Modification of the radiation definition in odd dimensions

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Motivation

- Theories with extra dimensions:
  - String theory,
  - Arkani-Hamed — Dimopoulos — Dvali model,
  - Randall — Sundrum model,
  - Dvali — Gabadadze — Porrati model.
- Gravitational-wave and multi-messenger astronomy.
- Holographic approach to the description of quark-gluon plasma.
- Field-theoretical models in condensed matter physics.

Huygens principle violation in odd dimensions

Retarded Green's function $G_{\text{ret}}^{n+1}(x)$ of the massless wave equation in $(n+1)$-dimensional space-time is given by the following equation

$$\Box\, G_{\text{ret}}^{n+1}(x) = \delta^{n+1}(x)$$

$$G_{\text{ret}}^{n+1}(x) = 0 \iff x^0 < 0,$$

where Minkowski metric is $\eta_{\mu\nu} = \text{diag}(1, -1, \ldots, -1)$ and $\Box = \partial^\mu \partial_\mu$ is D'Alembert operator. Solutions of this equation are given by two recurrent formulae[1], depending on the dimensionality of space-time,

$$G_{\text{ret}}^{2\nu+2}(x) = \frac{(-1)^{\nu-1}}{2(2\pi)^\nu} \frac{d^{\nu-1}}{(rdr)^{\nu-1}} \frac{\delta(t-r)}{r}, \ \nu = 1,2,\ldots$$

$$G_{\text{ret}}^{2\nu+1}(x) = \frac{(-1)^{\nu-1}}{(2\pi)^\nu} \frac{d^{\nu-1}}{(rdr)^{\nu-1}} \theta(t) \frac{\theta(t^2-r^2)}{\sqrt{t^2-r^2}}, \ \nu = 1,2,\ldots$$

where $t = x^0$, $r = |\mathbf{x}|$. Thus, in odd dimensions, retarded fields:

- propagate in space with all velocities lower than the speed of light;
- depend on the source's history of motion;
- diverge on the light cone $t^2 - r^2 = 0$.

[1a] J. Hadamard "Lectures on Cauchy's Problem in Linear Partial Differential Equations". Dover Publications, 2014
[1b] R. Courant, D. Hilbert "Methods of Mathematical Physics: Partial Differential Equations". Wiley Classics Library. Wiley, 2008
[1c] D. Ivanenko and A. Sokolov, Sov. Phys. Doklady, 36, 37, (1940); "Classical field theory" (in Russian), Moscow, 1948

Covariant retarded quantities

Consider a pointlike particle moving along a world line $z^\mu(\tau)$ with velocity $v^\mu = dz^\mu/d\tau$, and denote the coordinates of the observation point as x^μ. The retarded proper time[2] $\hat{\tau}$ is determined by equation

$$\left(x^\mu - z^\mu(\hat{\tau})\right)^2 = 0,\ x^0 \geq z^0(\hat{\tau})\,.$$

All hatted quantities will correspond to the retarded proper time $\hat{\tau}$. We introduce the following vectors:

- lightlike vector $\hat{R}^\mu = x^\mu - \hat{z}^\mu$,
- spacelike unit vector $\hat{u}^\mu$ orthogonal to the particle's velocity $\hat{v}^\mu$,
- lightlike vector $\hat{c}^\mu = \hat{v}^\mu + \hat{u}^\mu$,

with the following properties

$$\hat{v}^2 = -\,\hat{u}^2 = 1; \quad \hat{c}^2 = 0; \quad \hat{c}\hat{v} = -\,\hat{c}\hat{u} = 1; \quad \hat{v}\hat{u} = 0,$$

$$\hat{R}^\mu = \hat{\rho}\hat{c}^\mu; \quad \hat{\rho} = \hat{v}\hat{R}; \quad \hat{R}^2 = 0.$$

Note that Lorentz invariant quantity $\hat{\rho}$ is proportional to the spatial distance $\hat{\rho} \sim r$ far from the charge.

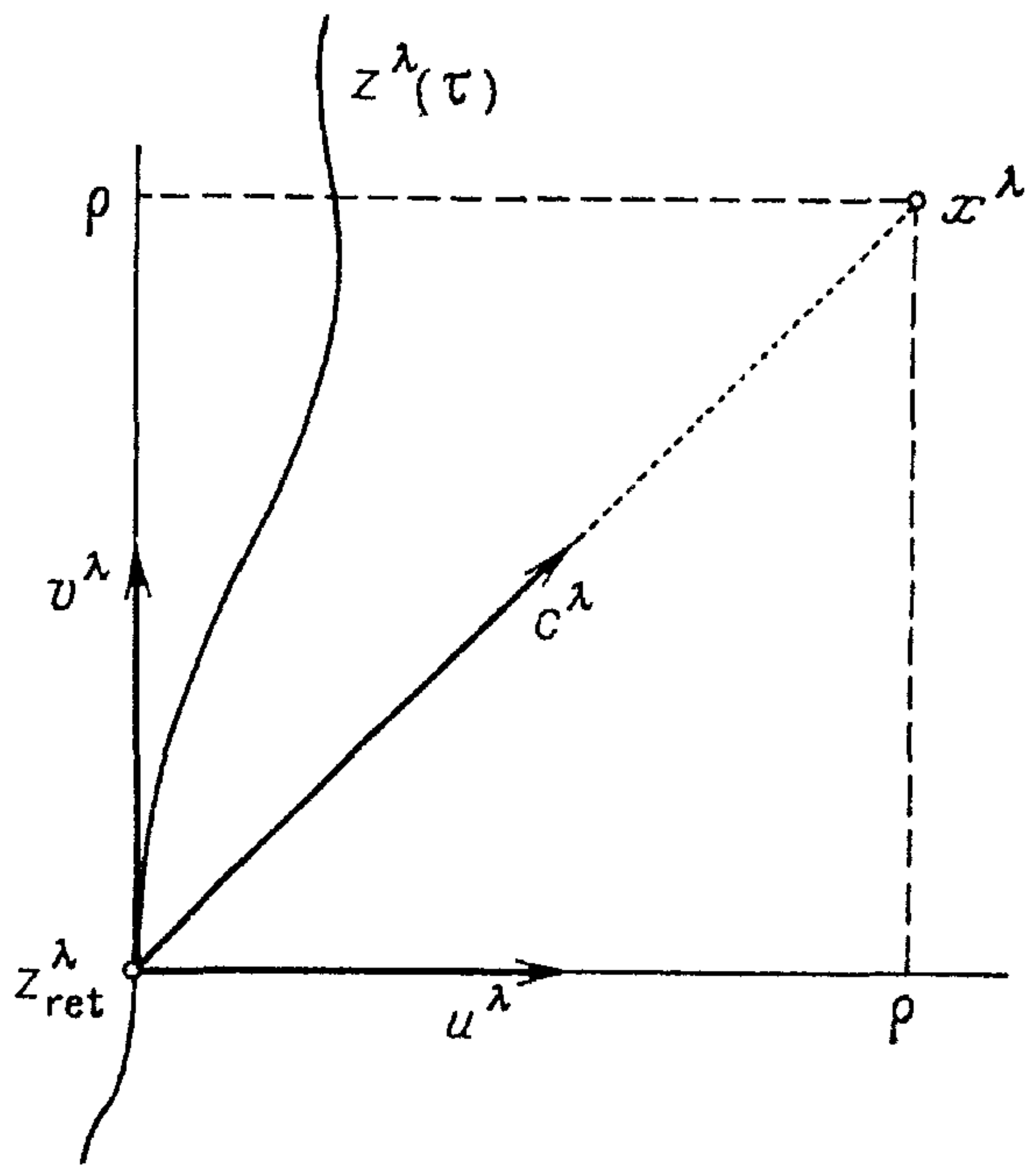

Fig. 1. Covariant retarded quantities[3].

[2a]F. Rohrlich, Il Nuovo Cimento **21** (5), 811 (1961), [2b]C. Teitelboim, Phys. Rev. D **1**, 1572 (1970)
[3]B.P. Kosyakov, Phys. Usp. **35** (2), 135 (1992)

Four-dimensional electrodynamics

Teitelboim considered the four-dimensional electrodynamics with pointlike source

$$\partial_\mu F^{\mu\nu} = 4\pi j^\nu(x), \quad j^\mu(x) = e\int d\tau\, v^\mu(\tau)\delta^4(x - z(\tau)), \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$

and demonstrated that the retarded solution of Maxwell's equations could be represented as

$$F^{\mu\nu} = F^{\mu\nu}_{\rm I} + F^{\mu\nu}_{\rm II}, \quad F^{\mu\nu}_{\rm I} = -\frac{2e}{\hat\rho^3}\hat v^{[\mu}\hat R^{\nu]} \sim 1/\hat\rho^2, \quad F^{\mu\nu}_{\rm II} = \frac{2e}{\hat\rho^2}\left(\hat a^{[\mu}\hat R^{\nu]} + \frac{\hat a^\alpha \hat R_\alpha}{\hat\rho}\hat v^{[\mu}\hat R^{\nu]}\right) \sim 1/\hat\rho, \quad a^\mu = \frac{d^2 z^\mu}{d\tau^2}.$$

The on-shell energy-momentum tensor of the electromagnetic field could be arranged as

$$T^{\mu\nu} = T^{\mu\nu}_{\rm Coul} + T^{\mu\nu}_{\rm mix} + T^{\mu\nu}_{\rm rad}$$

$$T^{\mu\nu}_{\rm Coul} \sim 1/\hat\rho^4, \quad T^{\mu\nu}_{\rm mix} \sim 1/\hat\rho^3, \quad T^{\mu\nu}_{\rm rad} \sim 1/\hat\rho^2$$

where the last term $T^{\mu\nu}_{\rm rad}$ has the properties, corresponding to the radiated part of energy-momentum:

- it is separately conserved $\partial_\nu T^{\mu\nu}_{\rm rad} = 0$;
- it is proportional to the direct product of two lightlike vectors $T^{\mu\nu}_{\rm rad} \sim \hat c^\mu \hat c^\nu$, $\hat c_\mu T^{\mu\nu}_{\rm rad} = 0$;
- It gives positive definite energy-momentum flux through the distant sphere.

Rohrlich-Teitelboim definition of radiation

In D dimensions, the on-shell energy-momentum tensor could be expanded in powers of $1/\hat{\rho}$ in analogous manner[4]

$$T^{\mu\nu} = T^{\mu\nu}_{\text{Coul}} + T^{\mu\nu}_{\text{mix}} + T^{\mu\nu}_{\text{rad}}$$

$$T^{\mu\nu}_{\text{Coul}} \sim \frac{A^{\mu\nu}(z)}{\hat{\rho}^{2D-4}}, \quad T^{\mu\nu}_{\text{mix}} \sim \frac{B^{\mu\nu}(z)}{\hat{\rho}^{2D-5}} + \ldots + \frac{C^{\mu\nu}(z)}{\hat{\rho}^{D-1}}, \quad T^{\mu\nu}_{\text{rad}} \sim \frac{D^{\mu\nu}(z)}{\hat{\rho}^{D-2}},$$

where the mixed part $T^{\mu\nu}_{\text{mix}}$ is absent in case of $D = 3$ and consists of more than one term for $D > 4$. The most long-range part $T^{\mu\nu}_{\text{rad}}$ has properties, which allow us to associate it with the radiated energy-momentum:

- it is separately conserved $\partial_\mu T^{\mu\nu}_{\text{rad}} = 0$;
- it is proportional to the direct product of two lightlike vectors $T^{\mu\nu}_{\text{rad}} \sim \hat{c}^\mu \hat{c}^\nu$, $\hat{c}_\mu T^{\mu\nu}_{\text{rad}} = 0$;
- It gives positive definite energy-momentum flux through the distant sphere $S_{\text{sph}} \sim r^{D-2}$.

The same structure of the on-shell energy-momentum tensor holds also in the scalar theory.

[4a]B.P. Kosyakov, Theor. Math. Phys. **119** (1), 493 (1999)
[4b]D.V. Gal'tsov and P.A. Spirin, Grav. Cosmol. **12**, 1 (2006)
[4c]D.V. Gal'tsov and P.A. Spirin, Grav. Cosmol. **13**, 241 (2007)
[4d]P.A. Spirin, Grav. Cosmol. **15** (1), 82 (2009)

Radiation power in odd dimensions

The flux of the radiated energy passing per unit time through the distant $(2\nu-1)$-dimensional sphere of radius r in $(2\nu+1)$-dimensional space-time is determined by the integral

$$W_{2\nu+1} = \int T^{0i}_{\text{rad}}\, n^i\, r^{2\nu-1}\, d\Omega_{2\nu-1},$$

where $d\Omega_{2\nu-1}$ is an angular element and $\mathbf{n}$ is a unit spacelike vector directed to the observation point.

The odd-dimensional flux of radiated energy has the following properties:

- it propagates in space with the speed o light (due to the structure $T^{\mu\nu}_{\text{rad}} \sim \hat{c}^{\mu}\hat{c}^{\mu}$, $\hat{c}_{\mu}T^{\mu\nu}_{\text{rad}} = 0$), despite the fact that the retarded field propagate in space with all velocities lower than the speed of light;
- it depends on the whole source's history of motion preceding the retarded moment of proper time $\hat{\tau}$.

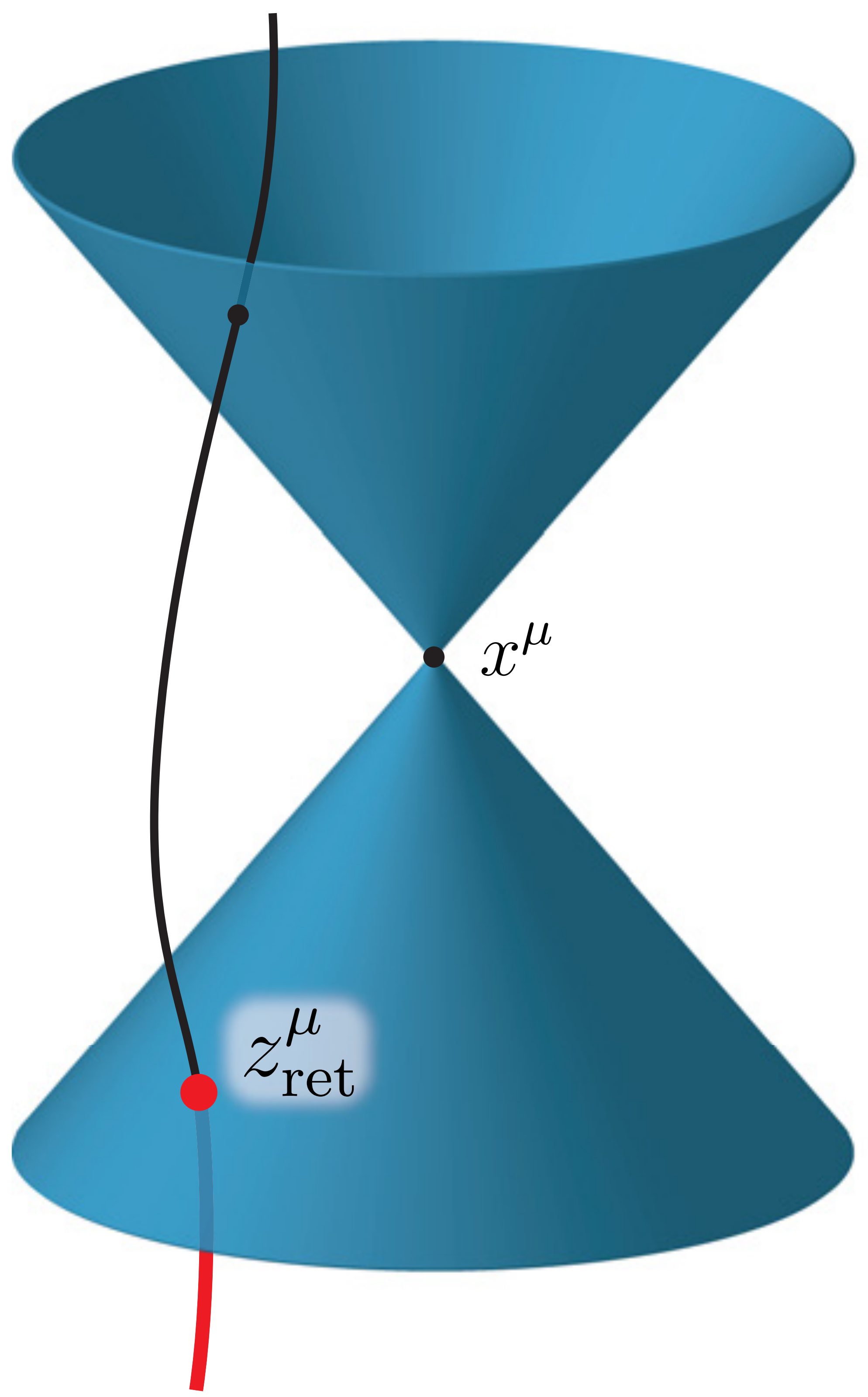

Fig. 2. Dependence of the radiation power on the source's history of motion[5].

[5]L. Barack, A. Pound, Rep. Prog. Phys. 82, 016904 (2019)

The setup

The action of the massless scalar field $\varphi(x)$ interacting with the massive pointlike scalar charge[6] moving along the world line $z^\mu(\tau)$ in the $(n+1)$-dimensional space-time is given by the integral

$$S = -\int (m + g\varphi(z))\sqrt{\dot{z}_\alpha \dot{z}^\alpha}\, d\tau + \frac{1}{2\Omega}\int \partial_\mu \varphi(x)\partial^\mu \varphi(x)\, d^{n+1}x, \quad \dot{z}^\mu(\tau) = dz^\mu(\tau)/d\tau,$$

where m is the charge's mass, g is the particle's scalar charge and $\Omega = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the $(n-1)$-dimensional unit sphere. The equation of motion of the scalar field has the following form

$$\Box\, \varphi(x) = -\,\Omega j(x) \quad \rightarrow \quad \varphi(x) = -\,\Omega \int d^{n+1}x' G^{n+1}_{\mathrm{ret}}(x - x') j(x'),$$

$$j(x) = g\int d\tau \sqrt{\dot{z}^\alpha \dot{z}_\alpha}\, \delta^{n+1}(x - z(\tau))\,.$$

The canonical energy-momentum tensor of the free massless scalar field is

$$T_{\mu\nu}(x) = \frac{1}{\Omega}\left(\partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu}\partial_\alpha \varphi \partial^\alpha \varphi\right).$$

In what follows we will consider the radiation of charge moving along the circular trajectory with constant velocity.

[6]R. A. Breuer et. al., Phys. Rev. D 8, 4309 (1973)

Static limit

Let us consider the field of $(4+1)$-dimensional static charge. Five-dimensional retarded Green's function is given by

$$G_{\rm ret}^{4+1}(X) = \frac{\theta(X^0)}{2\pi^2}\left(\frac{\delta(X^2)}{(X^2)^{1/2}} - \frac{1}{2}\frac{\theta(X^2)}{(X^2)^{3/2}}\right).$$

Assume that the source is switched on for the finite interval of proper time $\tau \in [a; b]$, $a < 0$ $b > 0$. Then, retarded field takes the following form

$$z^\mu(\tau) = [\tau, 0, 0, 0, 0] \;\; \rightarrow \;\; \varphi(x) = \frac{g}{2}\int_a^b d\tau \left(\frac{\theta(t-\tau-r-\varepsilon)}{[(t-\tau)^2-r^2]^{3/2}} - \frac{\delta(t-\tau-r-\varepsilon)}{r[(t-\tau)^2-r^2]^{1/2}}\right), \; \varepsilon = +0.$$

Performing integration, in the limit $a, b \rightarrow \pm\infty$, we get the finite static field

$$\varphi(t,r) = \frac{g}{2}\begin{cases} 0, \quad t < a + r, \\ -\dfrac{(t-a)}{r^2[(t-a)^2-r^2]^{1/2}}, \quad t \in [a+r, b+r), \\ \dfrac{(t-b)}{r^2[(t-b)^2-r^2]^{1/2}} - \dfrac{(t-a)}{r^2[(t-a)^2-r^2]^{1/2}}, \quad t \geq b + r. \end{cases} \quad \rightarrow \quad \varphi(x)\Big|_{a,b\rightarrow\pm\infty} = -\frac{g}{2r^2}.$$

Spectral distribution of the radiated energy

Using the scalar field's energy-momentum conservation law

$$\partial^\nu T_{\mu\nu}(x) = -\,\partial_\mu \varphi(x)\cdot j(x) \;\to\; P_\mu = \int d^{n+1}x\,\partial^\nu T_{\mu\nu} = \int\limits_\Omega d\sigma^\nu\, T_{\mu\nu} = -\int d^{n+1}\,\partial_\mu\varphi\cdot j(x),$$

we find the total energy-momentum radiated by the scalar field. Inserting the Fourier transforms of the retarded scalar field and scalar current

$$\varphi_{\rm ret}(x) = \int \frac{d^{n+1}k}{(2\pi)^{n+1}} e^{-ikx}\tilde{\varphi}_{\rm ret}(k), \quad \tilde{\varphi}_{\rm ret}(k) = \frac{\Omega\tilde{j}(k)}{k^2 + i\varepsilon k^0}, \quad j(x) = \int \frac{d^{n+1}k}{(2\pi)^{n+1}} e^{-ikx}\tilde{j}(k),$$

we come to the spectral-angular distribution of the total radiated energy

$$P_\mu = \frac{\Omega}{(2\pi)^n}\int d^{n+1}k\,k_\mu\,|\,\tilde{j}(k)\,|^2\,\theta(k^0)\delta(k^2) \;\to\; \frac{dE_{n+1}}{d\omega d\Omega} = \frac{\Omega\omega^{n-1}g^2}{2(2\pi)^n\gamma^2}\left|\int\limits_{-\infty}^{+\infty} dt\exp\{i\omega t - i\mathbf{k}\mathbf{z}(t)\}\right|^2,$$

where we changed the integration variable as $t = \gamma\tau$ and γ is the Lorentz-factor of the moving charge. In case of periodic motion of charge we find the spectral-angular distribution of the radiation power

$$W_{n+1} = \frac{\Omega\omega_0^{n-1}g^2}{2(2\pi)^{n-2}\gamma^2}\sum_{l=1}^{+\infty} l^{n-1}\int d\theta d\zeta\ldots\sin\theta\sin^2\zeta\ldots J_l^2\left(vl\cdot f(\theta,\zeta,\ldots)\right), \quad f(\theta,\zeta,\ldots) = \sin(\theta)\sin(\zeta)\ldots.$$

$(2+1)$-dimensional theory: radiated energy

The $(2+1)$-dimensional retarded scalar field is given by the integral

$$G^{2+1}_{\text{ret}}(X) = \frac{\theta\left(X^0\right)}{2\pi}\frac{\theta\left(X^2\right)}{\sqrt{X^2}} \quad \to \quad \varphi^{\text{ret}}(x) = -\,g\int_{-\infty}^{\hat{\tau}} d\tau \frac{\theta\left(X^0(z)\right)\cdot\theta\left(X^2(z)\right)}{\sqrt{X^2(z)}}, \qquad X^\mu(z) = x^\mu - z^\mu(\tau)\,.$$

By analogy with Teitelboim's calculations, we find the most long range (with respect to $\hat{\rho}$) part of the scalar field's gradient, which we denote as $\varphi^{\text{rad}}_\mu(x)$

$$\partial_\mu\varphi^{\text{ret}} \quad\to\quad \varphi^{\text{rad}}_\mu = \frac{g\hat{c}_\mu}{2^{1/2}\hat{\rho}^{1/2}}\int_{-\infty}^{\hat{\tau}} d\tau\left(\frac{1}{2(Z\hat{c})^{3/2}} - \frac{\delta(\tau-\hat{\tau})}{(Z\hat{c})^{1/2}}\right), \qquad X^\mu(z) = Z^\mu + \hat{\rho}\hat{c}^\mu, \quad Z^\mu = \hat{z}^\mu - z^\mu(\tau)\,.$$

Inserting this part of the gradient into the scalar field's energy-momentum tensor, we find the radiated part of the energy-momentum

$$T^{\text{rad}}_{\mu\nu} = \frac{g^2\hat{c}_\mu\hat{c}_\nu}{16\pi\hat{\rho}}A^2(x), \qquad A(x) = \lim_{\varepsilon\to 0}\int_{-\infty}^{\hat{\tau}-\varepsilon} d\tau\left(\frac{1}{(Z\hat{c})^{3/2}} - \frac{1}{(\hat{\tau}-\tau)^{3/2}}\right).$$

$(2+1)$-dimensional theory: circular motion

In case of circular motion with constant velocity the charge's world line takes the following form

$$z^\mu(\tau) = \left[\gamma\tau, R_0\cos(\omega_0\gamma\tau), R_0\sin(\omega_0\gamma\tau)\right], \quad \gamma = E/m = 1/\sqrt{1-v^2}, \quad v = R_0\omega_0,$$

where R_0 is the radius of the circular trajectory and ω_0 is frequency of orbital motion. Far from the circle $r \gg R_0$, we find

$$\hat{\tau} = \frac{t-r}{\gamma}, \quad \hat{\rho} = \gamma r\left(1 + v\sin(\omega_0\gamma\hat{\tau} - \phi)\right), \quad \hat{c}^\mu = \frac{r}{\hat{\rho}}[1, \cos\phi, \sin\phi],$$

where we introduced the polar coordinates for the observation point $x^\mu = [t, r\cos\phi, r\sin\phi]$.

By the use of obtained expressions, we come to the integral amplitude of $T^{\mathrm{rad}}_{\mu\nu}$ in the following form

$$A(x) = (\omega_0\gamma)^{1/2}\int\limits_0^{+\infty} ds\left\{\frac{(1-v\cos a)^{3/2}}{\left(s - v\sin a - v\sin(s-a)\right)^{3/2}} - \frac{1}{s^{3/2}}\right\}, \quad a = \omega_0\gamma\hat{\tau} - \phi + \pi/2, \quad s = \omega_0\gamma(\hat{\tau} - \tau).$$

$(2+1)$-dimensional theory: non-relativistic limit

In the non-relativistic limit $v \ll 1$, we calculate the integral amplitude $A(x)$ up to the leading order in the charge's velocity

$$A(x) = \frac{3}{2}\omega_0^{1/2} v \int\limits_0^{+\infty} ds \frac{\sin a + \sin(s-a) - s \cdot \cos a}{s^{5/2}} + O(v^2)\,.$$

Integrating by parts twice, we can explicitly demonstrate the convergence of the obtained integral

$$A(x) = \omega_0^{1/2} v \left(\frac{\sin(a) + \sin(\varepsilon - a)}{\varepsilon^{3/2}} - \frac{\cos(a)}{\varepsilon^{1/2}} + 2\frac{\cos(\varepsilon - a) - \cos(a)}{\varepsilon^{1/2}} \right) - 2\omega_0^{1/2} v \int\limits_{\varepsilon}^{+\infty} ds \frac{\sin(s-a)}{s^{1/2}}\,.$$

The remaining integral is the linear combination of Fresnel integrals: $\int\limits_0^{+\infty} ds \frac{\sin(s)}{s^{1/2}} = \int\limits_0^{+\infty} ds \frac{\cos(s)}{s^{1/2}} = \sqrt{\frac{\pi}{2}}\,.$

Choosing the retarded proper time as $\hat{\tau} = 3\pi/2\omega_0\gamma$ and introducing new angular variable $\alpha = 2\pi - a$, we find the radiation power, to leading order in v, in the following form

$$\frac{dW_{2+1}}{d\alpha} = \frac{g^2\omega_0 v^2}{4} \sin^2\left(\alpha + \frac{\pi}{4}\right),\ \alpha \in [0, 2\pi) \quad \rightarrow \quad W_{2+1}^{\text{nonrel}} = \frac{\pi}{4} g^2 \omega_0 v^2\,.$$

$(2+1)$-dimensional theory: ultrarelativistic limit

In the ultrarelativistic limit $\gamma \gg 1$, the main contributions to the integral amplitude are given by[7]:

$$A(x) = (\omega_0\gamma)^{1/2} \int_0^{+\infty} ds \left\{ \frac{(1 - v\cos a)^{3/2}}{\big(s - v\sin a - v\sin(s-a)\big)^{3/2}} - \frac{1}{s^{3/2}} \right\}, \quad a = \omega_0\gamma\hat{\tau} - \phi + \pi/2, \quad s = \omega_0\gamma(\hat{\tau} - \tau)\,.$$

- small interval of proper time $\delta s \sim 1/\gamma$ before the retarded time $s = 0$;
- small interval of angular variable $\delta a \sim 1/\gamma$ around the direction of motion at retarded time $a = 0$.

The radiated energy flux is beamed in the direction of the charge motion and its dependence on the charge's history of motion is effectively localised at the retarded proper time $\hat{\tau}$.

We calculate the integral amplitude up to the leading order in Lorentz-factor γ

$$A(x) = \gamma\omega_0^{1/2} \int_0^{+\infty} dx\, F(x), \quad F(x) = \frac{1}{x^{3/2}} \left[\frac{(\hat{a}^2 + 1)^{3/2}}{(x^2/3 - \hat{a}x + \hat{a}^2 + 1)^{3/2}} - 1 \right], \quad x = \gamma s, \quad \hat{a} = \gamma a\,.$$

To make the convergence of the obtained integral explicit, we integrate by parts twice and obtain

$$\int_0^{+\infty} dx\, F(x) = \int_0^{+\infty} dx\, \frac{x^{1/2}}{(x^2/3 - \hat{a}x + \hat{a}^2 + 1)^{5/2}} \left[4 - \frac{15(2x/3 - \hat{a})^2}{x^2/3 - \hat{a}x + \hat{a}^2 + 1} \right].$$

[7] E. Shuryak, et. al., Phys. Rev. D 85 (2012) 104007

$(2+1)$-dimensional theory: ultrarelativistic limit

Thus, the radiation power, up to the leading order in γ, is given by the following integral

$$W_{2+1} = \frac{g^2\omega_0\gamma^2}{4\pi}\int\limits_{-\infty}^{+\infty} d\hat{a}\frac{A^2(x)}{(\hat{a}^2+1)^2}\,.$$

The integral amplitude here is just the numerical factor, independent of any physical parameter of the system, and could be found by the numerical calculations. Numerical integration gives value $4\pi/\sqrt{3}$, and the radiation power takes the following form

$$W_{2+1}^{\text{synch}} = \frac{g^2\omega_0\gamma^2}{\sqrt{3}}\,.$$

$(2+1)$-dimensional theory: spectral calculation

Let us consider the non-relativistic limit and use the spectral-angular distribution of the radiation power of the periodically moving charge, which in $(2+1)$-dimensional space-time takes the form

$$W_{2+1} = \frac{\pi\omega_0 g^2}{\gamma^2} \sum_{l=1}^{+\infty} l\, J_l^2(vl)\,.$$

We use the following approximation of the Bessel functions of integer order of small argument ($vl \ll 1$, due to $v \ll 1$)

$$J_n(x)\,|_{x\to 0} = \frac{x^n}{2^n n!}\,.$$

Then, up to the leading order in charge's velocity, we obtain the following result for the radiation power

$$W_{2+1}^{\text{nonrel}} = \frac{\pi}{4} g^2 \omega_0 v^2,$$

which is the same as obtained by the calculations in the wave zone.

$(2+1)$-dimensional theory: spectral calculation

In the ultrarelativistic limit, we start with the spectral-angular distribution of the total radiated energy

$$\frac{dE_{2+1}}{d\omega d\Omega} = \frac{\omega g^2}{4\pi\gamma^2} \int\limits_{-\infty}^{+\infty} dt_1 \int\limits_{-\infty}^{+\infty} dt_2\, e^{i\omega(t_1-t_2)-i\mathbf{k}(\mathbf{z}(t_1)-\mathbf{z}(t_2))}\,.$$

Transforming the integration variables, we obtain the spectral-angular distribution of the radiation power

$$\frac{dW_{2+1}}{d\omega d\Omega} = \frac{dE_{2+1}/dt}{d\omega d\Omega} = \frac{\omega g^2}{4\pi\gamma^2} \int\limits_{-\infty}^{+\infty} dt_0\, e^{-i\omega t_0 - i\mathbf{k}(\mathbf{z}(t-t_0/2)-\mathbf{z}(t+t_0/2))}, \quad t_1 = t - t_0/2,\, t_2 = t + t_0/2$$

which in the case of circular motion of charge could be represented, up to the leading order of Lorentz-factor γ, as

$$\frac{dW_{2+1}}{d\omega} = \frac{\omega g^2}{4\pi\gamma^2} \int\limits_{0}^{2\pi} d\phi \int\limits_{-\infty}^{+\infty} dt_0 \exp\left\{-i\omega t_0\left(\frac{1}{2}\left(a^2+1/\gamma^2\right) + \frac{\omega_0^2 t_0^2}{24}\right)\right\}, \quad a = \omega_0 t - \phi + \pi/2.$$

$(2+1)$-dimensional theory: spectral calculation

Using the definition of Airy function[8] and performing integration, we come to the radiation power in following form

$$\mathrm{Ai}(u) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} dt' \exp\left\{ i\left(ut' + \frac{t'^3}{3} \right) \right\} \to W_{2+1} = \frac{3}{2^{1/3}}\pi g^2\omega_0\gamma^2 \int\limits_0^{+\infty} dx\, x\, \mathrm{Ai}^2\left(\frac{x}{2^{2/3}}\right).$$

The remaining integral is given by formula[8]

$$\int\limits_0^{+\infty} ds\, s\, \mathrm{Ai}^2(s) = \frac{1}{6\sqrt{3}\pi}.$$

Finally, we obtain the same result as found by calculations in the wave zone

$$W_{2+1}^{\mathrm{synch}} = \frac{g^2\omega_0\gamma^2}{\sqrt{3}}.$$

[8]V. Olivier and S. Manuel, “Airy Functions And Applications To Physics (2nd Edition)” (World Scientific Publishing Company, 2010)

$(4+1)$-dimensional theory: results

In five dimensions, the radiated part of the energy-momentum is given by the sum of three terms

$$G^{4+1}_{\text{ret}}(X) = \frac{\theta(X^0)}{2\pi^2}\left(\frac{\delta(X^2)}{(X^2)^{1/2}} - \frac{1}{2}\frac{\theta(X^2)}{(X^2)^{3/2}}\right) \to T^{\text{rad}}_{\mu\nu}(x) = \frac{g^2\hat{c}_\mu\hat{c}_\nu}{64\pi^2\hat{\rho}^3}A^2(x),$$

$$A(x) = \int\limits_{-\infty}^{\hat{\tau}} d\tau \left\{\frac{3}{2}\frac{1}{(Z\hat{c})^{5/2}} - \frac{3}{2}\frac{1}{(\hat{\tau}-\tau)^{5/2}} - \frac{\hat{a}\hat{c}}{(\hat{\tau}-\tau)^{3/2}}\right\}.$$

After calculations, analogous to those in three dimensions, we obtain the radiation power in non-relativistic and ultrarelativistic limits

$$W^{\text{nonrel}}_{4+1} = \frac{\pi}{32}g^2\omega_0^3 v^2, \quad W^{\text{synch}}_{4+1} = \frac{g^2\omega_0^3\gamma^6}{\sqrt{27}} \to W^{\text{synch}}_{n+1} = g^2\left(\frac{\omega_0\gamma^2}{\sqrt{3}}\right)^{n-1}.$$

The angular distribution of the radiation power of the non-relativistic charge in $(4+1)$-dimensional theory is given by the formula

$$\frac{dW_{4+1}}{d\Omega} = \frac{g^2\omega_0^3 v^2}{16\pi}\sin^2\theta\sin^2\zeta\sin^2\left(\alpha - \frac{\pi}{4}\right), \quad \alpha = 2\pi - a \to \frac{W^{\text{nonrel}}_{4+1}|_{\zeta=\pi/2}}{W^{\text{nonrel}}_{4+1}} = \frac{8}{3\pi} \approx 0.85.$$

Contribution of the tail parts

Let us consider the $(2+1)$-dimensional scalar non-relativistic charge moving along the x-axis with the acceleration in form of the Gaussian function

$$a^{\mu}(\tau) = \left\{0; \frac{1}{A}\exp\left[-\frac{(\tau-a)^2}{2b^2}\right]; 0\right\} \;\to\; z^{\mu}(\tau) = \left\{\tau; \frac{b^2}{A}\exp\left[-\frac{(\tau-a)^2}{2b^2}\right] + \sqrt{\frac{\pi}{2}}\frac{b}{A}(\tau-a)\left(1+\mathrm{erf}\left[\frac{\tau-a}{\sqrt{2}b}\right]\right); 0\right\}.$$

Here we assume that the charge was at rest at the origin of coordinates at $\tau \to -\infty$. The condition of non-relativistic motion is $b/A \ll 1$. Introducing the dimensionless variables $x = \tau/a$, $\beta = b/a$ and $\alpha = A/a$, we numerically calculate the radiation power, which could be represented in the following form

$$W_{2+1}^{\mathrm{Gauss}} = \frac{g^2}{16\pi a}\int_0^{2\pi} d\phi \frac{A^2(\phi)}{(1-\mathbb{B})^3}, \quad A(\phi) = \int_{-\infty}^{x_{\mathrm{ret}}} dx \left\{\frac{(1-\mathbb{B})^{3/2}}{\left(\mathbb{C}-\cos(\phi)[\mathbb{D}+\mathbb{E}]\right)^{3/2}} - \frac{1}{(x_{\mathrm{ret}}-x)^{3/2}}\right\}, \quad x_{\mathrm{ret}} = \hat{\tau}/a,$$

$$\mathbb{B} = \sqrt{\frac{\pi}{2}}\frac{\beta}{\alpha}\cos(\phi)\left\{1+\mathrm{erf}\left[\frac{x_{\mathrm{ret}}-1}{\sqrt{2}\beta}\right]\right\}, \quad \mathbb{C} = (x_{\mathrm{ret}}-x)\left\{1-\sqrt{\frac{\pi}{2}}\frac{\beta}{\alpha}\cos(\phi)\right\},$$

$$\mathbb{D} = \frac{\beta^2}{\alpha}\left\{\exp\left[-\frac{(x_{\mathrm{ret}}-1)^2}{2\beta^2}\right] - \exp\left[-\frac{(x-1)^2}{2\beta^2}\right]\right\} \quad \mathbb{E} = \sqrt{\frac{\pi}{2}}\frac{\beta}{\alpha}\left\{(x_{\mathrm{ret}}-1)\mathrm{erf}\left[\frac{x_{\mathrm{ret}}-1}{\sqrt{2}\beta}\right] - (x-1)\mathrm{erf}\left[\frac{x-1}{\sqrt{2}\beta}\right]\right\}.$$

Contribution of the tail parts

$$a^{\mu}(\tau)=\left\{0;\frac{1}{A_0}\exp\left[-\frac{(\tau-a)^2}{2b^2}\right];0\right\}\ \rightarrow\ z^{\mu}(\tau)=\left\{\tau;\frac{b^2}{A_0}\exp\left[-\frac{(\tau-a)^2}{2b^2}\right]+\sqrt{\frac{\pi}{2}}\frac{b}{A_0}(\tau-a)\left(1+\mathrm{erf}\left[\frac{\tau-a}{\sqrt{2}b}\right]\right);0\right\}$$

$$x=\tau/a,\ \beta=b/a$$
$$\alpha=A_0/a,\ \beta/\alpha\ll 1$$

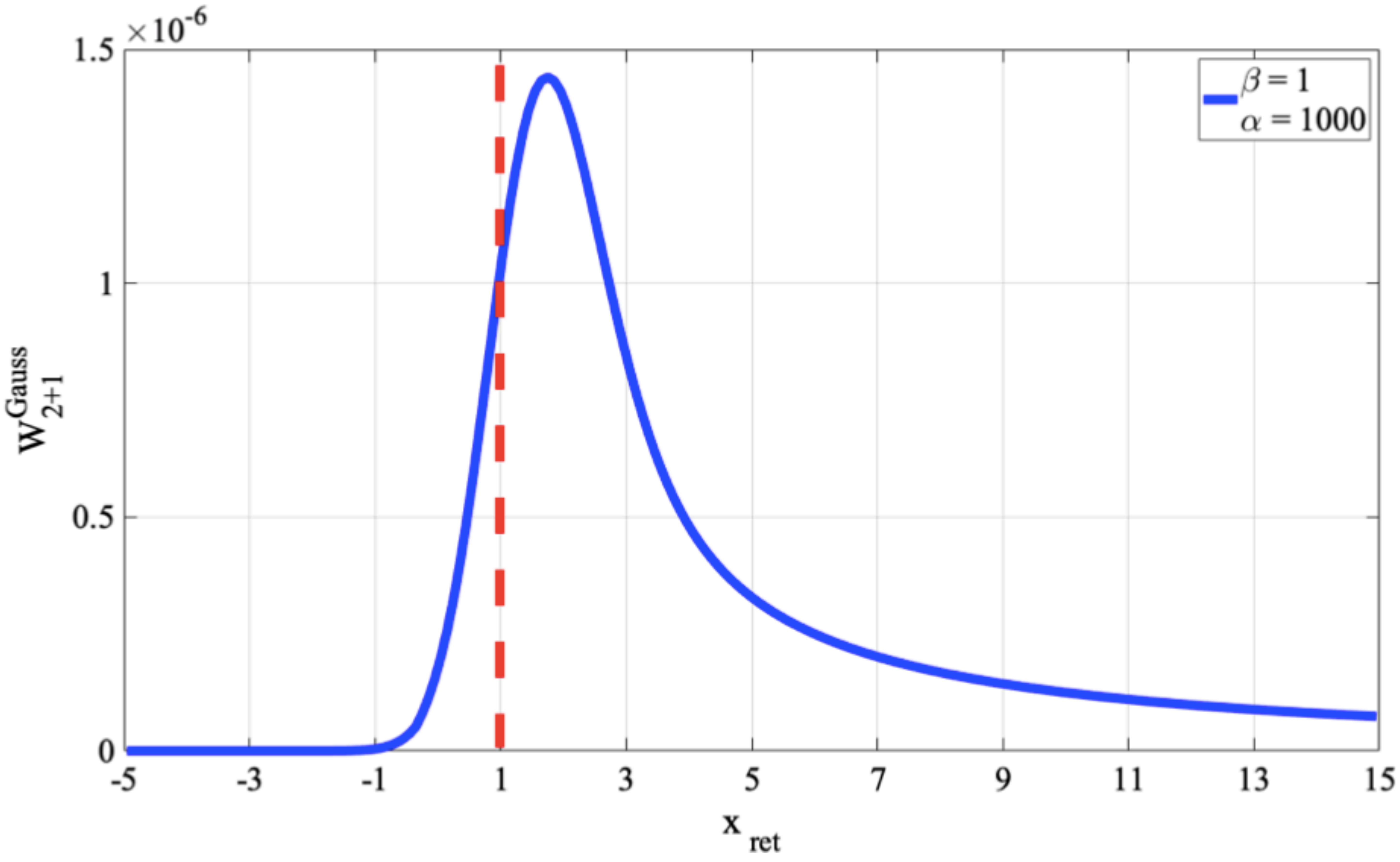

Fig. 4. Dependence of the radiation power of the non-relativistic charge on the observation time.

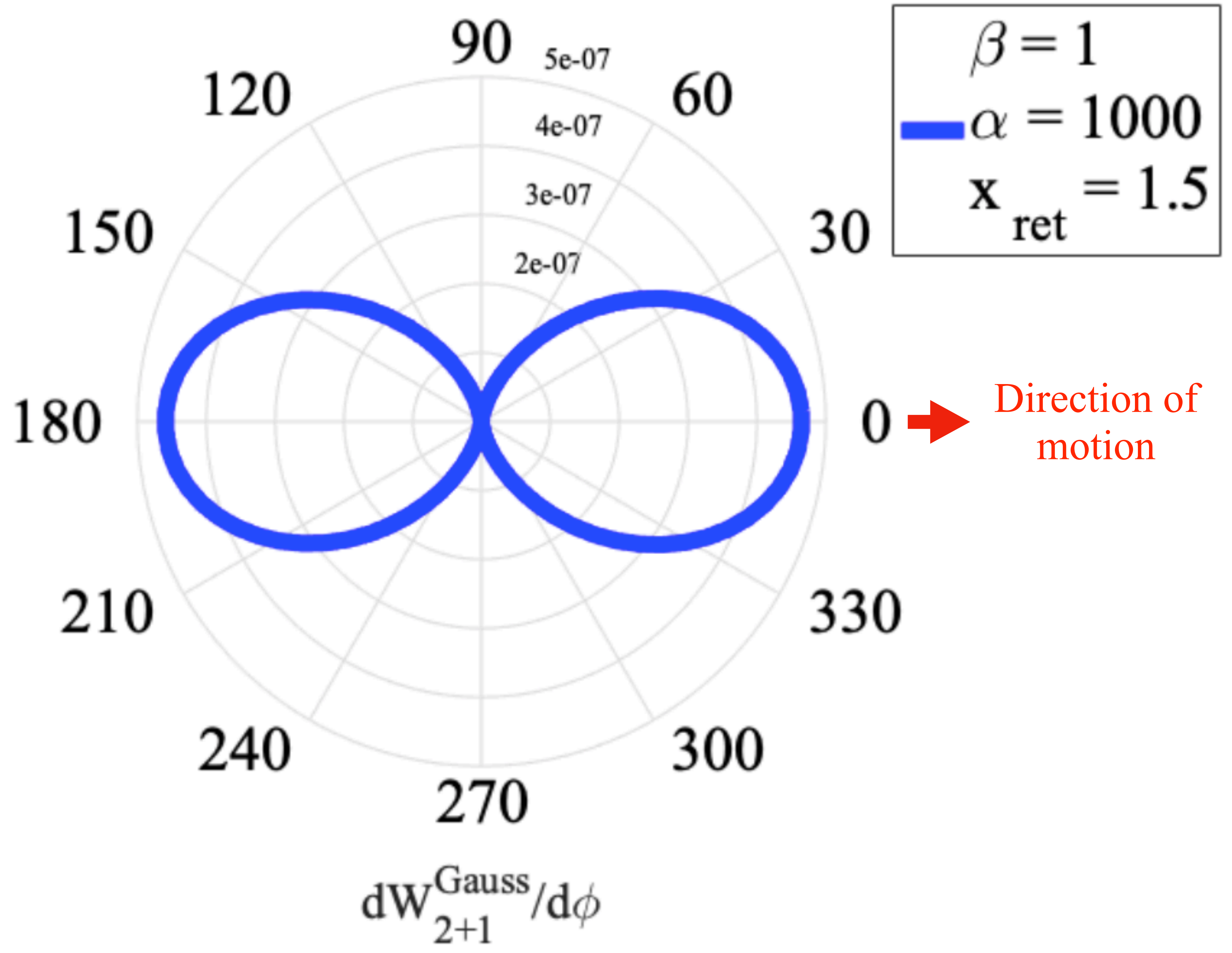

Fig. 5. Angular distribution of the radiation power of the non-relativistic charge.

Thank you for your attention!