Some aspects of massive gravity and Horndeski theory

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Plan

- A brief introduction to massive gravity
- New theory of free massive gravitons in an arbitrary spacetime.
- Anisotropy screening in Horndeski cosmologies
- Palatini versions of the Horndeski theory

Motivations for massive gravity

Cosmic acceleration $\Rightarrow$ dark energy problem.

- either Λ-term, very natural phenomenologically,

$$G_{\mu\nu} = \kappa T_{\mu\nu} \quad \rightarrow \quad G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

  but unnatural from the QFT viewpoint

- or modification of gravity (many options). Massive gravity:

$$\text{Newton} \quad \frac{1}{r} \quad \rightarrow \quad \text{Yukawa} \quad \frac{1}{r} e^{-mr}$$

  $m \sim 1/(\text{Hubble radius}) \sim 10^{-33}$ eV. If $r <$ Hubble, then Yukawa=Newton, usual physics. Screening for $r \geq$ Hubble $\Rightarrow$ gravity is weaker at large distance = cosmic acceleration.

- From QFT viewpoint small m is more natural (multiplicative renormalization) than small Λ (additive renormalization).

Fierz-Pauli massive gravity

Linear massless gravitons – linearized GR

$\mathcal{L} = \frac{1}{2\kappa} R\sqrt{-g} + \mathcal{L}_{\mathrm{matter}}$ /$\kappa = 8\pi G$, signature $(-+++)$ /

$$\boxed{G_{\mu\nu} = \kappa T_{\mu\nu}}$$

If $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ then /check this/

$$\begin{aligned} - & \frac{1}{2}\left\{ \Box h_{\mu\nu} - \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu} + \eta_{\mu\nu}(\partial^\alpha \partial^\beta h_{\alpha\beta} - \Box h) + \partial_{\mu\nu} h \right\} \\ \equiv & -\frac{1}{2}(\Box h_{\mu\nu} + \ldots) = \kappa T_{\mu\nu} \end{aligned}$$

so that

$$\boxed{\Box h_{\mu\nu} + \ldots = -2\kappa T_{\mu\nu}}$$

Gauge invariance $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ does not change the l.h.s. $\Rightarrow$ Bianchi identities

$$0 \equiv \partial^\mu(\Box h_{\mu\nu} + \ldots) \quad \Rightarrow \quad \partial^\mu T_{\mu\nu} = 0$$

DoF counting

Gauge invariance $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ implies that one can impose gauge conditions. With $\mathbf{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu}$ one requires

$$\partial^\mu \mathbf{h}_{\mu\nu} = 0 \qquad\qquad \text{4 gauge conditions}$$

and the equations reduce to

$$\Box \mathbf{h}_{\mu\nu} = -2\kappa T_{\mu\nu}$$

Suppose $T_{\mu\nu} = 0$. Then $\mathbf{h}_{\mu\nu}$ are harmonic, and there is still residual gauge freedom generated by harmonic $\Box \xi_\mu = 0 \Rightarrow$ one can impose 4 more conditions $\Rightarrow$ $2 = 10 - 4 - 4$ DoF. If $T_{\mu\nu} = 0$

$$\mathbf{h} = 0, \quad \mathbf{h}_{0k} = 0 \qquad \Rightarrow \qquad \mathbf{h}_{00} = 0, \quad \partial_i \mathbf{h}_{ik} = 0$$

the solution is

$$\mathbf{h}_{\mu\nu}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & D_+ & D_\times & 0 \\ 0 & D_\times & -D_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik(t-z)}$$

Linear massive gravitons – Fierz and Pauli /1939/

$\Box\phi = 0 \Rightarrow \Box\phi = m^2\phi$. Similarly for gravitons $/h = \eta^{\mu\nu} h_{\mu\nu}/$

$$\Box h_{\mu\nu} + \ldots = m^2 (h_{\mu\nu} - \alpha\, h\, \eta_{\mu\nu}) - 2\kappa T_{\mu\nu}$$

$\Rightarrow$ no gauge invariance anymore. Taking the divergence gives 4 constraints

$$m^2 (\partial^\mu h_{\mu\nu} - \alpha \partial_\nu h) = 0$$

Taking the trace and using the 4 constraints gives

$$2(\alpha - 1)\,\Box h = m^2 (1 - 4\alpha)\, h - 2\kappa T$$

$\Rightarrow$ for $\alpha = 1$ one gets the fifth constraint

$$h = -\frac{2\kappa}{3m^2}\, T$$

$\Rightarrow$ $10 - 5 = 5$ DoF=graviton polarizations.

FP equations

$$
\begin{aligned}
\Box h_{\mu\nu} \;&-\; \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu} + \eta_{\mu\nu}(\partial^\alpha \partial^\beta h_{\alpha\beta} - \Box h) \\
&+\; \partial_{\mu\nu} h = m^2 (h_{\mu\nu} - \; h\,\eta_{\mu\nu}) - 2\kappa T_{\mu\nu}
\end{aligned}
$$

are equivalent to

$$
\begin{aligned}
\Box h_{\mu\nu} - \partial_{\mu\nu} h &= m^2 (h_{\mu\nu} - h\,\eta_{\mu\nu}) - 2\kappa T_{\mu\nu} \\
\partial^\mu h_{\mu\nu} &= \partial_\nu h \\
h &= -\frac{2\kappa}{3m^2}\, T
\end{aligned}
$$

They describe free massive gravitons in flat space. Each graviton has 5 degrees of freedom = 5 spin polarizations.

Theory is NOT invariant under $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

Veltman-van Dam-Zakharov (VdVZ) discontinuity

Massless case

$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{4\kappa}\, h_{\mu\nu}$ with gauge condition $\partial^{\mu}\left(h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu}\right) = 0$

$$\Box h_{\mu\nu} = -\sqrt{\kappa}\left(T_{\mu\nu} - \frac{T}{2}\,\eta_{\mu\nu}\right)$$

in Fourier representation $\Box \to -k^2$ hence

$$h_{\mu\nu}(k) = \frac{\sqrt{\kappa}}{k^2}\left(T_{\mu\nu} - \frac{T}{2}\,\eta_{\mu\nu}\right) = \sqrt{\kappa}\, D_{\mu\nu\alpha\beta}\, T^{\alpha\beta}$$

where

$$D_{\mu\nu\alpha\beta}(k) = \frac{1}{2k^2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta} - \eta_{\alpha\beta}\eta_{\mu\nu}\right)$$

assuming $k^{\mu} = (0, \vec{k})$ the scattering amplitude

$$\mathcal{M}(\vec{k}) = (\sqrt{\kappa})^2\, T_1^{\mu\nu}(\vec{k}) D_{\mu\nu\alpha\beta}(\vec{k})\, T_2^{\mu\nu}(-\vec{k})$$

Massless case

$$T_a^{\mu\nu}(\vec{x}) = M_a \delta_0^\mu \delta_0^\nu \delta(\vec{x} - \vec{x}_a) \qquad \Rightarrow T_a^{\mu\nu}(\vec{k}) = M_a \delta_0^\mu \delta_0^\nu \exp(i\vec{k}\vec{x}_a)$$

where $a = 1, 2$. The interaction potential

$$V = -\frac{1}{(2\pi)^3} \int d^3k \mathcal{M}(\vec{k}) = -\frac{GM_1 M_2}{r}$$

Massive case

FP equations

$$\begin{aligned} \Box h_{\mu\nu} - \partial_{\mu\nu} h &= m^2 (h_{\mu\nu} - h\,\eta_{\mu\nu}) - 2\kappa T_{\mu\nu} \\ \partial^\mu h_{\mu\nu} &= \partial_\nu h, \qquad h = -\frac{2\kappa}{3m^2}\,T \end{aligned}$$

after rescaling $h_{\mu\nu} \to \sqrt{4\kappa} h_{\mu\nu}$ reduce to

$$(\Box - m^2)\,h_{\mu\nu} = -\sqrt{\kappa}\left(T_{\mu\nu} - \frac{T}{3}\,\eta_{\mu\nu} + \frac{1}{3m^2}\,\partial_\mu \partial_\nu T\right)$$

hence

$$h_{\mu\nu}(k) = \frac{\sqrt{\kappa}}{k^2 + m^2}\left(T_{\mu\nu} - \frac{T}{3}\,\eta_{\mu\nu} - \frac{1}{3m^2}\,k_\mu k_\nu\; T\right) = \sqrt{\kappa}\,D_{\mu\nu\alpha\beta}\,T^{\alpha\beta}$$

and computing the potential yields

$$V = -\frac{4}{3}\frac{GM_1 M_2}{r}\,e^{-mr} \quad \to \quad -\frac{4}{3}\frac{GM_1 M_2}{r}$$

$\Rightarrow$ extra force due to the scalar graviton.

VdVZ solution

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}R'^2(r)dr^2 + r^2 e^{\mu(r)}d\Omega^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$$

where $R(r) = re^{\mu(r)/2}$. Expanding for small ν, λ, μ yields $h_{\mu\nu}$ and the FP equations admit exact solution

$$\nu = -\frac{2C}{r}\, e^{-mr}, \qquad \lambda = \frac{C}{r}\,(1 + mr)\, e^{-mr}$$

$$\mu = C\,\frac{1 + mr + (mr)^2}{m^2 r^3}\; e^{-mr}$$

In the near zone, for $r \ll 1/m$, this reduces to the VdVZ solution

$$\nu = -\frac{2C}{r}, \qquad \lambda = \frac{C}{r}, \qquad \mu = \frac{C}{r(mr)^2}$$

hence $\nu + \lambda \neq 0$ (in GR $\lambda = -\nu = r_g/r$) $\Rightarrow$ either Newton law is wrong or the light bending is wrong, depending on choice of C. The mass m in the denominator suggests that non-linear corrections are important (Vainshtein).

Non-linear Fierz-Pauli – the bimetric theory

Non-linear FP

$$S = \frac{1}{\kappa}\int \sqrt{-g}\left(\frac{1}{2}R(g) - m^2\,U(g,f)\right)d^4x + S_{\rm mat}$$

where U is a scalar function of $g_{\mu\nu}$. One cannot construct a scalar using only $g_{\mu\nu}$. However, if there is a second fixed non-dynamical reference metric $f_{\mu\nu} = \eta_{\mu\nu}$ then one defines

$$\hat{\mathcal{S}} = \hat{1} - \hat{g}^{-1}\hat{f} \qquad \Rightarrow \qquad \mathcal{S}^{\mu}_{\ \nu} = \delta^{\mu}_{\nu} - g^{\mu\sigma}f_{\sigma\nu}$$

and then one can choose any function (infinitely many options)

$$U = U([\hat{\mathcal{S}}], [\hat{\mathcal{S}}^2], [\hat{\mathcal{S}}^3], \det\hat{\mathcal{S}}).$$

In the weak field limit $g_{\mu\nu} = f_{\mu\nu} + h_{\mu\nu}$ and $\mathcal{S}_{\mu\nu} = h_{\mu\nu} + \ldots$ The correct FP limit for small $\hat{\mathcal{S}}$ is achieved if

$$U = \frac{1}{8}\left([\hat{\mathcal{S}}^2] - [\hat{\mathcal{S}}]^2\right) + \mathcal{O}(\mathcal{S}^3)$$

One can allow for diffeomorphisms by setting

$$f_{\mu\nu} = \eta_{AB}\partial_\mu\Phi^A\partial_\nu\Phi^B$$

where Φ^A are Stueckelberg scalars.

Equations

$$G_{\mu\nu} = m^2 T_{\mu\nu} \quad \Rightarrow \quad \nabla^\mu T_{\mu\nu} = 0$$

where

$$T_{\mu\nu} = 2\frac{\partial U}{\partial g_{\mu\nu}} - U g_{\mu\nu}$$

VdVZ and Vainshtein mechanism

Vainshtein /1972/

Let us consider a non-linear FP

$$S = \frac{1}{\kappa} \int \left(\frac{1}{2} R - \frac{m^2}{8} (\mathcal{S}^{\alpha}{}_{\beta} \mathcal{S}^{\beta}{}_{\alpha} - (\mathcal{S}^{\alpha}{}_{\alpha})^2) \right) \sqrt{-g}\, d^4 x + S_{\mathrm{mat}}$$

with $\mathcal{S}^{\mu}_{\nu} = \delta^{\mu}_{\nu} - g^{\mu\alpha} \eta_{\alpha\nu}$

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} R'^2 dr^2 - R^2 d\Omega^2$$

with $R = r\, e^{\mu/2}$ and compute non-linear corrections to the VdVZ. At large r, one looks for solutions of $G_{\mu\nu} = m^2 T_{\mu\nu}$ in the form

$$\nu(r) = \sum_{n \geq 1} \kappa^n \nu_n(r), \qquad \lambda(r) = \sum_{n \geq 1} \kappa^n \lambda_n(r), \qquad \mu(r) = \sum_{n \geq 1} \kappa^n \mu_n(r).$$

the $n = 1$ terms being the VdVZ solution

Large r solution

$$
\begin{aligned}
\nu &= -\frac{2r_g}{r}\left(1 + c_1\,\frac{r_g}{m^4 r^5} + \dots\right)\\
\lambda &= \frac{r_g}{r}\left(1 + c_2\,\frac{r_g}{m^4 r^5} + \dots\right)\\
\mu &= \frac{r_g}{m^2 r^3}\left(1 + c_3\,\frac{r_g}{m^4 r^5} + \dots\right)
\end{aligned}
$$

Leading terms are the VdVZ solution. For $m \sim 1/(\text{Hubble radius})$ the non-linear terms become small at

$$
r \gg r_V = \left(r_g/m^4\right)^{1/5} \sim 400\,000 \text{ light years}
$$

The VdVZ problem therefore arises only for $r \gg r_V$.

What happens for $r < r_V$?

Small r solution – expansion over m^2

$$\nu(r) = \sum_{n\geq 0} m^{2n}\nu_n(r), \quad \lambda(r) = \sum_{n\geq 0} m^{2n}\lambda_n(r), \quad \mu(r) = \sum_{n\geq 0} m^{2n}\mu_n(r),$$

it is assumed that ν_0, λ_0 are small, their equations are linearized, while μ_0 is not small and its equation is fully non-linear. For $r_V \gg r \gg r_g$ one finds

$$\begin{aligned}
\nu &= -\frac{r_g}{r}\left(1 + a_1\,(mr)^2\sqrt{r/r_g} + \ldots\right) \\
\lambda &= \frac{r_g}{r}\left(1 + a_2\,(mr)^2\sqrt{r/r_g} + \ldots\right) \\
\mu &= \sqrt{\frac{ar_g}{r}}\left(1 + a_3\,(mr)^2\sqrt{r/r_g} + \ldots\right)
\end{aligned}$$

so ν, λ show the GR behavior. Corrections are small for $r \ll r_V \Rightarrow$ one recovers GR in the non-linear regime.

Summary

- Free massive gravitons are described by the linear Fierz-Pauli theory.
- This theory gives different from GR predictions in the $m \to 0$ limit due to the additional attraction mediated by the scalar graviton (VdVZ problem).
- In non-linear generalizations of the FP theory the scalar graviton is strongly coupled by non-linear effects within the Vainshtein radius

  $$r_V = \left(\frac{r_g}{m^4}\right)^{1/5}$$

  This pushes the VdVZ effect to the region $r \gg r_V$ and restores GR for $r < r_V$.
- This suggests that theories with massive gravitons can agree with observations.

Boulware-Deser problem:
non-linear effects bring back the
ghost $=$ sixth DoF.

Fierz and Pauli with 6 DoF

$$\Box h_{\mu\nu} + \ldots = m^2 (h_{\mu\nu} - \alpha \, h \, \eta_{\mu\nu}) - 2\kappa T_{\mu\nu}$$

Taking the divergence gives 4 constraints

$$m^2 (\partial^\mu h_{\mu\nu} - \alpha \partial_\nu h) = 0$$

Taking the trace gives

$$2(\alpha - 1) \Box h = m^2 (1 - 4\alpha) \, h - 2\kappa T$$

$\Rightarrow$ for $\alpha = 1$ one gets the fifth constraint

$$h = -\frac{2\kappa}{3m^2} \, T$$

$\Rightarrow$ $10 - 5 = 5$ DoF=graviton polarizations. However, if $\alpha \neq 1$ then there are 6 DoF. The additional mode is a ghost: its kinetic energy is negative.

Boulware-Deser problem /1972/

The ghost can be removed in the linear FP theory by choosing $\alpha = 1$. However, it comes back in the non-linear FP. Therefore the latter make no sense.

This stopped all developments of massive gravity for almost 40 years.

Hamiltonian formulation

The Lagrangian

$$\mathcal{L} = \left(\frac{1}{2} R - m^2 U\right) \sqrt{-g}$$

after the ADM decomposition

$$\begin{aligned} ds_g^2 &= -N^2 dt^2 + \gamma_{ik}(dx^i + N^i dt)(dx^k + N^k dt) \\ ds_f^2 &= -dt^2 + \delta_{ik} dx^i dx^k \end{aligned}$$

becomes

$$\mathcal{L} = \frac{1}{2}\sqrt{\gamma}\, N \left(K_{ik}K^{ik} - K^2 + R^{(3)}\right) - m^2 \mathcal{V}(N^\mu, \gamma_{ik}) + \text{total derivative}$$

where $\mathcal{V} = \sqrt{\gamma}\, N\, \mathcal{U}$ and the second fundamental form

$$K_{ik} = \frac{1}{2N}\left(\dot{\gamma}_{ik} - \nabla_i^{(3)} N_k - \nabla_k^{(3)} N_i\right)$$

Variables are γ_{ik} and $N^\mu = (N, N^k)$.

Hamiltonian

Canonical momenta

$$\pi^{ik} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ik}} = \frac{1}{2}\sqrt{\gamma}(K^{ik} - K\gamma^{ik}), \qquad \boxed{p_{N_\mu} = \frac{\partial \mathcal{L}}{\partial \dot{N}^\mu} = 0} \quad \text{constraints}$$

$\Rightarrow$ N^ν are non-dynamical $\Rightarrow$ phase space is spanned by 12 variables (π^{ik}, γ_{ik}) = 6 DoF. Hamiltonian

$$\boxed{H = \pi^{ik}\dot{\gamma}_{ik} - \mathcal{L} = N^\mu \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \mathcal{V}(N^\mu, \gamma_{ik})}$$

with

$$\mathcal{H}_0 = \frac{1}{\sqrt{\gamma}}\left(2\pi_{ik}\pi^{ik} - (\pi^k_k)^2\right) - \frac{1}{2}\sqrt{\gamma}R^{(3)}, \quad \mathcal{H}_k = -2\nabla^{(3)}_i \pi^i_k$$

Secondary constrints

$$-\dot{p}_{N_\mu} = \frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\mu, \gamma_{ik})}{\partial N^\mu} = 0$$

Degrees of freedom, $m = 0$

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\mu, \gamma_{ik})}{\partial N^\mu} = 0 \qquad (\star)$$

- If $m = 0$ this gives 4 constraints

$$\mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) = 0$$

They are first class

$$\{\mathcal{H}_\mu, \mathcal{H}_\nu\} \sim \mathcal{H}_\alpha$$

and generate gauge symmetries, one can impose 4 gauge conditions, there remain 4 independent phase space variables

$$12 - 4 - 4 = 4 = 2 \times (2 \text{ DoF}) \quad \Rightarrow \quad 2 \text{ graviton polarizations}$$

Energy vanishes on the constraint surface (up to a surface term)

$$H = N^\mu \mathcal{H}_\mu = 0$$

Degrees of freedom, $m \neq 0$

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\mu, \gamma_{ik})}{\partial N^\mu} = 0 \qquad (\star)$$

- If $m \neq 0$ this gives 4 equations for laps and shifts whose solution is $N^\mu(\pi^{ik}, \gamma_{ik})$. No constraints arise $\Rightarrow$ there are

  $$12 = 2 \times (6 \text{ degrees of freedom})$$

  Inserting $N^\mu = N^\mu(\pi^{ik}, \gamma_{ik})$ back to the Hamiltonian

  $$H = N^\mu \mathcal{H}_\mu + m^2 \mathcal{V}(N^\mu, \gamma_{ik})$$

  yields $H(\pi^{ik}, \gamma_{ik})$ whose kinetic energy part is not positive-definite $\Rightarrow$ the energy is unbounded from below. This is related to the sixth DoF=ghost. Its contribution vanishes on flat background if $\alpha = 1$, but it comes back on arbitrary background.

Non-linear Fierz-Pauli theory cures the VdVZ but brings the ghost back /Boulware-Deser 1972/

Ghost-free massive gravity

Ghost-free massive gravity /2010/

One has

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\mu, \gamma_{ik})}{\partial N^\mu} = 0 \qquad (\star)$$

If $\mathcal{V} = \sqrt{-g}\,U$ is linear in N then

$$\text{rank}\left(\frac{\partial^2 \mathcal{V}(N^\mu, \gamma_{ik})}{\partial N^\nu \partial N^\mu}\right) = 3$$

$\Rightarrow$ the 4 equations $(\star)$ determine only 3 shifts $N^k = N^k(\pi^{ik}, \gamma_{ik})$, the lapse N remains undetermined, the 4-th equation reduces to a constraint

$$\mathcal{C}(\pi^{ik}, \gamma_{ik}) = 0 \qquad \Rightarrow \qquad \dot{\mathcal{C}} = \{\mathcal{C}, H\} \equiv S = 0.$$

The two constraints $\mathcal{C}, \mathcal{S}$ remove one DoF, there remain 5.
How to make $\mathcal{V}$ to be linear in $\mathcal{N}$?

The idea

$$
\begin{aligned}
ds_g^2 &= -N^2 dt^2 + \gamma_{ik}(dx^i + N^i dt)(dx^k + N^k dt) = \eta_{ab} e^a_\mu e^b_\nu \\
ds_f^2 &= -dt^2 + \delta_{ik} dx^i dx^k = \eta_{ab} f^a_\mu f^b_\nu
\end{aligned}
$$

N is contained only in $e^0 = N dt$. One chooses

$$
\begin{aligned}
& \int \mathcal{U} \sqrt{-g} d^4x = \int \epsilon_{abcd} \left(\frac{b_0}{4!} e^a \wedge e^b \wedge e^c \wedge e^d \right. \\
+ & \frac{b_1}{3!} e^a \wedge e^b \wedge e^c \wedge f^d + \frac{b_2}{2!2!} e^a \wedge e^b \wedge f^c \wedge f^d \\
+ & \left. \frac{b_3}{3!} e^a \wedge f^b \wedge f^c \wedge f^d + \frac{b_4}{4!} f^a \wedge f^b \wedge f^c \wedge f^d \right) \Big\}
\end{aligned}
$$

Due to asymmetry, e^0 enters each term not more then once – the expression is linear in N.

dRGT theory

Explicitely

$$S = M_{\rm Pl}^2 \int \left(\frac{1}{2} R - m^2\, U \right) \sqrt{-g}\, d^4x$$

$$U = b_0 + b_1 \sum_a \lambda_a + b_2 \sum_{a<b} \lambda_a \lambda_b + b_3 \sum_{a<b<c} \lambda_a \lambda_b \lambda_c + b_4 \lambda_0 \lambda_1 \lambda_2 \lambda_3$$

where b_k are parameters and λ_a are eigenvalues of the matrix

$$\boxed{\gamma^\mu{}_\nu = e^\mu_a f^a_\nu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}}$$

/de Rham, Gabadadze, Tolley 2010/

Field equations

$$G_{\mu\nu} = m^2 T_{\mu\nu}$$

with

$$\begin{aligned} T_{\mu\nu} &= -b_0\, g_{\mu\nu} + b_1 \{\gamma_{\mu\nu} - [\gamma]\, g_{\mu\nu}\} \\ &+ b_2 \frac{f}{e} \{(\gamma^{-2})_{\mu\nu} - [\gamma^{-1}](\gamma^{-1})_{\mu\nu}\} \\ &- b_3 \frac{f}{e} (\gamma^{-1})_{\mu\nu} \end{aligned}$$

$\gamma^{\mu}{}_{\alpha}\gamma^{\alpha}{}_{\nu} = g^{\mu\alpha} f_{\alpha\nu}$ and $\gamma_{\mu\nu} = g_{\mu\sigma}\gamma^{\sigma}{}_{\nu}$.

Bigravity

Bigravity

$$S = \frac{1}{2\kappa_1}\int R(g)\sqrt{-g}d^4x + \frac{1}{2\kappa_2}\int R(f)\sqrt{-f}d^4x - \frac{m^2}{\kappa_1+\kappa_2}\int \mathcal{U}\sqrt{-g}d^4x + S_{\rm mat}[g,\Psi_g] + S_{\rm mat}[f,\Psi_f]$$

with the same potential as before

$$\mathcal{U} = b_0 + b_1\sum_a \lambda_a + b_2\sum_{a<b}\lambda_a\lambda_b + b_3\sum_{a<b<c}\lambda_a\lambda_b\lambda_c + b_4\lambda_0\lambda_1\lambda_2\lambda_3$$

There is interchange symmetry

$$g_{\mu\nu}\leftrightarrow f_{\mu\nu} \quad \kappa_1\leftrightarrow\kappa_2 \quad b_k\leftrightarrow b_{4-k} \quad T^{\rm mat}_{\mu\nu}(g)\leftrightarrow T^{\rm mat}_{\mu\nu}(f)$$

7 DoF = one massive + one massless graviton

/Hassan and Rosen 2012/

Field equations

$$G_{\mu\nu}(g) = m^2 \cos^2\eta \, T_{\mu\nu}(g,f) + \kappa_1 \, T^{\rm mat}_{\mu\nu}(g)$$

$$G_{\mu\nu}(f) = m^2 \sin^2\eta \, \mathcal{T}_{\mu\nu}(g,f) + \kappa_2 \, T^{\rm mat}_{\mu\nu}(f)$$

with $\tan^2\eta = \kappa_2/\kappa_1$ and

$$T^\mu_{\ \nu} = g^{\mu\alpha} T_{\alpha\nu} = \tau^\mu_{\ \nu} - \mathcal{U}\,\delta^\mu_\nu$$

$$\mathcal{T}^\mu_{\ \nu} = f^{\mu\alpha} \mathcal{T}_{\alpha\nu} = -\frac{\sqrt{-g}}{\sqrt{-f}}\,\tau^\mu_{\ \nu}$$

with

$$\begin{aligned}
\tau^\mu_{\ \nu} \ &= \ \{b_1\,\mathcal{U}_0 + b_2\,\mathcal{U}_1 + b_3\,\mathcal{U}_2 + b_4\,\mathcal{U}_3\}\,\gamma^\mu_{\ \nu} \\
&- \ \{b_2\,\mathcal{U}_0 + b_3\,\mathcal{U}_1 + b_4\,\mathcal{U}_2\}\,(\gamma^2)^\mu_{\ \nu} \\
&+ \ \{b_3\,\mathcal{U}_0 + b_4\,\mathcal{U}_1\}\,(\gamma^3)^\mu_{\ \nu} \\
&- \ \{b_4\,\mathcal{U}_0\}\,(\gamma^4)^\mu_{\ \nu}
\end{aligned}$$

In the limit where $\kappa_2 \to 0$ and $f_{\mu\nu} \to \eta_{\mu\nu}$ the theory reduces to the dRGT massive gravity $\Rightarrow$ dRGT is contained in the bigravity.

Flat space

$$g_{\mu\nu} = f_{\mu\nu} = \eta_{\mu\nu}$$

is a solution if

$$\begin{aligned} b_0 &= 4c_3 + c_4 - 6, \quad b_1 = 3 - 3c_3 - c_4, \quad b_2 = 2c_3 + c_4 - 1 \\ b_3 &= -(c_3 + c_4), \quad b_4 = c_4 \end{aligned}$$

Small fluctuations $g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$ and $f_{\mu\nu} = \eta_{\mu\nu} + \delta f_{\mu\nu}$

$$h^{\mathrm{m}}_{\mu\nu} = \cos\eta\,\delta g_{\mu\nu} + \sin\eta\,\delta f_{\mu\nu} \qquad h^{0}_{\mu\nu} = \cos\eta\,\delta f_{\mu\nu} - \sin\eta\,\delta g_{\mu\nu}$$

fulfill

$$\begin{aligned} (\Box + \ldots)h^{\mathrm{m}}_{\mu\nu} &= m^2(h^{\mathrm{m}}_{\mu\nu} - h^{\mathrm{m}}\eta_{\mu\nu}) \\ (\Box + \ldots)h^{0}_{\mu\nu} &= 0 \end{aligned}$$

$\Rightarrow$ theory contains a massive graviton and a massless one (7 DoF)

Cosmologies and black holes

Proportional solutions

Setting

$$f_{\mu\nu} = C^2 g_{\mu\nu}$$

$\Rightarrow$ constants C must fulfil fourth order algebraic equation with coefficients depending on b_k, η. Equations reduce to

$$G_{\mu\nu}(g) + \Lambda g_{\mu\nu} = 0 \quad \text{with} \quad \Lambda = m^2 \cos^2 \eta \, F(C)$$

$C = 1$ is always a solution of $(*)$ in which case $\Lambda = 0 \Rightarrow$ one recovers vacuum GR for $f_{\mu\nu} = g_{\mu\nu}$ $\Rightarrow$ all vacuum solutions: black holes etc. However, Schwarzschild becomes unstable.

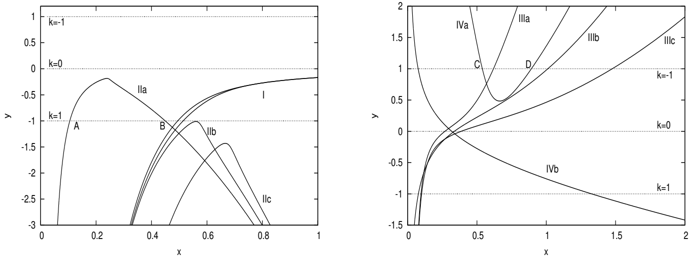
$C \neq 0 \Rightarrow \Lambda \neq 0$. For $\Lambda > 0$ one obtains de Sitter – accelerating universe with $\Lambda \sim m^2$ – acceleration driven by graviton mass.

$$ds_g^2 = -dt^2 + e^{2\Omega}\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \qquad /k = 0, \pm 1/$$

$$ds_f^2 = -\mathcal{A}^2 dt^2 + e^{2\mathcal{W}}\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

Amplitudes $\mathcal{A}, \mathcal{W}$ can be expressed in terms of $\mathbf{a} = e^{\Omega}$

$$\dot{\mathbf{a}}^2 + \mathbf{U}(\mathbf{a}) = -k$$



Various solutions.

Bianchi types /K.i. Maeda, M.S.V. 2013/

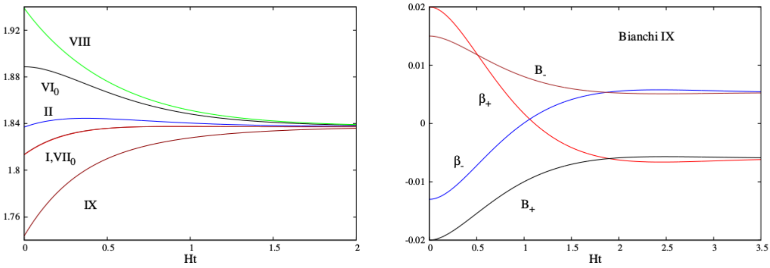
$$ds_g^2 = -dt^2 + dl_g^2 \qquad ds_f^2 = -\mathcal{A}^2(t)dt^2 + dl_f^2$$

$$\begin{aligned} dl_g^2 &= e^{2\Omega}\left(e^{2\beta_+ + 2\sqrt{2}\beta_-}(\omega^1)^2 + e^{2\beta_+ - 2\sqrt{2}\beta_-}(\omega^2)^2 + e^{-4\beta_+}(\omega^3)^2\right) \\ dl_f^2 &= e^{2\mathcal{W}}\left(e^{2\mathcal{B}_+ + 2\sqrt{2}\mathcal{B}_-}(\omega^1)^2 + e^{2\mathcal{B}_+ - 2\sqrt{2}\mathcal{B}_-}(\omega^2)^2 + e^{-4\mathcal{B}_+}(\omega^3)^2\right) \end{aligned}$$

$\langle\omega^a, e_b\rangle = \delta^a_b$ $[e_a, e_b] = C^c_{ab}e_c \Rightarrow$ Bianchi I,II,VI,VII,VIII,IX

Initial data at $t = t_0$: an anisotropic deformation of a finite size FLRW. f-sector is empty, g-sector contains radiation + dust. All solutions rapidly approach proportional backgrounds with constant $H = \dot{\Omega}$ and constant non-zero anisotropies= late time attractor.

Solutions



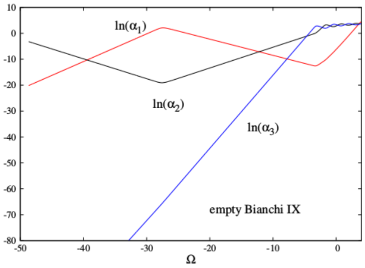
$\dot{\Omega}$ for all Bianchi types (left) and anisotropy parameters for Bianchi IX (right). At late time anisotropies oscillate around constant values $\beta_\pm = \beta_\pm(\infty) + const. \times e^{-3Ht}\cos(\omega t)$. The shear energy

$$\dot{\beta}_+^2 + \dot{\beta}_-^2 \sim e^{-3H} \sim 1/\mathbf{a}^3$$

behaves as a non-relativistic (dark ?) matter, while in GR it is $\sim 1/\mathbf{a}^6$.

Chaos

In the past solutions show singularity where e^{Ω} and $e^{\mathcal{W}}$ vanish, anisotropies oscillate near singularity.



Sequence of Kasner-type periods during which eigenvalues of the three-metric

$$\alpha_a \sim t^{p_a} \quad \text{with} \quad p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^1$$

$$1/\mathbf{a}^6 \quad \leftarrow \quad \text{shear energy} \;\; \dot\beta_+^2 + \dot\beta_-^2 \quad \rightarrow \quad 1/\mathbf{a}^3$$

Hairy black holes

M.S.V., Phys.Rev. D85 (2012) 124043
Brito, Cardoso, Pani, Phys.Rev. D88 (2013) 064006

Static bidiagonal metrics

$$
\begin{aligned}
ds_g^2 &= -Q^2 dt^2 + \frac{R'^2}{N^2}\, dr^2 + R^2 d\Omega^2 \\
ds_f^2 &= -q^2 dt^2 + \frac{U'^2}{Y^2}\, dr^2 + U^2 d\Omega^2
\end{aligned}
$$

6 functions Q, N, R, q, Y, U depend on r, one can impose 1 gauge condition ($R = r$). For black holes Q^2, q^2, N^2, Y^2 should have a simple zero at one place, $r = r_h$.

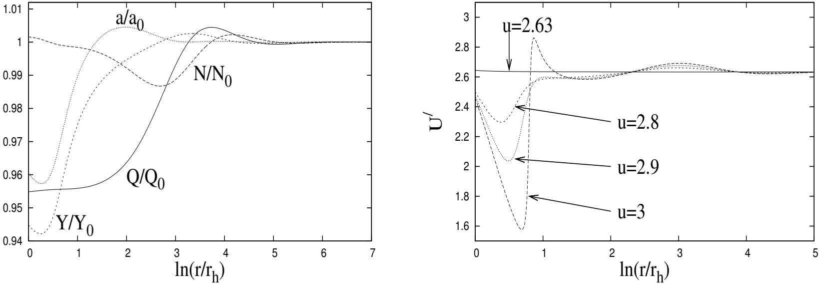
Event horizon at $r = r_h$

Equations reduce to a dynamical system for N, Y, U, one has

$$N^2 = \sum_{n\geq 1} a_n (r - r_h)^n, \quad Y^2 = \sum_{n\geq 1} b_n (r - r_h)^n, \quad U = u_h + \sum_{n\geq 1} c_n (r - r_h)^n$$

- Regular horizon is common for both metrics
- Black hole solutions comprise a two-parameter set labeled by r_h and u_h $\Rightarrow$ horizon radii measured by the two metric.
- Horizon surface gravities and temperatures are the same for both metrics.

Black holes with massive graviton hair



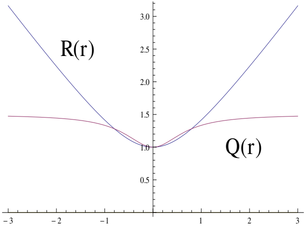
- For generic values of r_h, u_h solutions either show a curvature singularity at a finite distance away from r_h or approach asymptotically the AdS space /M.S.V. 2012/
- For specially fine-tuned r_h, u_h there are asymptotically flat black holes with $r_h \sim 1/m \Rightarrow$ they are cosmologically large /Brito, Cardoso, Pani 2013/
  This conclusion was questioned by /Torsello et al 2018/

Wormholes

/S.V.Sushkov and M.S.V. 2015/

Wormholes – bridges between universes

$$ds^2 = -Q^2(r)dt^2 + dr^2 + R^2(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2),$$



- $G_{\mu\nu} = 8\pi G T_{\mu\nu} \Rightarrow \rho + p < 0,\ p < 0 \Rightarrow$ violation of the null energy conditions $\Rightarrow$ vacuum polarization, or exotic matter (phantoms), or gravity modifications (Gauss-Bonnet, braneworld).
- The structure of $T_{\mu\nu}$ and $\mathcal{T}_{\mu\nu}$ in the bigravity theory generically violates the N.E.C. /Visser et al, 2012/

Wormholes – local solution

$$
\begin{aligned}
ds_g^2 &= -Q^2 dt^2 + dr^2 + R^2 d\Omega^2 \\
ds_f^2 &= -q^2 dt^2 + \frac{U'^2}{Y^2} dr^2 + U^2 d\Omega^2
\end{aligned}
$$

$$
\begin{aligned}
Y &= Y_1 r + Y_3 r^3 + \ldots \quad Q = Q_0 + Q_2 r^2 + \ldots \quad R = h + R_2 r^2 + \ldots \\
q &= q_0 + q_2 r^2 + \ldots \qquad U = uh + U_2 r^2 + \ldots
\end{aligned}
$$

Expanding the field equations gives in the leading order algebraic equations for Q_0 and q_0, whose solution exists if only $h \geq 1/\sqrt{3}$ (in units of $1/m$) $\Rightarrow$ wormholes are cosmologically large – in principle we may live inside a wormhole.

- Characteristic surfaces of the dRGT massive gravity theory can be locally timelike $\Rightarrow$ superluminal signals.
- This has also been detected in the Galileon models.
- It is unclear if this implies aucausality. It is also unclear if timelike characteristics can be global.

Massive spin-2 in curved space

C.Mazuet, M.S.V. JCAP 1807 (2018) 012

Massive fields in curved space

How to generalize wave equations to curved space ?

Spin-0: Klein-Gordon equation

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2)\Phi = 0$$

generalizes to curved space via simply

$$\eta_{\mu\nu} \Rightarrow g_{\mu\nu}, \qquad \partial_\mu \Rightarrow \nabla_\mu$$

which yields

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu - m^2)\Phi = 0$$

Similarly for spins $1/2, 1$ (spin $3/2$?).

The procedure fails for massive spin-2

Fierz-Pauli equations /1939/

$$E_{\mu\nu} \equiv \partial^\sigma\partial_\mu h_{\sigma\nu} + \partial^\sigma\partial_\nu h_{\sigma\mu} - \partial^\sigma\partial_\sigma h_{\mu\nu} - \partial_\mu\partial_\nu h + \eta_{\mu\nu}\left(\partial^\sigma\partial_\sigma h - \partial^\alpha\partial^\beta h_{\alpha\beta}\right) + m^2(h_{\mu\nu} - h\,\eta_{\mu\nu}) = 0,$$

imply 5 constraints

$$\mathcal{C}_\nu \equiv \partial^\mu E_{\mu\nu} = m^2(\partial^\mu h_{\mu\nu} - \partial_\nu h) = 0,$$
$$\mathcal{C}_5 \equiv \left(\partial^\mu\partial^\nu + \frac{m^2}{2}\,\eta^{\mu\nu}\right) E_{\mu\nu} = -\frac{3}{2}\,m^4 h = 0.$$

hence

$$(\Box - m^2)h_{\mu\nu} = 0, \qquad \partial^\mu h_{\mu\nu} = 0, \qquad h = 0.$$

Replacing $\eta_{\mu\nu} \Rightarrow g_{\mu\nu}$ abd $\partial_\mu \Rightarrow \nabla_\mu$ one finds that

$$\left(\nabla^\mu\nabla^\nu + \frac{m^2}{2}\,g^{\mu\nu}\right) E_{\mu\nu}$$

is not a constraint anymore (contains second derivatives) $\Rightarrow$ there are 6 DoF, unless if $R_{\mu\nu} = \Lambda g_{\mu\nu}$. A long standing problem.

Solution

The dRGT massive gravity /2010/

$$\begin{aligned} G_{\mu\nu}(g) \quad &+ \quad \beta_0\, g_{\mu\nu} + \beta_1([\gamma]\, g_{\mu\nu} - \gamma_{\mu\nu}) \\ &+ \quad \beta_2\, |\gamma|\, ([\gamma]\, \gamma_{\mu\nu} - \gamma^{-2}_{\mu\nu}) + \beta_3\, |\gamma|\, \gamma_{\mu\nu} = 0 \qquad (1) \end{aligned}$$

contains $g_{\mu\nu}$ and a reference metric $f_{\mu\nu}$ with $\gamma^{\mu}{}_{\nu} = \sqrt{g^{\mu\sigma} f_{\sigma\nu}}$.
The idea is to represent

$$g_{\mu\nu} = \eta_{ab} e^{a}{}_{\mu} e^{b}{}_{\nu}$$

then linearize (1) with respect to tetrad perturbations

$$e^{a}{}_{\mu} \to e^{a}{}_{\mu} + \delta e^{a}{}_{\mu}$$

and use (1) to determine $f_{\mu\nu}$. This gives a linear theory for a non-symmetric tensor

$$X_{\mu\nu} = \eta_{ab} e^{a}{}_{\mu} \delta e^{b}{}_{\nu}$$

which propagates 5 DoF for any $g_{\mu\nu}$.

Equations: $\Delta_{\mu\nu} + \mathcal{M}_{\mu\nu} = 0$

with the kinetic term

$$\begin{aligned}\Delta_{\mu\nu} &= \frac{1}{2}\nabla^\sigma\nabla_\mu(X_{\sigma\nu}+X_{\nu\sigma})+\frac{1}{2}\nabla^\sigma\nabla_\nu(X_{\sigma\mu}+X_{\mu\sigma})\\ &- \frac{1}{2}\Box(X_{\mu\nu}+X_{\nu\mu})-\nabla_\mu\nabla_\nu X - R^\sigma_\mu X_{\sigma\nu} - R^\sigma_\nu X_{\sigma\mu}\\ &+ g_{\mu\nu}\left(\Box X - \nabla^\alpha\nabla^\beta X_{\alpha\beta} + R^{\alpha\beta}X_{\alpha\beta}\right)\end{aligned}$$

and the mass term

$$\begin{aligned}\mathcal{M}_{\mu\nu} &= \beta_1\left(\gamma^\sigma{}_\mu X_{\sigma\nu} - g_{\mu\nu}\,\gamma^{\alpha\beta}\,X_{\alpha\beta}\right)\\ &+ \beta_2\,\{-\gamma^\alpha{}_\mu\gamma^\beta{}_\nu\,X_{\alpha\beta} - (\gamma^2)^\alpha{}_\mu\,X_{\alpha\nu} + \gamma_{\mu\nu}\,\gamma_{\alpha\beta}\,X^{\alpha\beta}\\ &\quad +[\gamma]\,\gamma^\alpha{}_\beta\,X_{\alpha\nu} + ((\gamma^2)_{\alpha\beta}\,X^{\alpha\beta} - [\gamma]\,\gamma_{\alpha\beta}\,X^{\alpha\beta})\,g_{\mu\nu}\}\\ &+ \beta_3\,|\gamma|\,\left(X_{\mu\sigma}(\gamma^{-1})^\sigma{}_\nu - [X](\gamma^{-1})_{\mu\nu}\right)\end{aligned}$$

$\gamma_{\mu\nu}$ is algebraically related to the background $g_{\mu\nu}$ via

$$\begin{aligned}G_{\mu\nu}(g) &+ \beta_0\,g_{\mu\nu} + \beta_1([\gamma]\,g_{\mu\nu} - \gamma_{\mu\nu})\\ &+ \beta_2\,|\gamma|\,([\gamma]\,\gamma_{\mu\nu} - \gamma^{-2}_{\mu\nu}) + \beta_3\,|\gamma|\,\gamma_{\mu\nu} = 0\end{aligned}$$

Constraints

There are 16 equations

$$E_{\mu\nu} \equiv \Delta_{\mu\nu} + \mathcal{M}_{\mu\nu} = 0$$

for 16 components of $X_{\mu\nu}$. The imply 11 conditions:

$$\Delta_{[\mu\nu]} = 0 \quad \Rightarrow \quad \mathcal{M}_{[\mu\nu]} = 0 \quad \Rightarrow \quad \text{6 algebraic constraints}$$

$$\mathcal{C}_\nu = \nabla^\mu E_{\mu\nu} = 0 \quad \Rightarrow \quad \text{4 vector constraints}$$

$$\begin{aligned} \mathcal{C}_5 \; &= \; \nabla_\mu((\gamma^{-1})^{\mu\nu}\mathcal{C}_\nu) + \frac{\beta_1}{2} E^\alpha_{\ \alpha} + \beta_2 \gamma^{\mu\nu} E_{\mu\nu} \\ &+ \; \beta_3 \frac{|\gamma|}{g^{00}} \left((\gamma^{-1})^{0\alpha}(\gamma^{-1})^{0\beta} - (\gamma^{-1})^{00}(\gamma^{-1})^{\alpha\beta} \right. \\ &\times \; \left. \left(E_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}(E^\sigma_{\ \sigma} - \frac{1}{g^{00}} E^{00}) \right) \right) = 0 \; \Rightarrow \; \text{scalar constraint} \end{aligned}$$

The number of DoF is $16 - 6 - 4 - 1 = 5$.

A simple case

The mass term is a non-linear function of the background $R_{\mu\nu}$.

$$\mathcal{M}_{\mu\nu} = B_0\, g_{\mu\nu} + B_1\, R_{\mu\nu} + B_2\, (R^2)_{\mu\nu} + B_3\, (R^3)_{\mu\nu}$$

where B_m are functions of scalar invariants of $R^{\mu}_{\ \nu}$ and of β_A.

$\mathcal{M}_{\mu\nu}$ is linear in Ricci if $\beta_2 = \beta_3 = 0 \Rightarrow$

$$\begin{aligned} \mathcal{M}_{\mu\nu} &= \gamma_{\mu\alpha} X^{\alpha}_{\ \nu} - g_{\mu\nu}\, \gamma_{\alpha\beta} X^{\alpha\beta} \\ \text{with} \quad \gamma_{\mu\nu} &= R_{\mu\nu} + \left(m^2 - \frac{R}{6} \right) g_{\mu\nu} \end{aligned}$$

Massive spin-2 in Einstein spaces

$R_{\mu\nu} = \Lambda g_{\mu\nu}$ then $X_{\mu\nu} = X_{\nu\mu}$, everything reduces to

$$\Delta_{\mu\nu} + M_{\rm H}^2 (X_{\mu\nu} - X g_{\mu\nu}) = 0$$

with $M_{\rm H}^2 = \Lambda/3 + m^2$ from where $\nabla^\mu X_{\mu\nu} = \nabla_\nu X$ and

$$\Box X_{\mu\nu} \quad - \quad \nabla_\mu \nabla_\nu X + 2 R_{\mu\alpha\nu\beta} X^{\alpha\beta} - \Lambda X g_{\mu\nu} = M_{\rm H}^2 (X_{\mu\nu} - X g_{\mu\nu})$$

Taking the trace yields $\boxed{(2\Lambda - 3M_{\rm H}^2) X = 0}$

- $M_{\rm H}^2 > 2\Lambda/3 \Rightarrow X = 0 \Rightarrow 5$ DoF.
- $M_{\rm H}^2 = 2\Lambda/3$ Partially massless limit: $\Rightarrow X \neq 0$ BUT local

  $$X_{\mu\nu} \to X_{\mu\nu} + (\nabla_\mu \nabla_\nu + \Lambda/3\, g_{\mu\nu})\Omega \quad \Rightarrow \quad 10 - 4 - 2 = 4 \;\; \text{DOF}$$

- $M_{\rm H}^2 < 2\Lambda/3 \Rightarrow X = 0 \Rightarrow 5$ DoF BUT the scalar polarization becomes ghost.

Higuchi bound: System is stable for $M_{\rm H}^2 > 2\Lambda/3$.

Massive spin-2 in the expanding universe

The background geometry

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\mathbf{x}^2$$

where $a(t)$ fulfills the background Einstein equations

$$3\,\frac{\dot{a}^2}{a^2} = \frac{\boldsymbol{\rho}}{M_{\mathrm{Pl}}^2} \equiv \rho, \qquad 2\,\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -\frac{\boldsymbol{p}}{M_{\mathrm{Pl}}^2} \equiv -p,$$

where $\boldsymbol{\rho}, \boldsymbol{p}$ are the energy density and pressure of the background matter. We wish to solve the equations

$$\Delta_{\mu\nu} + \mathcal{M}_{\mu\nu} = 0.$$

for the two special models.

Fourier decomposition

$$X_{\mu\nu}(t,\mathbf{x}) = a^2(t)\sum_{\mathbf{k}} X_{\mu\nu}(t,\mathbf{k})e^{i\mathbf{kx}}$$

where the Fourier amplitude splits into the sum of the tensor, vector, and scalar harmonics,

$$X_{\mu\nu}(t,\mathbf{k}) = X^{(2)}_{\mu\nu} + X^{(1)}_{\mu\nu} + X^{(0)}_{\mu\nu}$$

One obtains the effective action in each sector

$$I_{(2)} = \int K\left(\dot{\mathrm{D}}^2 - \frac{c^2}{k^2}\,\mathrm{D}^2\right)\,a^3 dt$$

where $D = T_\pm$ for tensors, $D = V_\pm$ for vectors, only one scalar mode $D = S \Rightarrow$ altogether 5 DoF ! The kinetic term K and sound speed c^2 are functions of the background scale factor a and momentum k and m.

Stability

The kinetic term and the sound speed squared should be positive

$$K > 0, \qquad c^2 > 0$$

(no ghosts and tachyons). These conditions are fulfilled

- at all times after the inflation if $M \geq 10^{13}$ GeV
- at present if $M \geq 10^{-33}$ eV
- Assuming that $X_{\mu\nu}$ couples only to gravity and hence massive spin-2 particles do not have other decay channels, it follows that they could be a part of Dark Matter (DM) at present

Summary

- A consistent theory of a free massive spin-2 field propagating 5 DoF in arbitrary spacetimes is constructed.
- This allows for the first time to consistently consider a model of Dark Matter made of massive spin-2 particles.

Horndeski theory

Galileons in the decoupling limit: $M_{\rm Pl}\to\infty$, $m\to 0$, $\Lambda_3=(M_{\rm Pl}m^2)^{1/3}$=const.

$$g_{\mu\nu}=\eta_{\mu\nu}+\frac{1}{M_{\rm Pl}}h_{\mu\nu} \qquad \partial_\mu\Phi^\alpha=\delta^\alpha_\mu+\frac{1}{mM_{\rm Pl}}\,\partial_\mu A^\alpha+\frac{1}{m^2M_{\rm Pl}^2}\,\partial_\mu\partial^\alpha\phi$$

with $h_{\mu\nu}=\mathbf{h}_{\mu\nu}+a_1\,\phi\eta_{\mu\nu}+a_2\,\partial_\mu\phi\partial_\nu\phi$ one obtains (if $A_\mu=0$)

$$\mathcal{L}_{\Lambda_3}=\mathcal{L}_0(\mathbf{h}_{\mu\nu})+\sum_{n=2}^{5}\frac{d_n}{\Lambda_3^{3(n-2)}}\,\mathcal{L}^{(n)}_{\rm Gal}[\phi]+\frac{q}{\Lambda_3^6}\,\mathbf{h}^{\mu\nu}X^{(3)}_{\mu\nu}(\phi)$$

where the Galileon terms (shift inv. $\phi\to\phi+\phi_0$) $/\Pi_{\mu\nu}=\partial_{\mu\nu}\phi/$

$$\begin{aligned}
\mathcal{L}^{(2)} &= (\partial\phi)^2,\\
\mathcal{L}^{(3)} &= (\partial\phi)^2[\Pi],\\
\mathcal{L}^{(4)} &= (\partial\phi)^2([\Pi]^2-[\Pi^2]),\\
\mathcal{L}^{(5)} &= (\partial\phi)^2([\Pi]^3-3[\Pi][\Pi^2]+3[\Pi^3])
\end{aligned}$$

contain second derivatives, but equations are second order.

Gregory Walter Horndeski (Canada), 1974

$$S_{\mathrm{H}}[g_{\mu\nu},\Phi] = \int L_{\mathrm{H}}\,\sqrt{-g}\,d^4x$$

$$\begin{aligned} L_{\mathrm{H}} &= G_2(X,\Phi) + G_3(X,\Phi)\,\Box\Phi \\ &+ G_4(X,\Phi)\,R + \partial_X G_4(X,\Phi)\,\delta^{\mu\nu}_{\alpha\beta}\,\nabla^\alpha_\mu\Phi\nabla^\beta_\nu\Phi \\ &+ G_5(X,\Phi)\,G_{\mu\nu}\nabla^{\mu\nu}\Phi - \frac{1}{6}\,\partial_X G_5(X,\Phi)\,\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\,\nabla^\alpha_\mu\Phi\nabla^\beta_\nu\Phi\nabla^\gamma_\rho\Phi \end{aligned}$$

with $X \equiv \frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi$, $\quad \delta^{\lambda\rho}_{\nu\alpha} = 2!\,\delta^\lambda_{[\nu}\delta^\rho_{\alpha]}$, $\quad \delta^{\lambda\rho\sigma}_{\nu\alpha\beta} = 3!\,\delta^\lambda_{[\nu}\delta^\rho_\alpha\delta^\sigma_{\beta]}$.

The most general theory with second order field equations.

The GW170817 event shows that GW propagate with the speed of light $\Rightarrow$ one has to have $\partial_X G_4 = G_5 = 0$

DHOST generalizatons with higher order equations but still with 3 propagating modes – free from *Ostrogradsky ghost*.

Anisotropy screening in Horndeski cosmologies

A.A. Starobinsky, S.V. Sushkov, M.S.V.
Phys.Rev. D101 (2020) 064039

The red model

$$S = \frac{1}{2}\int (\mu\, R - (\alpha\, G_{\mu\nu} + \varepsilon\, g_{\mu\nu})\nabla^\mu\phi\nabla^\nu\phi - 2\Lambda)\,\sqrt{-g}\, d^4x$$

$$ds^2 = -dt^2 + \mathrm{a}_1^2\, dx_1^2 + \mathrm{a}_2^2\, dx_2^2 + \mathrm{a}_3^2\, dx_3^2$$

with $\mathrm{a}_1 = \mathrm{a}\, e^{\beta_+ + \sqrt{3}\beta_-}$, $\mathrm{a}_2 = \mathrm{a}\, e^{\beta_+ - \sqrt{3}\beta_-}$, $\mathrm{a}_3 = \mathrm{a}\, e^{-2\beta_+}$. Equations

$$3\mu\mathcal{H}^2 \equiv 3\mu\left(\frac{\dot{\mathrm{a}}^2}{\mathrm{a}^2} - \dot\beta_+^2 - \dot\beta_-^2\right) = \frac{1}{2}\,(\varepsilon - 9\alpha\,\mathcal{H}^2)\,\dot\phi^2 + \Lambda,$$

$$(2\mu + \alpha\dot\phi^2)\mathrm{a}^3\dot\beta_\pm = \mathcal{B}_\pm = const., \qquad \mathrm{a}^3\left(3\alpha\,\mathcal{H}^2 - \varepsilon\right)\dot\phi = C = const.$$

Isotropic case, $\mathcal{B}_\pm = 0$, then $\epsilon/(9\alpha) \leftarrow \dot{\mathrm{a}}^2/\mathrm{a}^2 \rightarrow \Lambda/(3\mu)$

$\Rightarrow$ kinetic inflation. Anisotropic case: if $C = \dot\phi = 0$ then

$$3\mu\,\frac{\dot{\mathrm{a}}^2}{\mathrm{a}^2} = \frac{\mathcal{B}_+^2 + \mathcal{B}_-^2}{\mathrm{a}^6} + \Lambda$$

$\Rightarrow$ anisotropy contribution is large at small a but small at large a.

Anisotropies

If $C \neq 0$ then

$$\dot{\phi} = \frac{C}{\mathrm{a}^3(3\alpha\mathcal{H}^2 - \varepsilon)}$$

in which case the anisotropy contribution is suppressed as $1/\mathrm{a}^6$ at large a and as a^6 at small a. As a result, the anisotropy is screened near singularity by the scalar charge – an unusual feature.

However, in the Bianchi IX case

$$ds^2 = -dt^2 + \frac{1}{4}\left(\mathrm{a}_1^2\,\omega_1 \otimes \omega_1 + \mathrm{a}_2^2\,\omega_2 \otimes \omega_2 + \mathrm{a}_3^2\,\omega_3 \otimes \omega_3\right),$$

where ω_a are the invariant forms on S^3, $d\omega_a + \epsilon_{abc}\,\omega_b \wedge \omega_c = 0$, one finds strong anisotropies and chaos near singularity.

Bianchi IX

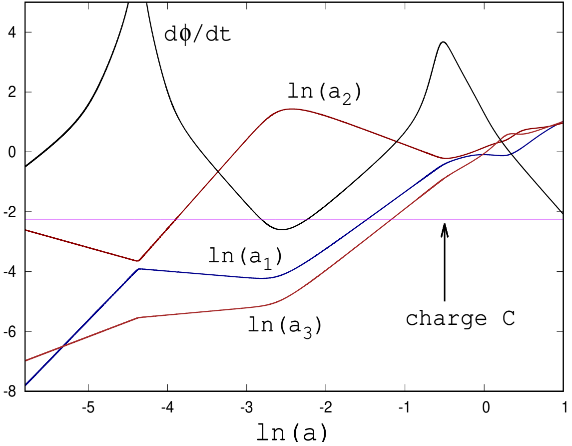


Figure: Solution in the Bianchi IX case shows a typical sequence of Kasner epochs during which $\ln(a_k) \propto \ln(a)$.

Varying the Horndeski Lagrangian in the Palatini approach

T. Helpin and M.S.V., JCAP 2001 (2020) 044

Metric-affine version of Horndeski action

$$S_{\rm P}[\Gamma^{\sigma}_{\alpha\beta}, g_{\mu\nu}, \Phi] = \int L_{\rm P}\, \sqrt{-g}\, d^4x$$

$$\begin{aligned}
L_{\rm P} &= G_2(X,\phi) + G_3(X,\phi)\,[\hat{\Phi}] \\
&+ G_4(X,\phi)\,\overset{(\Gamma)}{R} - \partial_X G_4(X,\phi)\,\left([\hat{\Phi}]^2 - [\hat{\Phi}^2]\right), \\
&+ G_5(X,\phi)\,[\hat{G}\hat{\Phi}] + \frac{1}{6}\,\partial_X G_5(X,\phi)\,\left([\hat{\Phi}]^3 - 3[\hat{\Phi}][\hat{\Phi}^2] + 2[\hat{\Phi}^3]\right) \\
&+ \Delta L_{\rm P}(Q^{\mu\nu}{}_{\alpha}) \qquad \text{/infinitely many possibilities/}
\end{aligned}$$

$$\overset{(\Gamma)}{R}{}_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\alpha}, \qquad \overset{(\Gamma)}{R} = g^{\mu\nu}\overset{(\Gamma)}{R}{}_{\mu\nu}$$

$$\hat{G} = g^{\mu\sigma}\overset{(\Gamma)}{R}{}_{\sigma\nu} - \frac{1}{2}\overset{(\Gamma)}{R}\,\delta^\mu_\nu, \qquad \hat{\Phi} = g^{\alpha\sigma}\overset{(\Gamma)}{\nabla}{}_\sigma \overset{(\Gamma)}{\nabla}{}_\beta\,\phi, \qquad Q^{\mu\nu}{}_\alpha \equiv \overset{(\Gamma)}{\nabla}{}_\alpha g^{\mu\nu}.$$

Result depends on whether the red terms are included or not.

The Kinetic Gravity Brading (KGB) theory

Varying gives second order equations if only

$$\boxed{G_4 = G_4(\Phi), \qquad G_5 = 0}$$

same condition insures that GW speed is one !

This leads to

$$S_{\mathrm{P}}[\Gamma^{\sigma}_{\alpha\beta}, g_{\mu\nu}, \phi] = \int \left(G_4(\Phi)\overset{(\Gamma)}{R} + G_3(\Phi, X)\overset{(\Gamma)}{\Box}\Phi + K(\Phi, X) \right) \sqrt{-g}\, d^4x$$

$$G_4(\Phi) \equiv e^{\omega}, \qquad G_3(X, \Phi) \equiv \gamma e^{\omega}, \qquad K(X, \Phi) \equiv \kappa e^{\omega}$$

Varying with respect to $\Gamma^{\alpha}_{\mu\nu}$

determines the non-metricity

$$Q^{\mu\nu}{}_{\alpha} = \overset{(\Gamma)}{\nabla}_{\alpha} g^{\mu\nu} = g^{\mu\nu}\partial_\alpha\omega + \frac{2}{3}\gamma\left(g^{\mu\nu}\partial_\alpha\Phi + \delta_\alpha^{(\mu}\partial^{\nu)}\Phi\right)$$

which can be resolved to obtain

$$\begin{aligned}\Gamma^{\alpha}_{\mu\nu} = \left\{{\alpha \atop \mu\nu}\right\} \quad &+ \quad \frac{1}{2}\left(\delta^\alpha_\mu\partial_\nu\omega + \delta^\alpha_\nu\partial_\mu\omega - g_{\mu\nu}\partial^\alpha\omega\right) \\ &+ \quad \frac{1}{3}\gamma\left(\delta^\alpha_\mu\partial_\nu\Phi + \delta^\alpha_\nu\partial_\mu\Phi\right)\end{aligned}$$

$\Rightarrow$ not a metric connection. One can absorb the non-metricty to the effective energy-momentum tensor and express everything in terms of ordinary covariant derivatives; for example

$$\overset{(\Gamma)}{R}_{\mu\nu} = R_{\mu\nu} - \nabla_\mu\nabla_\nu\,\omega - \gamma\nabla_\mu\nabla_\nu\Phi + \frac{1}{2}\,\partial_\mu\omega\partial_\nu\omega + \ldots$$

Varying with respect to Φ, $g_{\mu\nu}$ yields equations

$$G_{\mu\nu}(g) + T_{\mu\nu} = 0, \qquad \nabla_\mu J^\mu = \Sigma \quad \text{where}$$

$$\begin{aligned} T_{\mu\nu} &= -\omega' \partial_\mu \partial_\nu \Phi - \gamma\, \partial_{(\mu} \Phi \partial_{\nu)} X \\ &+ \left(\kappa_X + \gamma \Box \Phi + 2\omega' X \gamma - 2\omega'' + \omega'^2 - 2\gamma' - \frac{2}{3}\, \partial_X(\gamma^2 X) \right) X_{\mu\nu} \\ &+ \left(\frac{1}{2} \langle \partial\Phi \partial X \rangle \gamma - \frac{1}{2}\, \kappa + \omega' \Box \Phi + (2\omega'' + \frac{1}{2}\omega'^2 + \gamma' + \frac{1}{3}\, \gamma^2) X \right) g_{\mu\nu} \end{aligned}$$

and

$$\begin{aligned} J^\mu &= \left\{ \partial_X K + (\omega' - \frac{2}{3}\gamma')(2X\partial_X + 1) G_3 - \partial_\Phi G_3 \right\} \partial^\mu \Phi \\ &\quad + \partial_X G_3 \left\{ \Box \Phi\, \partial^\mu \Phi - \partial^\mu X \right\}, \\ \Sigma &= \partial_\Phi K + \partial_\Phi G_4 \overset{(\Gamma)}{R} + \partial_\Phi G_3 \overset{(\Gamma)}{\Box} \Phi\,. \end{aligned}$$

Does this define a new theory ?

Equivalent metric theory

The metric-affine theory

$$S_{\rm P}[\Gamma^{\sigma}_{\alpha\beta}, g_{\mu\nu}, \phi] = \int \left(G_4 \overset{(\Gamma)}{R} + G_3 \overset{(\Gamma)}{\Box} \Phi + K \right) \sqrt{-g}\, d^4x$$

is equivalent to the metric theory

$$S_{\rm H}[g_{\mu\nu}, \phi] = \int \left(G_4\, R + G_3\, \Box \Phi + \tilde{K} \right) \sqrt{-g}\, d^4x$$

where

$$\tilde{K} = K + \left(2 G_3 \partial_\phi G_4 + 3 (\partial_\phi G_4)^2 - \frac{2}{3} G_3^2 \right) \frac{X}{G_4} \,.$$

Varying the KGB action in the metric-affine approach does not produce a new theory but gives a metric theory from the same KGB class. The GW speed is still equal to one.

Why ?

Non-dynamical connection

Solving *algebraic equations* for the connection yields

$$\Gamma^{\sigma}_{\rho\gamma} = \Gamma^{\sigma}_{\rho\gamma}\left(g_{\alpha\beta}, \phi\right)$$

injecting which to the action one obtains the metric action

$$S_{\rm P}[\Gamma^{\sigma}_{\rho\gamma}\left(g_{\alpha\beta}, \phi\right), g_{\mu\nu}, \phi] = \tilde{S}_{\rm H}[g_{\mu\nu}, \phi].$$

Let us vary the scalar field, $\phi \to \phi + \delta\phi$,

$$\delta S_{\rm P} = \frac{\delta S_{\rm P}}{\delta \Gamma^{\sigma}_{\rho\gamma}} \frac{\partial \Gamma^{\sigma}_{\rho\gamma}\left(g_{\alpha\beta}, \phi\right)}{\partial \phi}\, \delta\phi + \frac{\delta S_{\rm P}}{\delta \phi}\, \delta\phi = \delta \tilde{S}_{\rm H} = \frac{\delta \tilde{S}_{\rm H}}{\delta \phi}\, \delta\phi\,.$$

Since the connection is on-shell value, one has

$$\frac{\delta S_{\rm P}}{\delta \Gamma^{\sigma}_{\rho\gamma}} = 0, \qquad \Rightarrow \qquad \boxed{\frac{\delta S_{\rm P}}{\delta \phi} = \frac{\delta \tilde{S}_{\rm H}}{\delta \phi}}$$

equations obtained by varying the Palatini action $S_{\rm P}$ are the same as those obtained from the metric Horndeski action $\tilde{S}_{\rm H}$.

Conclusion

- Palatini versions of Horndeski theory with $G_5 = 0$ are always equivalent to metric theories.
- If $G_4 = G_4(\Phi)$ the Palatini approach yields the Horndeski theory with second derivatives. If $G_4 = G_4(\Phi, X)$ then the Palatini approach yields theories with higher derivatives, which can sometimes be equivalent to DHOST. It is unclear if they are always equivalent to DHOST.
- $G_5 \neq 0$ then connection becomes dynamical and starts propagating. This case remains totally unknown.

Ghost-free example

Theory with third derivatives

$$L_{\mathrm{P}} = \left(\sigma \overset{(\Gamma)}{R} + \xi\, G_{\mu\nu}\, \partial^\mu \Phi \partial^\nu \Phi\right) \sqrt{-g}\,,$$
$$= \overset{(\Gamma)}{R}_{\mu\nu}\, h^{\mu\nu} \sqrt{-h}$$

with

$$h_{\mu\nu} = \sqrt{\sigma^2 - \xi^2 X^2} \left(g_{\mu\nu} - \frac{\xi}{\sigma + \xi X}\, \partial_\mu \Phi \partial_\nu \Phi\right),$$

and this is simply vacuum GR for the effective metric $h_{\mu\nu}$ – higher derivatives are removed by disformal transformation.

Summary of part

- There are infinitely many metric-affine versions of the Horndeski Lagrangian which differ from each other by non-metricity terms

$$\Delta L_{\mathrm{P}}(Q^{\mu\nu}{}_{\alpha})$$

- Theories with $G_5 = 0$ are equivalent to some metric theories which are either ghost-free or contain ghost.

- Theories with $G_5 \neq 0$ contain a dynamical connecrion

It seems that Palatini version of Horndeski Lagrangian cannot give new ghost-free theories. However, it can give a new parametrisation of such theories.

A non-Horndeski example

- Other metric-affine theories, not of Horndeski type, can also be ghost-free, for example,

$$
\begin{aligned}
L_{\mathrm{P}} &= K(X,\phi) + G_3(X,\phi)[\hat{\Phi}] \\
&+ G_4(X,\phi) \overset{(\Gamma)}{R} - \partial_X G_4(X,\phi) \left([\hat{\Phi}]^2 - [\hat{\Phi}^2]\right) \\
&\quad - \frac{\partial_X G_4(X,\phi)}{X} (\nabla_\mu X - [\hat{\Phi}]\nabla_\mu \phi)\nabla^\mu X
\end{aligned}
$$