Current Algebra and Generalised Geometry

based on arXiv:1910.00029

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Here: Go back to the world-sheet theory and study how concepts of generalised geometry, in particular the so-called generalised fluxes, manifest theirselves there.

Overview

1. Review: T-duality and generalised geometry
2. Background fields and Poisson structure
3. Generalised fluxes and the current algebra
4. Non-commutativity and non-associativity in the current algebra
5. Summary and outlook

T-duality and generalised geometry

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  - spectrum invariant under $R \leftrightarrow \sqrt{\alpha'}/R$ (R: radius of S^1)
  - exchange of momentum and winding modes

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- string σ-model in background with isometry (x^1)

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rewrite $\mathcal{L}$: $\quad \partial_\pm x^1 \to k_\pm \qquad \mathcal{L} \to \mathcal{L} - \tilde{x}^1(\partial_+ k_- - \partial_- k_+)$,

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integrating out $k_\pm$ yields equivalent geometry: (Buscher rules)

$$\bar{E}_{11} = \frac{1}{E_{11}}, \quad \bar{E}_{1i} = \frac{E_{1i}}{E_{11}}, \quad \bar{E}_{1i} = -\frac{E_{i1}}{E_{11}}, \quad \bar{E}_{ij} = E_{ij} - \frac{E_{i1}E_{1j}}{E_{11}}.$$

- change of dilaton via path integral

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  - GL-transformations – $G + B \;\rightarrow\; A^T(G+B)A$
  - B-shifts – $B \rightarrow B + B_0$
  - β-shifts – $\beta \rightarrow \beta + \beta_0$, open string variables: $\frac{1}{G+B} = g + \beta$

$$\varphi_{GL} = \begin{pmatrix} A^T & \\ & A^{-1} \end{pmatrix}, \quad \varphi_B = \begin{pmatrix} \mathbb{1} & B_0 \\ & \mathbb{1} \end{pmatrix}, \quad \varphi_\beta = \begin{pmatrix} \mathbb{1} & \\ \beta_0 & \mathbb{1} \end{pmatrix}$$

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- 'non-geometry': target space not necessarily 'geometric' ($\leftrightarrow$ manifold, algebraic variety)
  - T-folds (also S-, U-) [Hull 05]
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  - non-commutative/non-associative target spaces
- characterised by generalised (non-geometric) fluxes
  arise e.g. as S- and T-duals of geometric (flux) backgrounds

Standard example: T-duality chain 1

[Shelton et al. 05]

$$\mathbf{H}_{abc} \xrightarrow{T_1} \mathbf{f}^{c}{}_{ab} \xrightarrow{T_2} \mathbf{Q}_{c}{}^{ab} \xrightarrow{T_3} \mathbf{R}^{abc}$$

- start with **H**-flux on T^3: $\mathbf{H} = \mathrm{d}B = h\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2, \qquad B = hx^3\mathrm{d}x^1 \wedge \mathrm{d}x^2,$$

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  - 'twisted torus': $(x^1, x^2, x^3) \sim (x^1 - hx^2, x^2, x^3 + 1)$
  - parallelisable (globally well-defined frame field $e^a{}_i\mathrm{d}x^i$)

$$\mathrm{d}e^c = -\frac{1}{2}\mathbf{f}^c{}_{ab}e^a \wedge e^b \longrightarrow \text{geometric } \mathbf{f}\text{-flux, here } \mathbf{f}^3{}_{12} = h$$

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  - monodromy of β in general given by

$$\mathbf{Q}_c{}^{ab} = \partial_c \beta^{ab} \longrightarrow \text{non-geometric } \mathbf{Q}\text{-flux} \;, \text{ here } \mathbf{Q}_3{}^{12} = h$$

Standard example: T-duality chain 3

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- (formal) T-duality along x^3:

$$\hat{ds}^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2, \qquad \beta^{12} = h\tilde{x}^3$$

  - $\tilde{x}^3$ 'winding' coordinate
  - no parameterisation in terms of standard coordinates
    $\rightarrow$ 'locally non-geometric'

$\mathbf{R}^{abc} = \tilde{\partial}^{[c}\beta^{ab]} \longrightarrow$ non-geometric $\mathbf{R}$-flux , here $\mathbf{R}^{123} = h$

Generalised fluxes

- Hamiltonian: $H \sim \int \mathrm{d}\sigma \mathbf{E}_I \mathbf{E}_J \mathcal{H}^{IJ}(G, B)$
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- Different parameterisations possible – examples:
  - 'geometric' frame (**H**-, **f**-flux): $E = \begin{pmatrix} e^T & \\ & e^{-1} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1} & B \\ & \mathbb{1} \end{pmatrix}$
  - 'non-geometric' frame (**f**, **Q**, **R**): $E = \begin{pmatrix} e^T & \\ & e^{-1} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1} & \\ \beta & \mathbb{1} \end{pmatrix}$

Doubled geometry

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- Example: pure **R**-flux background $\beta^{12} = h\tilde{x}^3$ on T^3
  violates strong constraint for the section (x^1, x^2, x^3)

Background fields and Poisson structure

Magnetically charged backgrounds [Jackiw 85]

- particle (charge q, mass m) in electric background $F = \mathrm{d}A$

$$H = \frac{1}{2m}(p - qA(x))^2 = \frac{1}{2m}\pi^2$$

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- background field = deformation of Poisson structure

here: $\Pi = \Pi_{can.} + qF$, Jacobi identity $\Leftrightarrow \mathrm{d}F = 0$

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Magnetically charged backgrounds [Jackiw 85]

- particle (charge q, mass m) in electric background $F = \mathrm{d}A$

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- generalisation to strings in NSNS backgrounds?

Integrable σ-models

- principal chiral model, $g: \Sigma \to G$, $j_\alpha = (g^{-1}\partial_\alpha g)^a t_a$
  G: Lie group, structure constants $f^c{}_{ab}$

$$H \sim \int \mathrm{d}\sigma \left((j_0)^2 + (j_1)^2 \right)$$

$$\{j_{0,a}(\sigma), j_{0,b}(\sigma')\} = -f^c{}_{ab} j_{0,c}(\sigma)\delta(\sigma - \sigma'), \quad \left\{j_1^a(\sigma), j_1^b(\sigma')\right\} = 0.$$

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- Is such a form generic?

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Key steps:

1. Phrase canonical current algebra (Poisson bracket of $\partial x(\sigma) = \partial_\sigma x(\sigma)$ and $p(\sigma)$)

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3. Generalise to magnetically charged backgrounds, or those of doubled geometry.

Lie vs. Courant bracket

$$\{\mathbf{E}_M(\sigma_1), \mathbf{E}_N(\sigma_2)\} = \frac{1}{2}\eta_{MN}(\partial_1 - \partial_2)\delta(\sigma_1 - \sigma_2) + \frac{1}{2}\omega_{MN}(\partial_1 + \partial_2)\delta(\sigma_1 - \sigma_2)$$

with: $\mathbf{E}_M(\sigma) = \begin{pmatrix} p_m(\sigma) \\ \partial x^m(\sigma) \end{pmatrix}$, $\omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ and η the $O(d,d)$-metric

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- without the ω-term: Courant bracket [Siegel 93, Alekseev/Strobl 04]
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  - contributes for winding strings:
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  - ensures, that the bracket is a Lie bracket
    (canonical Poisson structure)

Hamiltonian formulation in generalised flux frame 1

- Hamiltonian 'diagonalised' in generalised flux frame

$$H \sim \int \mathrm{d}\sigma \; \mathbf{E}_M \mathbf{E}_N \mathcal{H}^{MN}(G, B) = \int \mathrm{d}\sigma \; \mathbf{E}_A \mathbf{E}_B \delta^{AB}$$

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- all background information in current algebra:

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Hamiltonian formulation in generalised flux frame 2

- Jacobi identity: $\{\mathbf{E}_A(\sigma_1), \{\mathbf{E}_B(\sigma_2), \mathbf{E}_C(\sigma_3)\}\} + \text{c.p.} = 0$
  $\Rightarrow$ B.I. of generalised fluxes: $\partial_{[A}\mathbf{F}_{BCD]} - \frac{3}{4}\mathbf{F}^E{}_{[AB}\mathbf{F}_{CD]E} = 0$

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- if Courant bracket (without ω-term)
  e.g. for open strings on D-branes:
  Jacobi identity $\Rightarrow \mathbf{F}_{ABC}|_{\text{D-brane}} = 0.$

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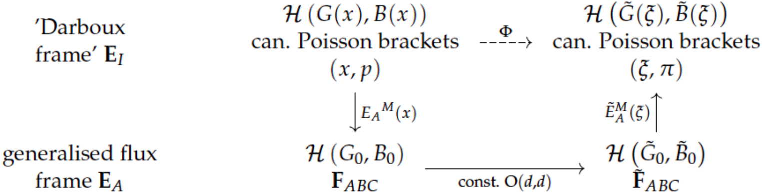

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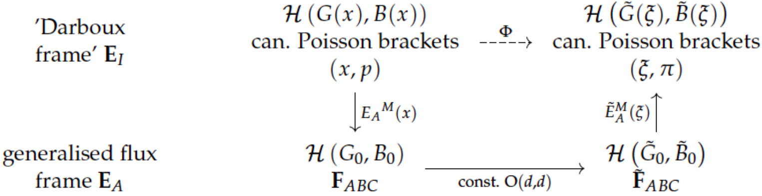

- examples:
  - abelian T-duality: $\mathbf{F}_{ABC} = 0$
  - Poisson-Lie T-duality: $\mathbf{F}_{ABC}$ structure const. of a Lie bialgebra
  - constructions for general constant $\mathbf{F}_{ABC}$ possible

Non-commutativity and non-associativity in the current algebra

Non-commutative interpretation of the current algebra

- $\mathbf{E}_I(\sigma) = \big(p_i(\sigma), \partial x^i(\sigma)\big) \longrightarrow \mathbf{E}_A(\sigma) = \big(\pi_a(\sigma), \partial y^a(\sigma)\big)$

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- without using e.o.m.
- easily applicable to any model

Examples

- open string in constant B-field [Chu/Ho 98, Seiberg/Witten 99]

$$\{y^\mu(\sigma_1), y^\nu(\sigma_2)\} = \begin{cases} -\beta^{\mu\nu}, & \sigma_1 = \sigma_2 = 1 \\ +\beta^{\mu\nu}, & \sigma_1 = \sigma_2 = 0 \\ 0 & \text{else.} \end{cases}$$

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  - only additional input: $y^a(\sigma) = y^a + w^a\sigma + \text{osc.}$
  - Jacobi identity of zero mode algebra only fulfilled, if ω-term included in current algebra

Non-associativity and violation of the section condition

- dual coordinate $\tilde{x}_i$ from $p_i = \partial \tilde{x}_i$:
  $\left\{ X^I(\sigma_1), X^J(\sigma_2) \right\} = -\eta^{IJ}\Theta(\sigma_1 - \sigma_2) \quad \text{with} \quad X^I = (x^i, \tilde{x}_i)$

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  - example: T^3 with pure **R**-flux background, $\beta^{12} = h\tilde{x}^3$
    $\{y^1, \{y^2, y^3\}\} + \text{c.p.} = -\mathbf{R}^{123} = -h$

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Thank you for your attention!