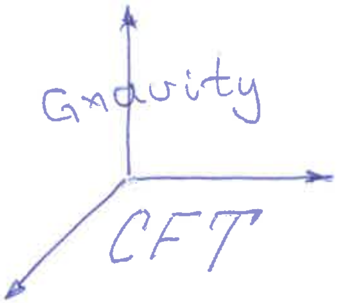
Large-N localization and quiver CFT

KZ, 2003.00993

AdS/CFT correspondence

AdS_5:



- large central charge:

$$c \sim N^2$$

$$(M_{Pl} R_{AdS})^3 \sim N^2$$

$N \gg 1 \Rightarrow$ gravity is classical

- strong coupling:

$$\left(\frac{R_{AdS}}{l_{string}}\right)^4 \sim \lambda$$

$\lambda \gg 1 \Rightarrow$ curvature is small

- anomaly matching:

$a = c \Rightarrow$ gravitational action is local (Einstein)

Simplest CFTs

$\mathcal{N}=2$ supersymmetry: CFT $\Longleftrightarrow$ $\beta_{1\text{-loop}}=0$ $\qquad \beta_{1\text{-loop}} \propto -\left(C(\text{adj})-C(R)\right) g_{YM}^3$

$C(\text{adj})=N$

$C(\text{fund})=1$

- Super-QCD:

$$\mathcal{L}=-\frac{1}{4}\,\mathrm{tr}\,F_{\mu\nu}^2+i\,\bar{\Psi}_f\,\not{D}\,\Psi_f \quad + \text{ superpartners}$$

$N_f=N_c$

$a\neq c$ holographic dual never weakly coupled

- $\mathcal{N}=4$ SYM:

$$\mathcal{L}=-\frac{1}{4}\,\mathrm{tr}\,F_{\mu\nu}^2+i\,\mathrm{tr}\,\bar{\Psi}\,\gamma^\mu\left[D_\mu,\Psi\right]+\text{superpartners}$$

$a=c$ has weakly-coupled holographic description at $N\to\infty$, $\lambda\gg1$

$\lambda=g_{YM}N^2$

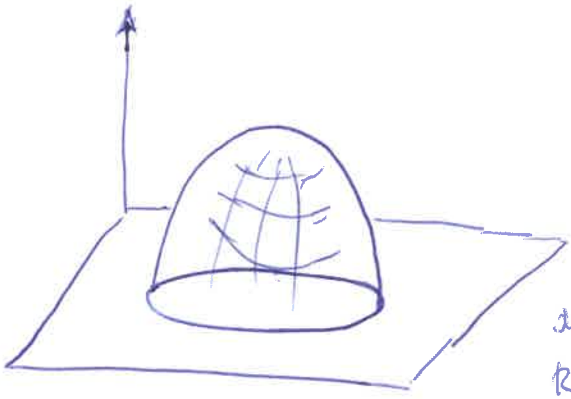
Wilson loops

$$W(C) = \left\langle \frac{1}{N} \operatorname{tr} \mathcal{P} \exp \oint_C ds \left(i \dot{x}^\mu A_\mu + |\dot{x}| \Phi \right) \right\rangle$$

SYM:

$$W(C) \overset{\lambda \gg 1}{\simeq} \sqrt{\frac{2}{\pi}}\, \lambda^{-\frac{3}{4}}\, e^{\sqrt{\lambda}}$$

Quantum fluct. Area law



Maldacena '98

Rey, Yee '98

Tension: $T = \frac{\sqrt{\lambda}}{2\pi}$

Area: $A = -2\pi$

Area law: $e^{-TA} = e^{\sqrt{\lambda}}$

Erickson, Semenoff, Z. '00

Drukker, Gross '00

Medina-Rincon, Tseytlin, Z. '19

SQCD:

$$W(C) \overset{\lambda \gg 1}{\simeq} \text{const} \frac{\lambda^3}{(\ln \lambda)^{\frac{3}{2}}}$$

Passerini, Z. '11

no holographic description

Quiver CFT

- SQCD with gauged flavor group

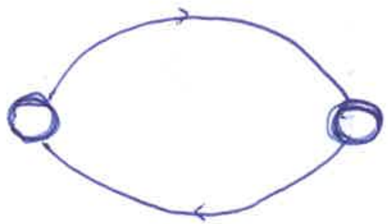
$$\mathcal{L} = -\frac{1}{4}\,\mathrm{tr}\left(F^{1}_{\mu\nu}\right)^2 - \frac{1}{4}\,\mathrm{tr}\left(F^{2}_{\mu\nu}\right)^2 + i\,\bar{\Psi}\,\not{D}\,\psi + \text{superpartners}$$

flavor (A^2_μ)

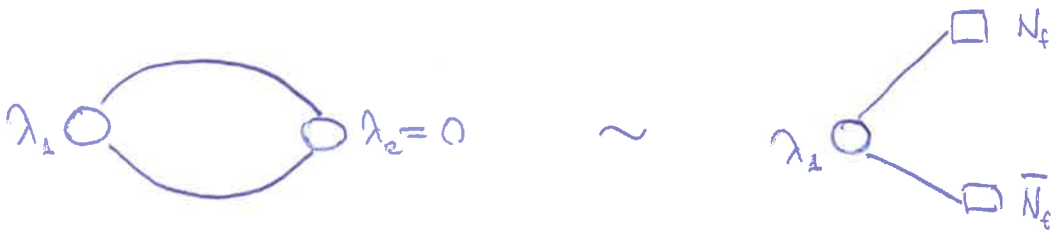
ψ_{if}

color (A^1_μ)

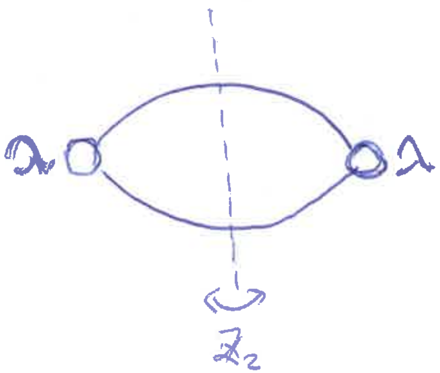
$SU(N)\times SU(N)$

$\lambda_1 = g_1^2 N$  $\lambda_2 = g_2^2 N$

- $g_2 = 0 \Rightarrow$ Flavor $SU(N)$ decouples : SQCD



- $g_1 = g_2 \Rightarrow$ Extra $\mathbb{Z}_2$



Orbifold equivalence

Global discrete symmetry: Γ

Gauge: $\Gamma' \subset SU(nN)$

Orbifold projection: $\Gamma \times \Gamma' \longrightarrow \Gamma_{\text{diagonal}}$

Th Planar diagrams of large-N parent $SU(nN)$ theory

and daughter orbifold $SU(N) \times \ldots \times SU(N)$ theory

are the same.

Bershadsky, Johansen '98

$\mathbb{Z}_2$ orbifold of $N=4$ SYM

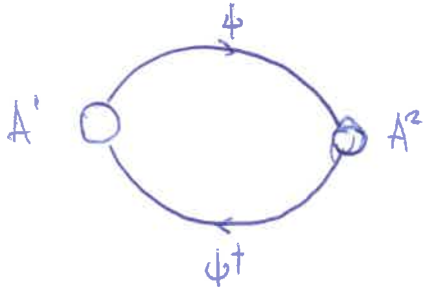
Global $\mathbb{Z}_2$: $$A_\mu \longrightarrow A_\mu$$ $$\psi \longrightarrow -\psi$$

Gauge $\mathbb{Z}_2$: $$A \longrightarrow \Omega^{-1} A\, \Omega \qquad \Omega = \begin{bmatrix} 1_{N\times N} & \\ & -1_{N\times N} \end{bmatrix}$$

Orbifold projection: $$A_\mu \longrightarrow \begin{bmatrix} A^1_\mu & \\ & A^2_\mu \end{bmatrix} \qquad \psi \longrightarrow \begin{bmatrix} & \phi \\ \phi^\dagger & \end{bmatrix}$$

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{tr}\left(F^1_{\mu\nu}\right)^2 - \frac{1}{4}\,\mathrm{tr}\left(F^2_{\mu\nu}\right)^2 + i\,\phi^\dagger \gamma^\mu D_\mu \phi + \text{superpartners}$$

bi-fundamental



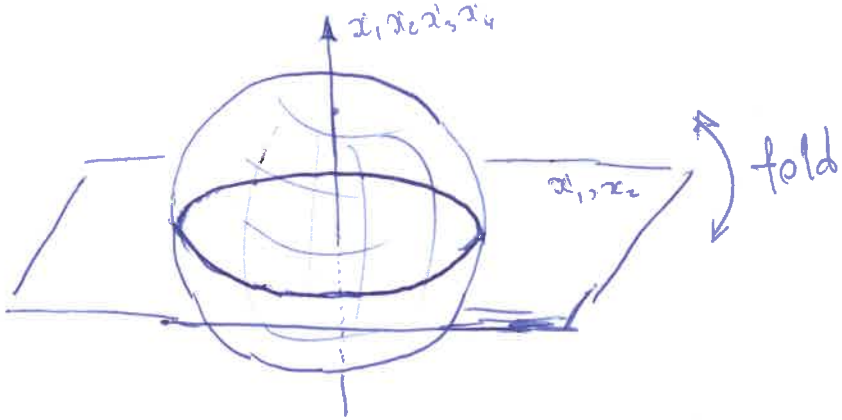
$= \quad \mathrm{SYM}/\mathbb{Z}_2$

same coupling : $\lambda_1 = \lambda_2 \equiv \lambda_{SYM}$

Lawrence, Nekrasov, Vafa '98

Holographic dual

$$\mathbb{Z}_2 : \quad \overbrace{\Phi_1 \quad \Phi_2 \quad \Phi_3 \quad \Phi_4}^{\text{hyper}} \quad \overbrace{\Phi_5 \quad \Phi_6}^{\text{vector}}$$
$$\qquad\quad - \quad\; - \quad\; - \quad\; - \quad\; + \quad\; +$$



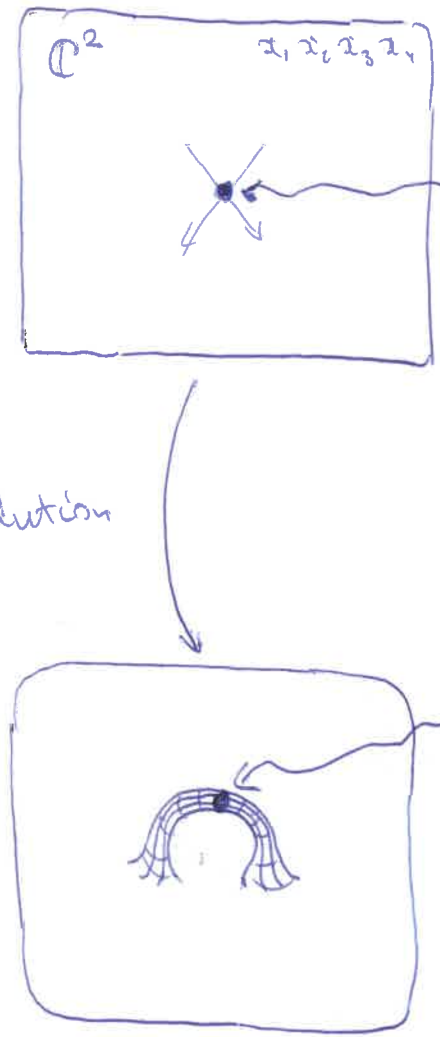
$$AdS_5 \times \left(S^5 / \mathbb{Z}_2 \right)$$

Kachru, Silverstein '98

Effective coupling:
$$\frac{2}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

String tension:
$$T = \frac{\sqrt{\lambda}}{2\pi}$$

Resolution



orbifold singularity

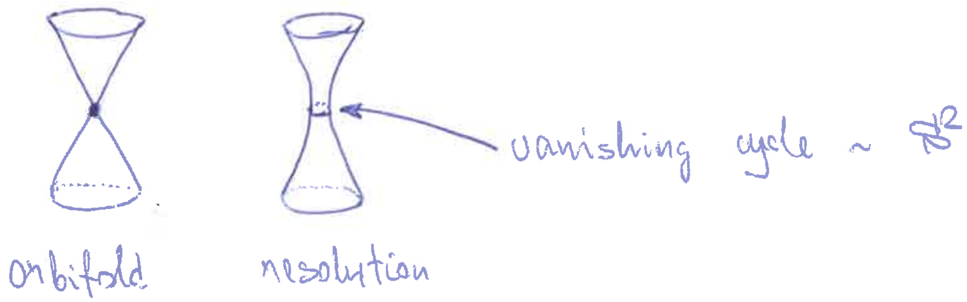
$\mathbb{C}^2/\mathbb{Z}_2$

resolution

non-contractable S^2 of size ε

Eguchi-Hanson space

B-flux



topological term

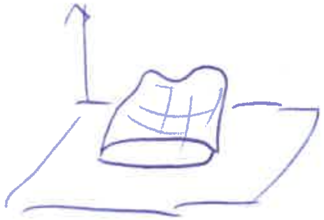
$$S_{str} = \int \left(\frac{\sqrt{\lambda}}{4\pi} G_{MN}\, dX^M \wedge * dX^N + \Theta\, n_i\, dn_j \wedge dn_k\, \varepsilon^{ijk} \right)$$

$$\boxed{\Theta = \pi - \pi \frac{\frac{1}{\lambda_1} - \frac{1}{\lambda_2}}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2}} = \frac{2\pi\lambda_1}{\lambda_1 + \lambda_2}}$$

Lawrence, Nekrasov, Vafa '98
Klebanov, Nekrasov '99
Gadde, Pomoni, Rastelli '09

- $0 \leq \Theta \leq 2\pi$
- SYM/$\mathbb{Z}_2$: $\lambda_1 = \lambda_2$ or $\Theta = \pi$

$$W(C) = \int \mathcal{D}X^{\mu}\; e^{-S_{str}[X]}$$



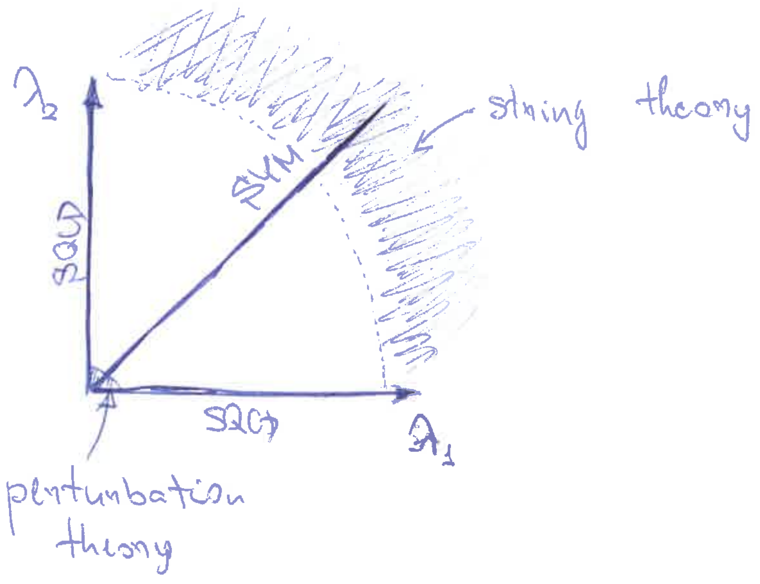
Instantons:

$$W = e^{\sqrt{\lambda}} \left(C_0 + C_1 e^{-\varepsilon\sqrt{\lambda} + i\theta} + C_2 e^{-2\varepsilon\sqrt{\lambda} + 2i\theta} + \ldots \right) \overset{\varepsilon \to 0}{=} F(\cos\theta)\, e^{\sqrt{\lambda}}$$

$$\theta = \frac{2\pi\lambda_1}{\lambda_1+\lambda_2} \qquad \frac{2}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

How trigonometric dependence on θ may arise in field theory?

Parameter space of quiver CFT:



Goal:

- compute circular Wilson loop in the stringy regime:

$$\lambda_1, \lambda_2 \longrightarrow \infty \qquad \frac{\lambda_1}{\lambda_2} - \text{fixed}$$

Circular Wilson loop in $\mathcal{N}=4$ SYM

$W(C) = 1 +$  $+$  $+$  $+ \dots$

$$\frac{\lambda}{16\pi^2}\oint ds_1 ds_2 \frac{|\dot{x}_1||\dot{x}_2| - \dot{x}_1\cdot\dot{x}_2}{(x_1-x_2)^2} = \frac{\lambda}{16\pi^2}\oint ds_1 ds_2 \cdot \frac{1}{2} = \frac{\lambda}{8}$$

$$W(C) = \left\langle \frac{1}{N}\,\mathrm{tr}\, e^{2\pi\Phi} \right\rangle$$

$$Z = \int d^{N^2}\Phi \; e^{-\frac{8\pi^2 N}{\lambda}\mathrm{tr}\,\Phi^2}$$

correctly accounts for combinatorics

$$W(C) = \frac{2}{\sqrt{\lambda}}\, I_1(\sqrt{\lambda}) \overset{\lambda\gg 1}{\simeq} \sqrt{\frac{2}{\pi}}\,\lambda^{-\frac{3}{4}}\, e^{\sqrt{\lambda}}$$

Erickson, Semenoff, Z'00

Drukker, Gross '00

Localization

In $\mathcal{N}=2$ theory, interactions are important, but still captured by a matrix integral (in the eigenvalue representation).

In quiver CFT:

$$\mathcal{Z} = \int \prod_{a=1}^{2} \prod_{i=1}^{N} d a_{ai} \frac{\prod_{a} \prod_{i<j} (a_{ai} - a_{aj})^2 H^2(a_{ai} - a_{aj})}{\prod_{ij} H^2(a_{1i} - a_{2j})} e^{-\sum_a \frac{8\pi^2 N}{\lambda_a} \sum_i a_{ai}^2}$$

$$H(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}}$$

$$W_a = \left\langle \frac{1}{N} \sum_i e^{2\pi a_{ai}} \right\rangle$$

Pestun'07

- At $\lambda_1 = \lambda_2$ the integral is <u>not</u> Gaussian. What about orbifold equivalence?

Saddle-point equations

$$\rho_a(x) = \left\langle \frac{1}{N} \sum_i \delta(x - a_{ai}) \right\rangle$$

$$\int_{-\mu_1}^{\mu_1} dy\, \rho_1(y) \left(\frac{1}{x-y} - K(x-y) \right) + \int_{-\mu_2}^{\mu_2} dy\, \rho_2(y)\, K(x-y) = \frac{8\pi^2}{\lambda_1} x$$

$$\int_{-\mu_2}^{\mu_2} dy\, \rho_2(y) \left(\frac{1}{x-y} - K(x-y) \right) + \int_{-\mu_1}^{\mu_1} dy\, \rho_1(y)\, K(x-y) = \frac{8\pi^2}{\lambda_2} x$$

$$K(x) = -\frac{H'(x)}{H(x)} = x\left(\psi(1+ix) + \psi(1-ix) + 2\gamma \right)$$

- At $\lambda_1 = \lambda_2$, ansatz $\rho_1(x) = \rho_2(x)$ goes thru

$$\Downarrow$$

$$\rho_1 = \rho_2 = \frac{2}{\pi \mu^2} \sqrt{\mu^2 - x^2} \qquad \mu = \frac{\sqrt{\lambda}}{2\pi}$$

Solution of Gaussian model

Strong coupling

"Locking":

$$\lambda_1, \lambda_2 \gg 1 \qquad \frac{\lambda_1}{\lambda_2} = \mathcal{O}(1) \qquad \Longrightarrow \qquad \rho_1(x) \approx \rho_2(x)$$

$$\rho_{1,2}(x) = \frac{2}{\pi \mu^2} \sqrt{\mu^2 - x^2}$$

$$\mu = \frac{\sqrt{\lambda}}{2\pi}$$

$$\frac{2}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

Suyama, Rey '10

Mitev, Pomoni '14

$W_a \sim e^{\sqrt{\lambda}}$ consistent w. AdS/CFT since $T = \frac{\sqrt{\lambda}}{2\pi}$

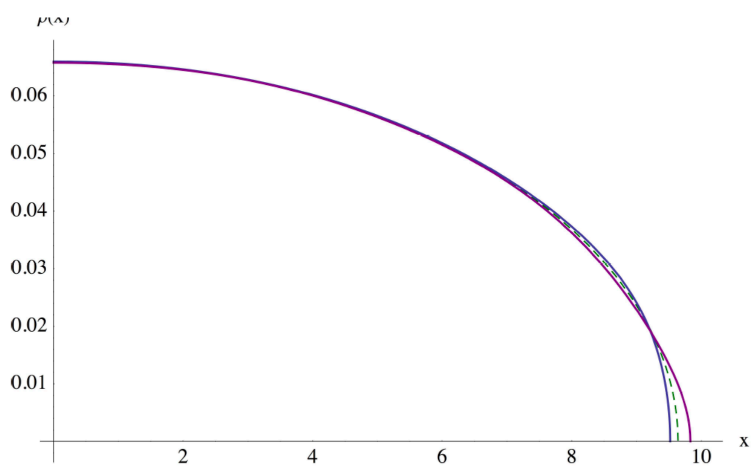


Figure 2: The eigenvalue densities ρ_1 (purple line) and ρ_2 (blue line) obtained by numerically solving (2.6), (2.7) for $\lambda_1 = 5320$, $\lambda_2 = 2797$. The dashed line is the Wigner distribution with the effective coupling $\lambda = 3667$. The density for the gauge

Strong-coupling expansion

Ansatz:

$$\rho_a(x) = A\sqrt{\mu_a^2 - x^2} + \frac{2\mu_a A B_a}{\sqrt{\mu_a^2 - x^2}} + \frac{4\mu_a^2 A C_a}{(\mu_a^2 - x^2)^{\frac{3}{2}}} + \dots$$

$$\mathcal{O}(\sqrt{\lambda}) \qquad \mathcal{O}(1) \qquad \mathcal{O}(1/\sqrt{\lambda})$$

7 parameters: $A,\ \mu_a,\ B_a,\ C_a$ $\qquad a = 1,2$

Normalization:

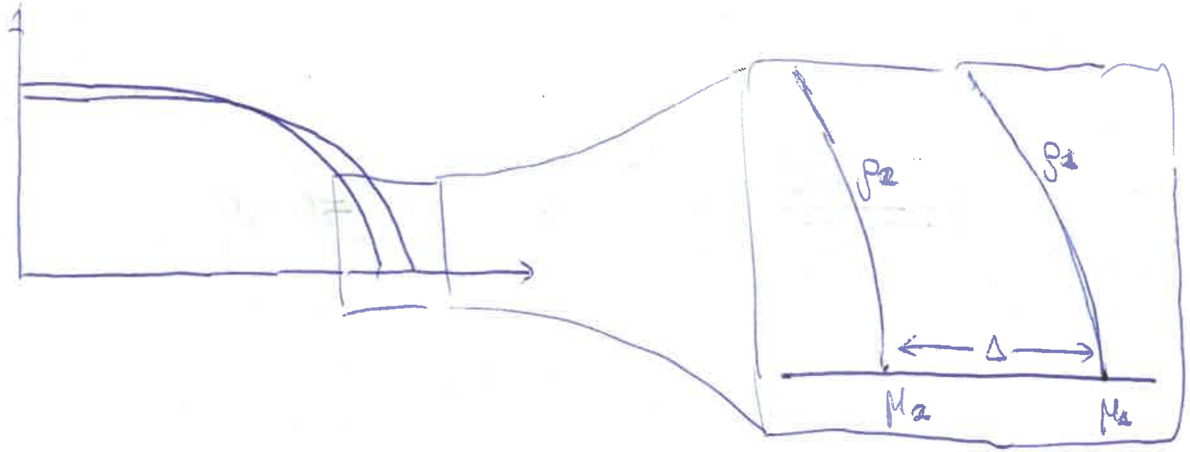
$$B_a = \frac{1}{2\pi A \mu_a} - \frac{\mu_a}{4}$$

Saddle-point eqs:

$$1 + \frac{\mu_{1,2}^2 - \mu_{2,1}^2}{2} - \frac{2}{\pi A}\ln\frac{\mu_{1,2}}{\mu_{2,1}} + 8\left(C_{1,2} - C_{2,1}\right) = \frac{8\pi}{A\lambda_{1,2}}$$

- $\#\text{unknowns} = 7 - 2 - 2 = 3$
- wrong boundary conditions $\quad \rho_{LO} \sim \sqrt{\mu - x}$ ✓ $\quad \rho_{NLO} \sim \frac{1}{\sqrt{\mu - x}}$ × $\quad \rho_{NNLO} \sim \frac{1}{(\mu - x)^{3/2}}$ ×

Boundary



$$\mu_{1,2} = \frac{\sqrt{\lambda}}{2\pi} + \alpha \pm \frac{\Delta}{2}$$

$$\boxed{\rho_a(x) \simeq A\sqrt{2\mu_a}\, f_a(\mu_a - x)}$$

$f_a(\xi)$ – scaling functions:

$$f_a(\xi) \sim J_a\sqrt{\xi} \qquad \xi \to 0$$

$$f_a(\xi) \sim \sqrt{\xi} + \frac{B_a}{\sqrt{\xi}} + \frac{C_a}{\xi^{3/2}} \qquad \xi \to \infty \quad \text{(matching to the bulk)}$$

$$\int_0^{\infty} d\eta \, R_{ab}(\xi-\eta) f_b(\eta) = g_a(\xi)$$

known source term

$$R(\xi) = \begin{bmatrix} \frac{1}{3} - K(\xi) & K(\xi-\Delta) \\ K(\xi+\Delta) & \frac{1}{3} - K(\xi) \end{bmatrix}$$

$$K(\xi) = \xi \left(\psi(1+i\xi) + \psi(1-i\xi) + 2\gamma \right)$$

Warm-up: integral equations of convolution type

$$\int_{-\infty}^{+\infty} d\eta \; R(\xi-\eta)\, f(\eta) = g(\xi)$$

Solved by Fourier transform:

$$R(\omega)\, f(\omega) = g(\omega)$$

$$f(\omega) = \frac{g(\omega)}{R(\omega)}$$

Wiener – Hopf problem

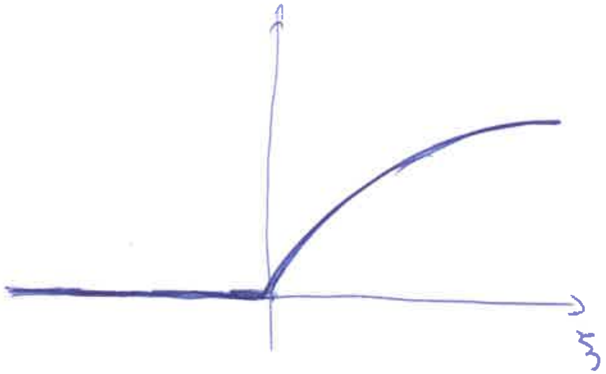
$$\int_0^\infty d\eta \, R(\xi-\eta)\, f(\eta) = g(\xi) \qquad \text{at} \quad \xi > 0$$

$$R(\omega)\, f(\omega) = g(\omega) + \underbrace{X(\omega)}_{\text{another unknown function}}$$

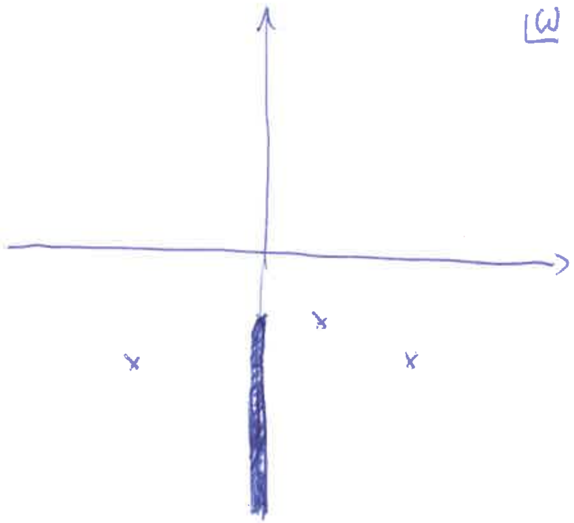
such that $X(\xi) = 0$ at $\xi > 0$.

Analyticity

$f(\xi)$:



$f(\omega)$



Projection:

$$\mathcal{F}_\pm(\omega) = \pm \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi i} \, \frac{\mathcal{F}(\nu)}{\nu - \omega \mp i0}$$

analytic in the upper/lower half-plane.

Wiener - Hopf problem : solution

$$R(\omega)\, \underbrace{f_+(\omega)}_{\text{unknown}} = g(\omega) + \underbrace{\chi_-(\omega)}_{\text{unknown}}$$

- Suppose $\boxed{G_- R = G_+}$

then : $$G_+ f_+ = G_- g + G_- \chi$$

- Apply + projection : $G_+ f_+ = (G_- g)_+$

- Multiply by G_+^{-1} :

$$\boxed{f_+ = G_+^{-1} (G_- g)_+}$$

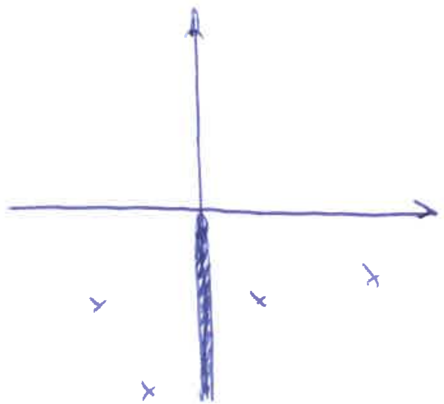
Riemann – Hilbert

$$R(\omega) = 2\pi i \,\mathrm{sign}\,\omega \,\coth\frac{\omega}{2} \begin{bmatrix} \coth\omega & -\dfrac{e^{i\Delta\omega}}{\sinh\omega} \\ -\dfrac{e^{-i\Delta\omega}}{\sinh\omega} & \coth\omega \end{bmatrix}$$

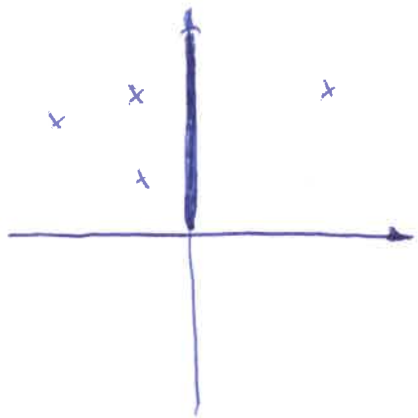
Q: find G_+ and G_- such that

$$G_-(\omega) R(\omega) = G_+(\omega)$$

G_+:



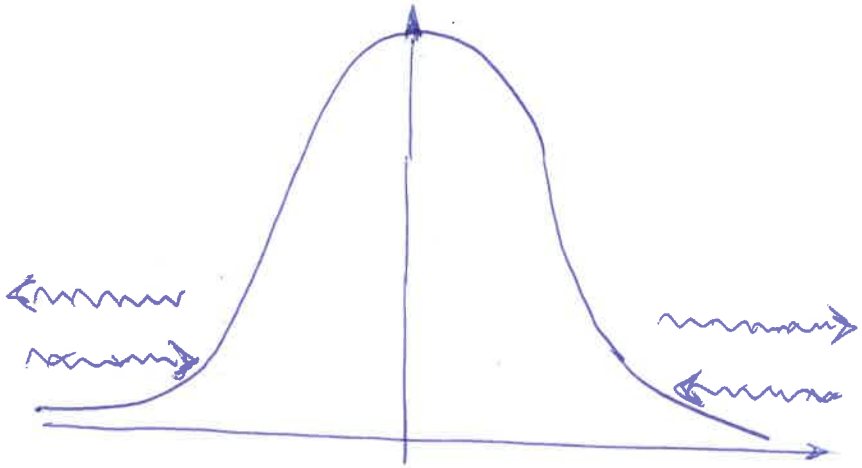
G_-



Antipov'15

Scattering in Quantum Mechanics

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = k^2\,\psi$$



Jost functions: $\psi \sim e^{\pm ikx}$

are half-plane analytic in k.

S-matrix:

$$W[\psi_R^- \quad \psi_L^-]\; S = W[\psi_L^+ \quad \psi_R^+]$$

Wronskian

G^- $\qquad$ R $\qquad$ G^+

Pöschl–Teller potential

$$V(x) = \frac{1}{4\cosh^2 x}$$

$$\phi_a^s = e^{-iaskx}\sqrt{1+e^{ax}}\;{}_2F_1\left(\tfrac{1}{2}-isk,\tfrac{1}{2};\,1-isk;\,-e^{ax}\right) \qquad \begin{matrix} a=\pm \\ s=\pm \end{matrix}$$

S-matrix:

$$S(k) = i\,\frac{B\left(\frac{1}{2}+ik,\frac{1}{2}\right)}{B\left(\frac{1}{2}-ik,\frac{1}{2}\right)}\begin{bmatrix} \tanh \pi k & -\dfrac{i e^{-ikx}}{\cosh \pi k} \\ -\dfrac{i e^{+ikx}}{\cosh \pi k} & \tanh \pi k \end{bmatrix}$$

$$\Big\downarrow \qquad k \to \frac{\omega}{\pi} + \frac{i}{2} \qquad x \to -\frac{\pi\Delta}{2}$$

$$R(\omega) = \begin{bmatrix} \coth \omega & -\dfrac{e^{i\Delta\omega}}{\sinh \omega} \\ -\dfrac{e^{-i\Delta\omega}}{\sinh \omega} & \coth \omega \end{bmatrix}$$

Solution for scaling functions

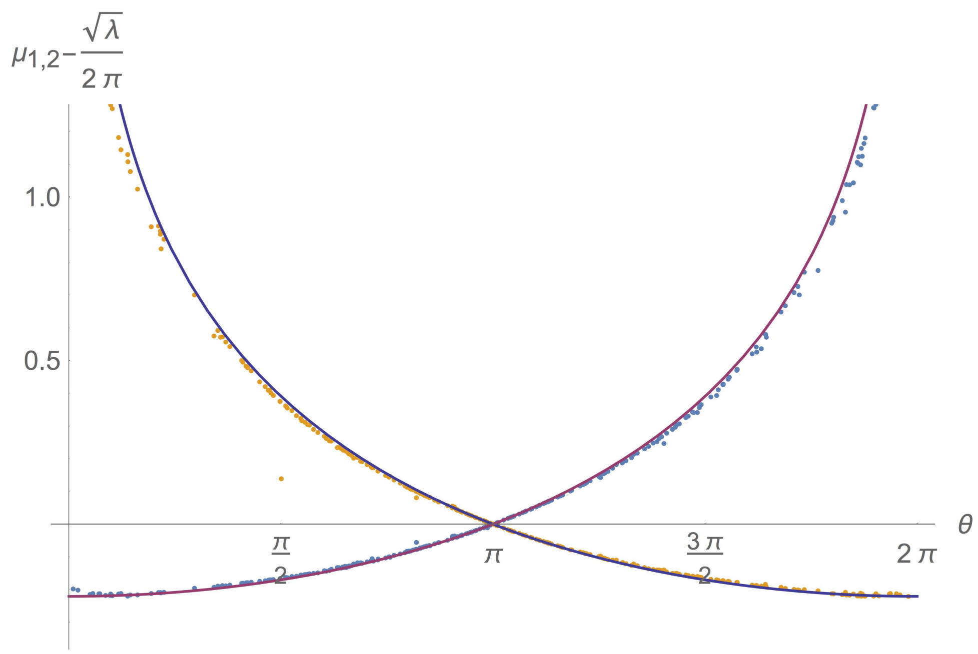
$$f_{1,2}(\omega) = \frac{i^{\frac{3}{2}} B\left(\frac{1}{2} - \frac{i\omega}{2\pi}, \frac{1}{2}\right)}{2\sqrt{\pi}\ \omega^{3/2}} \int_0^1 du \left(\frac{1 + e^{\pm\pi\Delta} u^2}{1-u^2} \right)^{\frac{i\omega}{\pi}} \left(1 - \frac{2i\omega}{\pi} \frac{1}{1 + e^{\pm\pi\Delta} u^2} \right)$$

- fixes remaining parameters from matching to the bulk:

$$\alpha = -\frac{1}{\pi} \ln \sin\frac{\theta}{2}$$

$$\Delta = \frac{2}{\pi} \ln \tan\frac{\theta}{2}$$

- trigonometric function of $\theta = \dfrac{2\pi\lambda_1}{\lambda_1 + \lambda_2}$?



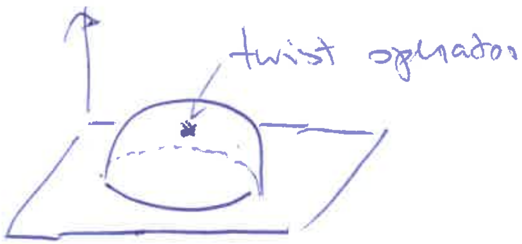
$\mu_{1,2}-\frac{\sqrt{\lambda}}{2\pi}$
1.0
0.5
θ
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

Wilson loops

$$W_{1,2} = 8\sqrt{\pi}\, f_{1,2}(2\pi i)\, \lambda^{-\frac{3}{4}}\, e^{\sqrt{\lambda} + 2\pi\alpha \pm \pi\Delta}$$

Twisted:

$$W_- = \frac{W_1 - W_2}{2 W_{SYM}}$$



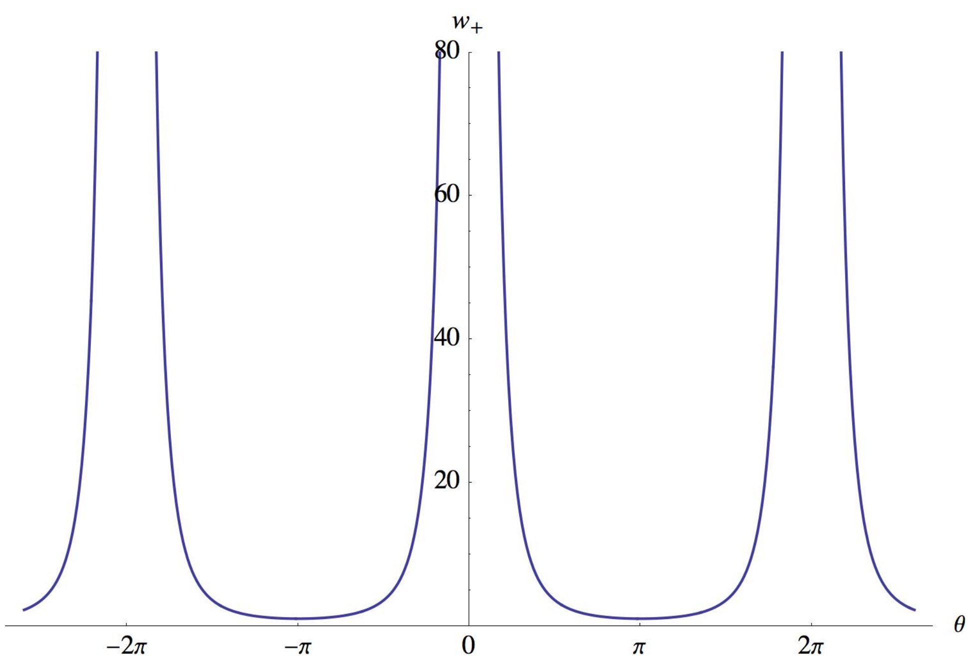
$$\boxed{W_- = -\frac{\pi \cos\frac{\theta}{2}}{2\sin^3\frac{\theta}{2}}}$$

Untwisted:

$$W_+ = \frac{W_1 + W_2}{W_{SYM}}$$

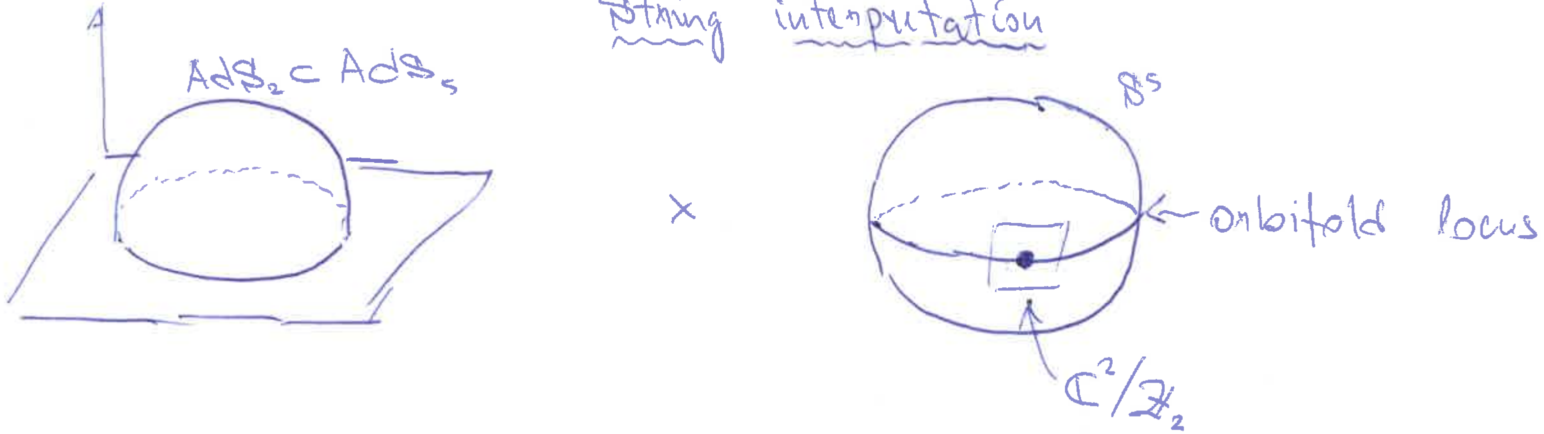
$$\boxed{W_+ = \frac{1 + \frac{\pi - \theta}{2}\cot\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}}$$

$$\theta = \frac{2\pi\lambda_1}{\lambda_1 + \lambda_2}$$



w+
80
60
40
20
θ
−2π
−π
0
π
2π

String interpretation



$$W_+ = \frac{W_1 + W_2}{2 W_{SYM}} = \mathcal{Z}_{\mathbb{C}^2/\mathbb{Z}_2}(\theta) \equiv \int \mathcal{D}X \, e^{-\int \sqrt{\lambda}\,(\partial X)^2 + i\theta\, dX \wedge dX}$$

$$X^M : \underbrace{AdS_2}_{\text{worldsheet}} \longrightarrow \underbrace{\mathbb{C}_2/\mathbb{Z}_2}_{\text{target}}$$

$$\mathcal{Z}_{\mathbb{C}^2/\mathbb{Z}_2}(\theta) = \lim_{\varepsilon \to 0} \sum_k \underbrace{\mathcal{A}_k}_{k\text{-instanton amplitude}} e^{-\sqrt{\lambda}\,\varepsilon |k| + i\theta k}$$

Localization prediction:

$$W_+ = \frac{1 + \frac{\pi - \theta}{2} \cot \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}}$$

$$\Omega_k \stackrel{?}{=} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-ik\theta} \frac{1 + \frac{\pi-\theta}{2} \cot \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} = \infty$$

Ill-defined...

- How to regularize?

- Problems w. $\varepsilon \to 0$ on the string side?

Conclusions

- Direct field theory predictions for $AdS_5 \times (S^5/\mathbb{Z}_n)$

  $n=2$: this talk

  $n>2$: Ouyang '20

- Severe singularities at $\theta \to 0$ and $\theta \to 2\pi$.

  / this corresponds to λ_1 or $\lambda_2 \to 0$

  so perhaps not too surprising /

- Orbifold σ-model on AdS_2 w. $\mathbb{C}^2/\mathbb{Z}_2$ target ?