Supergravity Excitations of Stringy Geometries

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O. L., arXiv:1907.03820
O. L., J.Tian, arXiv:2012.15083
work in progress

Motivation

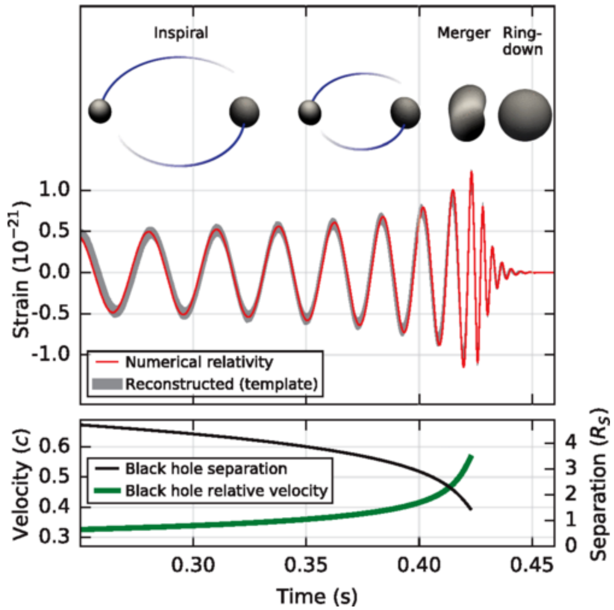
- Nontrivial geometries play an important role in string theory
  - insights into quantum gravity from black holes in various dimensions
  - insights into strong interactions via AdS/CFT
  - insights into stringy dynamics from integrable backgrounds
- Dynamics of stringy geometries
  - mass spectra of excitations
  - scattering amplitudes
  - topological defects (branes in curved spaces)
- Excitations of stringy geometries
  - light modes: pertubations from the supergravity multiplet
  - strings and branes in background fields
  - heavy excitations: new supergravity solutions
- Analysis of supergravity modes
  - systems of linear PDEs
  - black holes: separation of variables
  - integrable geometries: spectra from worldsheet techniques
- Goals of this work
  - separation of variables for all tensor fields in all rotating BHs
  - spectra and wavefunctions on large classes of WZW models

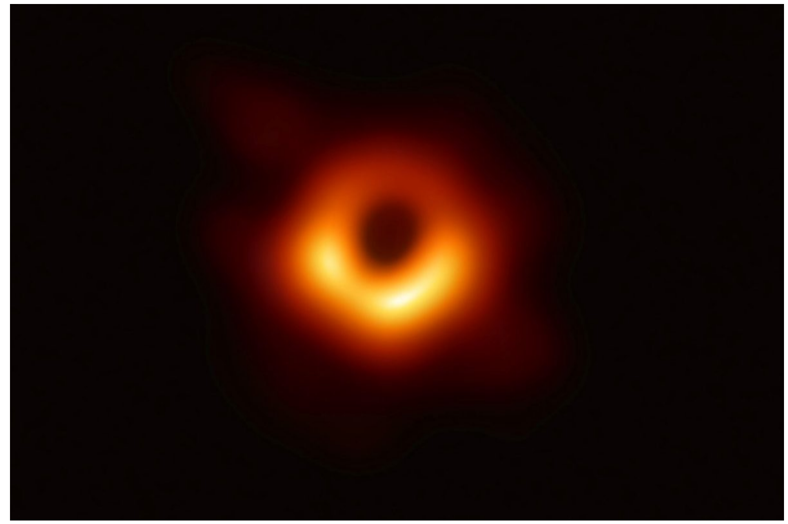
Outline

- Motivation
  - understanding of fields in the vicinity of black holes
  - exploration of integrable string theories and their dynamics
- Excitations of rotating black holes
  - review of the known results in 4D
  - Myers–Perry black holes and their symmetries
  - full separability of vector and tensor equations
  - progress towards understanding of gravitational waves
- Supergravity excitations of the WZW geometries
  - separation of variables in low–dimensional cases
  - vector excitations and T duality
  - spectra of scalar and vector modes from a group–theoretic construction
  - extension to λ–deformed backgrounds
- Similarities between separable models of black holes and WZW models
- Outlook

Excitations of rotating black holes

- Particles and fields provide insights into the nature of black holes





- Classical scattering and radiation
  - radiation from infalling particles
  - gravitational lenzing
  - gravitational waves
- Quantum fields
  - detailed study of the Hawking radiation
  - detection of the differences between the BH and its microstates
  - tests of AdS/CFT correspondence

Excitations of rotating black holes

- Particles and fields provide insights into the nature of black holes
- Classical scattering and radiation
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  - gravitational waves
- Quantum fields
  - detailed study of the Hawking radiation
  - detection of the differences between the BH and its microstates
  - tests of AdS/CFT correspondence
- Many excitations have been studied in the past
  - all fields in the static geometries: power of rotational symmetry
  - scalar fields in all dimensions
  - electromagnetic field and gravitons in 4D
- Goals of this work
  - finding the most general solution for photons and higher forms in all D
  - demonstrating separation of variables for gravitational waves
  - understanding the role of symmetries in the separation procedure
- Result: separation is controlled by eigenvectors of the Killing–Yano tensor

Electromagnetic field in the Kerr geometry

- Excitations the Schwarzschild geometry
  - $U(1)_t \times SO(3)$ symmetry $\Rightarrow$ spherical harmonics for all fields
  - system of ODEs for functions of r

$$ds^2 = -\left[1 - \frac{r_M}{r}\right] dt^2 + \frac{dr^2}{1 - \frac{r_M}{r}} + r^2 d\Omega^2$$

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0, \quad V^\mu \partial_\mu = \partial_t \quad \Rightarrow \quad \Psi \propto e^{i\omega t}$$

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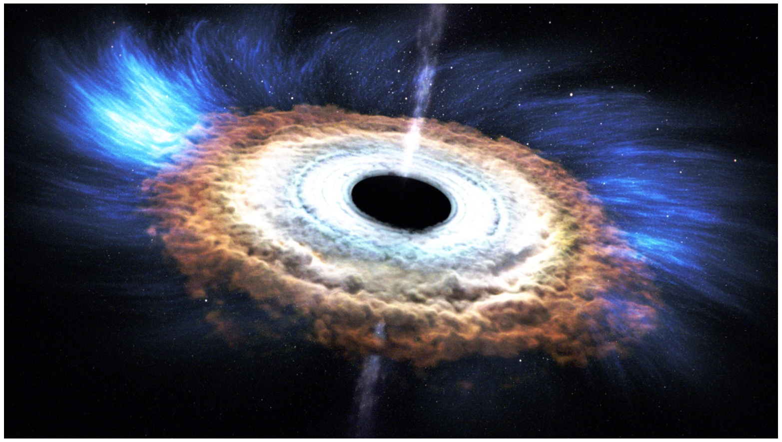
- Scalar excitations of the Kerr geometry
  - $U(1)_t \times U(1)_\phi$ $\Rightarrow$ system of PDEs for functions of (r, θ)
  - hidden symmetry $\Rightarrow$ full separation: $\Psi = e^{im\phi + i\omega t} R(r)\Theta(\theta)$ Carter '68

$$\nabla_\mu Y_{\nu\lambda} + \nabla_\nu Y_{\mu\lambda} = 0$$



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- Photons and gravitons: which components should separate?
- Electromagnetism in the Newman–Penrose formalism Newman-Penrose '62
  - define four null frames, $(l, n, m, \bar{m})$

$$l^\mu \partial_\mu = \frac{r^2 + a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi, \quad n^\mu \partial_\mu = \frac{r^2 + a^2}{2\Sigma}\partial_t - \frac{\Delta}{2\Sigma}\partial_r + \frac{a}{2\Sigma}\partial_\phi,$$

$$m^\mu \partial_\mu = \frac{1}{\sqrt{2}\rho}\left[ias_\theta \partial_t + \partial_\theta + \frac{i}{s_\theta}\partial_\phi\right], \quad \rho = r + iac_\theta, \ \Sigma = \rho\bar{\rho}, \ \Delta = r^2 + a^2 - 2Mr.$$

$$ds^2 = (-l_\mu n_\nu + m_\mu \bar{m}_\nu) dx^\mu dx^\nu$$

Electromagnetic field in the Kerr geometry

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- Photons and gravitons: which components should separate?
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  - define four null frames, $(l, n, m, \bar{m})$
  - gauge field is encoded in three complex scalars

$$F_{\mu\nu} = 2\left[\phi_1(n_{[\mu} l_{\nu]} + m_{[\mu}\bar{m}_{\nu]}) + \phi_2 l_{[\mu} m_{\nu]} + \phi_0 \bar{m}_{[\mu} n_{\nu]}\right] + cc.$$

  - first order PDEs for (ϕ_0, ϕ_1, ϕ_2)
- Kerr geometry: separable 2-nd order PDEs for ϕ_0 and $\bar{\rho}^2\phi_2$ Teukolsky '72
- Problems with the known solution:
  - the remaining eqn is not separable Starobinsky–Churilov '73, Teukolsky–Press '74
  - it is hard to recover the gauge potential Chandrasekhar '76
  - construction is based on F and $\star F$ $\Rightarrow$ hard to extend to $D > 4$
- New ansatz may solve these problems and lead to extensions to $D > 4$

New ansatz in four dimensions

OL '17

- Analysis of the explicit solution for A_μ
  - original expressions are very complicated — Chandrasekhar '83
  - simplifications in the frame components

$$l^\mu A_\mu = \frac{2ia}{r} l^\mu \partial_\mu [e^{i\omega t + im\phi} g_+(r) f_+(\theta)] + 2 l^\mu \partial_\mu H_+(r,\theta)$$

  - up to overall factors, no θ in (l^μ, n^μ), no r in $(m^\mu, \bar{m}^\mu)$

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  - up to overall factors, no θ in (l^μ, n^μ), no r in $(m^\mu, \bar{m}^\mu)$
- New proposal for a separable ansatz

$$l^\mu A_\mu = G_+(r) l^\mu \partial_\mu \Psi, \qquad n^\mu A_\mu = G_-(r) n^\mu \partial_\mu \Psi,$$
$$m^\mu A_\mu = F_+(\theta) m^\mu \partial_\mu \Psi, \quad \bar{m}^\mu A_\mu = F_-(\theta) \bar{m}^\mu \partial_\mu \Psi, \quad \Psi = e^{i\omega t + im\phi} R(r) S(\theta).$$

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  - simplifications in the frame components
  - up to overall factors, no θ in (l^μ, n^μ), no r in $(m^\mu, \bar{m}^\mu)$
- New proposal for a separable ansatz
- Most general separable solution of Maxwell's equations
  - functions $G_\pm$ and $F_\pm$ are completely determined by integrability conditions

$$l_\pm^\mu A_\mu = \pm \frac{ia}{r \pm i\mu a} \hat{l}_\pm \Psi, \quad m_\pm^\mu A_\mu = \mp \frac{1}{c_\theta \mp \mu} \hat{m}_\pm \Psi$$

  - Maxwell's equations $\Rightarrow$ "master equations" for (S, R):

$$\frac{D_\theta}{s_\theta} \frac{d}{d\theta} \left[\frac{s_\theta}{D_\theta} \partial_\theta S \right] + \left\{ -\frac{2\Lambda}{D_\theta} - (a s_\theta)^2 \left[\omega + \frac{m}{a s_\theta^2} \right]^2 + \Lambda \right\} S = 0$$

  - electromagnetism and scalar field are covered

scalar : $\quad D_r = 1, \quad D_\theta = 1, \quad \forall\Lambda;$

photon : $\quad D_r = 1 + \frac{r^2}{(\mu a)^2}, \quad D_\theta = 1 - \frac{c_\theta^2}{\mu^2}, \quad \Lambda = -\frac{1}{\mu}\left[a\omega + m - a\omega\mu^2\right]$

- Can this construction be extended to $D > 4$ and other fields?

Myers–Perry geometry

- General properties
  - $\left[\frac{D-1}{2}\right]$ rotations in D dimensions
  - different structures in odd and even D
  - $D = 2(n+1)$: $U(1)_t \times [U(1)]^n$ isometry
- Separable Klein–Gordon & Dirac eqns $\Rightarrow$ families of Killing(–Yano) tensors

$$\nabla_\mu Y^{(k)}_{\nu_1 \ldots \nu_k} + \nabla_{\nu_1} Y^{(k)}_{\mu \ldots \nu_k} = 0, \quad Y^{(D-2k)} = \star \Big[\wedge h \Big]^k$$

Frolov, Krtous, Kubiznak, Page '06-'08

$$h = \Lambda_r e^t \wedge e^r + \sum \Lambda_k e^{x_k} \wedge e^k$$

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- Ellipsoidal coordinates and "canonical" frames

$$e_t = -\sqrt{\frac{R^2}{FR(R-Mr)}}\left[\partial_t - \sum_k \frac{a_k}{r^2+a_k^2}\partial_{\phi_k}\right], \quad e_r = \sqrt{\frac{R-Mr}{FR}}\partial_r,$$

$$e_i = -\sqrt{\frac{H_i}{d_i(r^2+x_i^2)}}\left[\partial_t - \sum_k \frac{a_k}{a_k^2-x_i^2}\partial_{\phi_k}\right], \quad e_{x_i} = \sqrt{\frac{H_i}{d_i(r^2+x_i^2)}}\partial_{x_i}$$

$$R = \prod_{k=1}^{n}(r^2+a_k^2), \quad H_i = \prod_k (a_k^2 - x_i^2)$$

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- Separation of the wave equation
    - ansatz for the wavefunction: $\Psi = e^{i\omega t + i\sum m_i\phi_i}\Phi(r)\left[\prod X_i(x_i)\right]$ %pause
    - equations for Φ and X_i:

$$\frac{d}{dr}\left[(R-Mr)\frac{d\Phi}{dr}\right] + \frac{R^2}{R-Mr}\left[\omega - \sum_k \frac{a_k m_k}{r^2+a_k^2}\right]^2\Phi = P_{n-1}(r^2)\Phi,$$

OL '17

$$\frac{d}{dx_i}\left[H_i\frac{dX_i}{dx_i}\right] - H_i\left[\omega - \sum_k \frac{a_k m_k}{a_k^2-x_i^2}\right]^2 X_i = -P_{n-1}(-x_i^2)X_i$$

    - separation constants: coefficients of one polynomial P_{n-1}
- The ODEs should have counterparts for fields with higher spins

Maxwell's equations in the Myers–Perry geometry

OL '17

- Ansatz for the gauge field
  - lesson from 4D: we need $l_\pm^\mu$ and $m_\pm^\mu$
  - recall the frames

$$e_t = -\sqrt{\frac{R^2}{FR(R-Mr)}}\left[\partial_t - \sum_k \frac{a_k}{r^2+a_k^2}\partial_{\phi_k}\right], \quad e_r = \sqrt{\frac{R-Mr}{FR}}\partial_r,$$

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  - ... and combine them

$$l_\pm^\mu \partial_\mu = \frac{R}{\sqrt{\Delta}}\left\{\frac{\Delta}{R}\partial_r \pm \left[\partial_t - \sum_k \frac{a_k}{r^2+a_k^2}\partial_{\phi_k}\right]\right\}, \quad \Delta = R - Mr,$$

$$\left[m_\pm^{(j)}\right]^\mu \partial_\mu = \sqrt{H_j}\left\{\partial_{x_j} \pm i\left[\partial_t - \sum_k \frac{a_k}{a_k^2-x_j^2}\partial_{\phi_k}\right]\right\}$$

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  - recall the frames
  - ...and combine them

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$$\left[m_\pm^{(j)} \right]^\mu \partial_\mu = \sqrt{H_j} \left\{ \partial_{x_j} \pm i \left[\partial_t - \sum_k \frac{a_k}{a_k^2 - x_j^2} \partial_{\phi_k} \right] \right\}$$

  - Teukolsky's frames are eigenvectors of the KYT in 4D
  - impose an ansatz inspired by 4D

$$l_\pm^\mu A_\mu = \pm \frac{1}{r \pm i\mu} \hat{l}_\pm \Psi, \quad [m_\pm^{(j)}]^\mu A_\mu = \mp \frac{i}{x_j \pm \mu} \hat{m}_\pm^{(j)} \Psi\,,$$

Maxwell's equations in the Myers–Perry geometry

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- Ansatz for the gauge field

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- Separation of variables and the "master equations"

$$D_j\frac{d}{dx}\left[\frac{H_j}{D_j}X_j'\right]+\left\{\frac{2\Lambda}{D_j}-H_jW_j^2-\Lambda+P_{n-2}[-x_j^2]D_j\right\}X_j=0,$$

$$D_r\frac{d}{dr}\left[\frac{\Delta}{D_r}\dot{\Phi}\right]-\left\{\frac{2\Lambda}{D_r}-\frac{R^2W_r^2}{\Delta}-\Lambda+P_{n-2}[r^2]D_r\right\}\Phi=0,$$

$$\Omega=\omega-\sum\frac{m_ia_i}{\Lambda_i},\quad W_j=\omega-\sum\frac{m_ka_k}{a_k^2-x_j^2},\quad W_r=\omega-\sum\frac{m_ka_k}{a_k^2+r^2}.$$

- Separation constants in $d=2n+2$:
  $(n+1)$ angular mom/energy, $(n-1)$ parameters in P_{n-2}, μ.
- Equations cover scalar and photon

$$\text{scalar}:\qquad D_r=D_j=1,\quad \forall\Lambda;$$

$$\text{vector}:\qquad \begin{array}{l} D_j=1-\frac{x_j^2}{\mu^2}\\ D_r=1+\frac{r^2}{\mu^2}\end{array},\qquad \Lambda=\frac{\Omega}{\mu}\prod\Lambda_k,\quad \Lambda_i=(a_i^2-\mu^2).$$

Maxwell's equations in the Myers–Perry geometry

OL '17

- Ansatz for the gauge field

$$l_\pm^\mu A_\mu = \pm \frac{1}{r \pm i\mu} \hat{l}_\pm \Psi, \quad [m_\pm^{(j)}]^\mu A_\mu = \mp \frac{i}{x_j \pm \mu} \hat{m}_\pm^{(j)} \Psi\,,$$

- Separation of variables and the "master equations"

$$D_j \frac{d}{dx}\left[\frac{H_j}{D_j} X_j'\right] + \left\{\frac{2\Lambda}{D_j} - H_j W_j^2 - \Lambda + P_{n-2}[-x_j^2] D_j\right\} X_j = 0,$$

$$D_r \frac{d}{dr}\left[\frac{\Delta}{D_r}\dot{\Phi}\right] - \left\{\frac{2\Lambda}{D_r} - \frac{R^2 W_r^2}{\Delta} - \Lambda + P_{n-2}[r^2] D_r\right\} \Phi = 0,$$

$$\Omega = \omega - \sum \frac{m_i a_i}{\Lambda_i}, \quad W_j = \omega - \sum \frac{m_k a_k}{a_k^2 - x_j^2}, \quad W_r = \omega - \sum \frac{m_k a_k}{a_k^2 + r^2}.$$

- Separation constants in $d = 2n+2$:
  $(n+1)$ angular mom/energy, $(n-1)$ parameters in P_{n-2}, μ.
- Odd–dimensional case requires a separate analysis. The results are similar.
- Summary: separable solutions for $(d-2)$ polarizations in all d

Higher forms in the Myers–Perry geometry

OL '19

- The most general separable ansatz for the p–form potential

$$m_{\alpha_1}^{(l_1)\mu_1}\dots m_{\alpha_p}^{(l_p)\mu_p}A_{\mu_1\dots\mu_p}=F_{l_1\dots l_p}\left[\prod h_{\alpha_k}^{l_k}\right]m_{\alpha_1}^{(l_1)\mu_1}\dots m_{\alpha_p}^{(l_p)\mu_p}\partial_{\mu_1}\dots\partial_{\mu_p}\Psi,$$

$$F_{l_1\dots l_p}=\left[\prod_{k<l}^{p}[x_{l_k}^2-x_{l_l}^2]\right],\quad h_{\pm}^{l}=\frac{1}{[1+e_2x_l^2\pm ie_3x_l]\prod_{k=1}^{p-2}[1+q_kx_l^2]}\,.$$

- Parameters q_k are solutions of the algebraic equation

$$\left[\prod_i(1+qa_i^2)\right]\left[\omega-\sum_i\frac{qa_in_i}{1+qa_i^2}\right]=0\,.$$

- Separation constants:
  (e_2,e_3), angular mom/energy, coeffs. of the separation polynomial P_{n-3}.

$$D_i\frac{d}{dx_i}\left[\frac{H_i}{D_i}X_i'\right]+\left[-W_i^2H_i-(e_1-e_2x_i^2)(G_i-\frac{G_*}{2})+D_iP_{n-3}[x_i^2]\right]X_i$$
$$+\frac{ie_3x_i^2}{e_2}X_i\frac{d}{dx_i}\left[\frac{x_i}{x_*^2-x_i^2}\left\{\frac{H_iW_i}{x_i^2}-\frac{H_*W_*}{x_*^2}\right\}\right]=0.$$

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$$F_{l_1\dots l_p}=\left[\prod_{k<l}^{p}[x_{l_k}^2-x_{l_l}^2]\right],\quad h_\pm^l=\frac{1}{[1+e_2x_l^2\pm ie_3x_l]\prod_{k=1}^{p-2}[1+q_kx_l^2]}\,.$$

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- Separation constants:
  (e_2,e_3), angular mom/energy, coeffs. of the separation polynomial P_{n-3}.
- The separation ansatz works for the MP geometries with Λ as well.
- Summary: separable solutions for all p–forms in all dimensions
- Similar ansatz seems to be working for gravitational waves OL, work in progress

Excitations of the WZW models

- (Gauged) WZW models are exact CFTs on coset spaces

$$S = -\frac{k}{2\pi}\int d^2\sigma \eta^{\alpha\beta} \mathrm{tr}(g^{-1}\partial_\alpha g g^{-1}\partial_\beta g) + \frac{ik}{6\pi}\int \mathrm{tr}(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg)\,.$$

Witten, '84

- Scalar field on WZW geometries
  - group manifolds: unique equation, several $U(1)$ isometries
  - coset models G/H: ambiguity in the coupling to the dilaton:

$$e^{2\sigma\phi}\nabla^2(e^{-2\sigma\phi}\Psi) = -\Lambda\Psi$$

  - worldsheet Hamiltonian leads to the spectrum for $\sigma = 1$:

$$\Lambda = 2\pi\left[\frac{C_2(R)}{k+g_G} - \frac{C_2(r)}{k+g_H}\right]$$

Polychronakos–Sfetsos '10

- Questions addressed in this work
  - what are the eigenfunctions?
  - is there a separation of variables at least for some G and G/H?
  - what is the spectrum of vector fields and higher forms?
  - what is the spectrum for $\sigma \neq 1$?
  - can the results be extended to integrable deformations of the WZW models?

Separation of variables

OL, Tian '20

- Simple examples: product spaces
  - group model: $G = SO(4) = SU(2) \times SU(2)$

$$ds^2_{SU(2)} = \frac{k}{2\pi}\left[\frac{dy^2}{1-y^2} + d\alpha^2 + d\beta^2 + 2y d\alpha d\beta\right].$$

  - coset model $\frac{SO(4)}{SO(2)\times SO(2)} = \frac{SU(2)}{U(1)} \times \frac{SU(2)}{U(1)}$: separation for all σ.

$$ds^2_{SU(2)/U(1)} = \frac{k}{2\pi}\left[\frac{1+y}{1-y}d\tau^2 + \frac{dy^2}{1-y^2}\right], \quad e^{-2\phi} = (1-y).$$

  - separation of vector fields: starting with explicit components, it was shown that the the most general separable modes are (constant $(b_\pm, b_3, \tilde{b}_\pm, \tilde{b}_3)$)

$$e^\mu_\pm A_\mu = b_\pm e^\mu_\pm \partial_\mu Z, \quad e^\mu_3 A_\mu = b_3 e^\mu_3 \partial_\mu Z, \quad \tilde{e}^\mu_\pm A_\mu = \tilde{b}_\pm \tilde{e}^\mu_\pm \partial_\mu Z, \quad \tilde{e}^\mu_3 A_\mu = \tilde{b}_3 \tilde{e}^\mu_3 \partial_\mu Z$$

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OL, Tian '20

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  - group model: $G = SO(4) = SU(2) \times SU(2)$
  - coset model $\frac{SO(4)}{SO(2)\times SO(2)} = \frac{SU(2)}{U(1)} \times \frac{SU(2)}{U(1)}$: separation for all σ.
  - separation of vector fields: starting with explicit components, it was shown that the the most general separable modes are (constant $(b_\pm, b_3, \tilde{b}_\pm, \tilde{b}_3)$)

$$e^\mu_\pm A_\mu = b_\pm e^\mu_\pm \partial_\mu Z, \quad e^\mu_3 A_\mu = b_3 e^\mu_3 \partial_\mu Z, \quad \tilde{e}^\mu_\pm A_\mu = \tilde{b}_\pm \tilde{e}^\mu_\pm \partial_\mu Z, \quad \tilde{e}^\mu_3 A_\mu = \tilde{b}_3 \tilde{e}^\mu_3 \partial_\mu Z$$

- Full separation in the $\frac{SO(4)}{SO(2)} = \frac{SU(2)\times SU(2)}{U(1)_{diag}}$ coset
  - non–cyclic coordinates $(y, \tilde{y})$ are mixed, nontrivial dilaton and B field
  - this coset is an intermediate step in the sequence of T dualities

$$SO(4) \to \frac{SU(2)\times SU(2)}{U(1)_{diag}} \to \frac{SO(4)}{SO(2)\times SO(2)}$$

  - scalar equation is separable only for $\sigma = 1$ (eqn invariant under T duality)

$$e^{2\sigma\phi}\nabla^2(e^{-2\sigma\phi}\Psi) = -\Lambda\Psi$$

  - vector equation is separable only for $\sigma = 1$, $\zeta = \pm 1$

$$e^{2\phi}\nabla_\mu\left[e^{-2\phi}\mathcal{F}^{\mu\nu}\right] + 4\Lambda A^\nu = 0, \quad \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \zeta H_{\mu\nu\sigma}A^\sigma$$

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OL, Tian '20

- Simple examples: product spaces
  - group model: $G = SO(4) = SU(2) \times SU(2)$
  - coset model $\frac{SO(4)}{SO(2)\times SO(2)} = \frac{SU(2)}{U(1)} \times \frac{SU(2)}{U(1)}$: separation for all σ.
  - separation of vector fields: starting with explicit components, it was shown that the the most general separable modes are (constant $(b_\pm, b_3, \tilde{b}_\pm, \tilde{b}_3)$)

$$e_\pm^\mu A_\mu = b_\pm e_\pm^\mu \partial_\mu Z, \quad e_3^\mu A_\mu = b_3 e_3^\mu \partial_\mu Z, \quad \tilde{e}_\pm^\mu A_\mu = \tilde{b}_\pm \tilde{e}_\pm^\mu \partial_\mu Z, \quad \tilde{e}_3^\mu A_\mu = \tilde{b}_3 \tilde{e}_3^\mu \partial_\mu Z$$

- Full separation in the $\frac{SO(4)}{SO(2)} = \frac{SU(2)\times SU(2)}{U(1)_{diag}}$ coset
  - non–cyclic coordinates $(y, \tilde{y})$ are mixed, nontrivial dilaton and B field
  - this coset is an intermediate step in the sequence of T dualities
  - scalar equation is separable only for $\sigma = 1$ (eqn invariant under T duality)
  - vector equation is separable only for $\sigma = 1$, $\zeta = \pm 1$

$$e^{2\phi}\nabla_\mu \left[e^{-2\phi}\mathcal{F}^{\mu\nu}\right] + 4\Lambda A^\nu = 0, \quad \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \zeta H_{\mu\nu\sigma}A^\sigma$$

  - T duality leads to a twisted KYT $(\zeta = \pm 1)$

Chervonyi–OL '15

$$\nabla_n^\pm Y_{mpq} + \nabla_m^\pm Y_{npq} = 0, \quad \Gamma_{np}^{d\pm} = \Gamma_{np}^d \pm \frac{1}{2} H^d{}_{np}$$

Separation of variables

OL, Tian '20

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  - separable solution is built from the eigenvectors of the twisted KYT

$$e^\mu_\pm A_\mu = b_\pm(y) e^\mu_{1\pm} \partial_\mu Z, \quad \tilde{e}^\mu_\pm A_\mu = \tilde{b}_\pm(\tilde{y}) \tilde{e}^\mu_{2\pm} \partial_\mu Z, \quad e^\mu_0 A_\mu = b_0 e^\mu_0 \partial_\mu Z$$

- Separations of vector equations in black hole and WZW geometries follow the same pattern.

Spectra of WZW models beyond separation of variables

- $SO(5)$: partial separation

OL, Tian '20

  - group model: $G_L \times G_R$ isometries: $g \to h_L g h_R$

$$S = -\frac{k}{2\pi}\int d^2\sigma \eta^{\alpha\beta}\mathrm{tr}(g^{-1}\partial_\alpha g g^{-1}\partial_\beta g) + \frac{ik}{6\pi}\int \mathrm{tr}(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg)\,.$$

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  - $SO(5)$ has rank 2 $\Rightarrow$ 4 commuting Killing vectors in 10d space.
  - full separation (9 independent KT) is not there.
  - several infinite families of separable solutions depending on 6 coordinates

$$
\begin{aligned}
\Psi &= \frac{e^{2i[n_1(\alpha_L+\beta_L)+n_3(\alpha_R+\beta_R)]}}{(1+R^2)^q}\frac{(1+y_-^2)^{q+1}}{y_-^{n_1-n_3}} \\
&\times \quad F[-k,k+2q+3;2+2q;\frac{1}{1+R^2}]\,F\left[q+1-n_1,q+1+n_3;1-n_1+n_3;-y_-^2\right]. \\
\Lambda &= 2q(2+q)+4k(2q+3+k),\quad y_\pm=\frac{Y_1\pm Y_2}{1\mp Y_1Y_2}.
\end{aligned}
$$

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  - recall the scalar spectrum for $\sigma = 1$

$$e^{2\sigma\phi}\nabla^2(e^{-2\sigma\phi}\Psi) = -\Lambda\Psi, \quad \Lambda = 2\pi\left[\frac{C_2(R)}{k+g_G} - \frac{C_2(r)}{k+g_H}\right]$$

Polychronakos–Sfetsos '10

  - eigenfunctions are built by symmetrizing a product of g according to (R, r)

$$\Psi = \sum_P (-1)^{\sigma(P)} g_{i_1 j_{P[1]}} \dots g_{i_L j_{P[L]}} - (traces)$$

  - for $SO(6)/SO(5)$ and $SO(4)/SO(3)$ some polyn. structures were uncovered

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$$\boxed{1}: \quad \Psi^{[10]}_{[0]} = \frac{2}{Y^2} - 1,$$

$$\boxed{}: \quad \Psi^{[10]}_{[1]} = \frac{2}{Y^2}(2x+1) - 3 + 4\alpha\,, \qquad \begin{bmatrix} 12 & -4\kappa \\ -12\kappa & 4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \Lambda \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

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  - extensions to λ–deformed backgrounds were performed
- Scalar and vector spectra for general σ

  OL, Tian, to appear
  - group theory arguments don't work unless $\sigma = 1$
  - insights from $\sigma = 1$ wavefunctions: a simple limit of the geometry
  - full scalar spectrum for $SO(4)/SO(3)$ from the limit of PDEs:

$$\Lambda = \frac{2\pi}{k}\left[L_1(L_1 + 2\sigma) + L_2^2 - l_1(l_1 + \sigma)\right]$$

  - similar expressions for the vector field and for other cosets

Summary

- Motivation
  - solving eom for various fields in BH and WZW backgrounds
  - understanding the role of symmetries
- New approach to the Maxwell field in 4D
  - traditional method: simple "master equation", but it is hard to recover A_μ
  - construction is based on self–duality $\Rightarrow$ hard to extend to $D > 4$
  - new separable ansatz: gauge field is recovered algebraically
- Excitations of the Myers–Perry geometry
  - ansatze for all forms are based on eigenvectors of the KYT
  - universal separable equations for a scalar and the "master fields"
  - all polarizations with an appropriate number of separation constants
- Excitations of the WZW models
  - full or partial separation in low–dimensional cases
  - new examples of geometries with nontrivial KYT
  - separation of vector equation is similar to the BH case
  - algebraic construction of eigenvalues and eigenvectors for $SO(n+1)/SO(n)$
- Light modes probe dynamics of stringy geometries