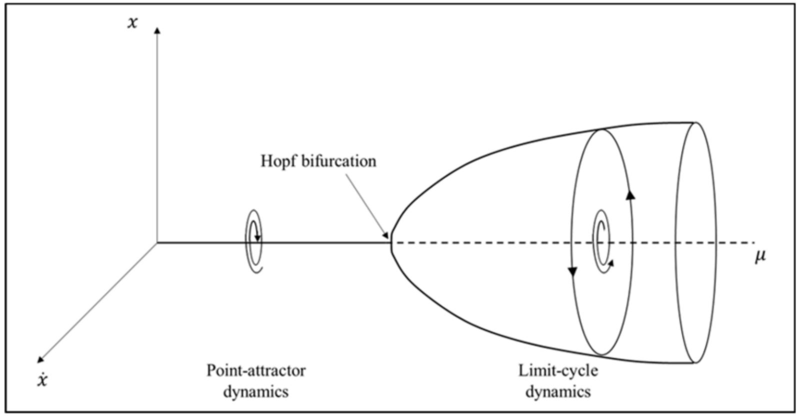
It could help us to understand
What is going on at N=4.475



Hopf bifurcation

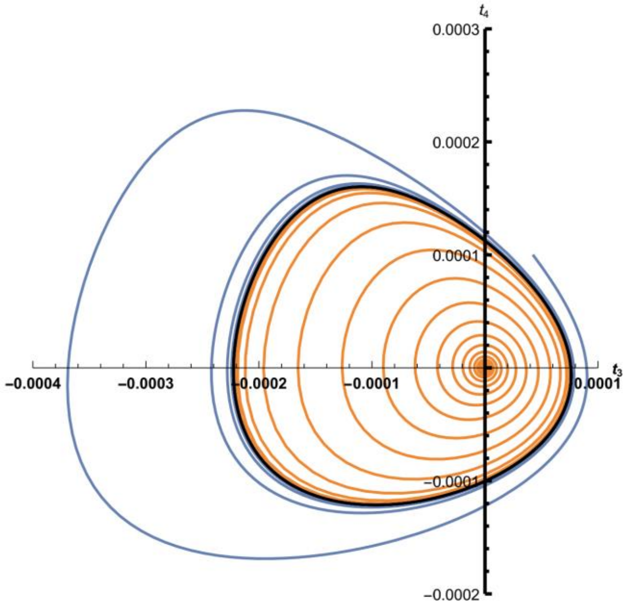
Namely we have a continuous parameter N and at N=4.465 we have a change in the behaviour of the system – a part of stable manifold transforms to the unstable one

Actually we can use the result of Eberhard Hopf in 1942
In this case very close to critical N limit cycles arise



Hopf bifurcation in QFT

And it is exactly what’s happening in our case
By picking the right coordinates with the use of central manifold theorem we can solve the equations numerically and get limit cycles for N=4.476



Limit Cycles in Higher orders

For a while we have considered only two loop approximation

Maybe they disappear if all orders of perturbation theory are considered

First of all we can notice that N_{crit} does not disappear for small enough \eps

The Hopf theorem requires only the existence of such a point

Therefore limit cycles will always emerge in such a system if they appear at the first non-vanishing order of perturbarion theory

Possible generalizations

We can try to gauge system so it would create a fictious “eps”

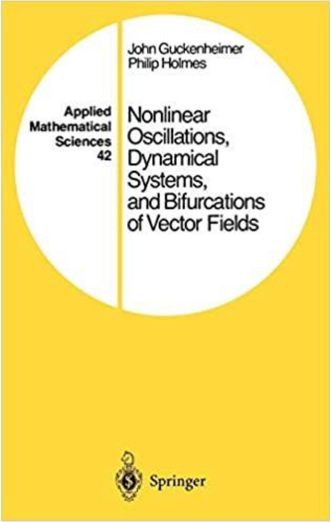
We can introduce two parameters N,M – that we could continuously change

Study at different dimensions d=2,3,4,6,8

Supersymmetric models?

Other type of bifurcations?

Tower of Efimov states?



Discrete Conformal Field Theory

The limit cycles in some sense share the same properties as usual CFT, but with discrete dilatation group $\mathbb{Z}$

We can try to solve the Callan–Symanzik equation

Discrete anomalous dimension?

Discrete Conformal field theory?

$$G(\lambda x) = \lambda^{\Delta} G(x)$$

Thank you for you attention!