Disforming the Kerr metric

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- TA, E. Babichev, C. Charmousis and M. Hassaine, *Disforming the Kerr metric*, *JHEP* **01** (2021) 018
- TA, E. Babichev and C. Charmousis, *Deformed black hole in Sagittarius A*, *Phys. Rev. D* **103** (2021) 124035

Introduction

- There is growing evidence that black holes exist in Nature: observation of stars around Sgr A* **[GRAVITY]**; detection of gravitational waves **[LIGO/VIRGO, 2015,...]**; imaging of M87* **[EHT, 2019]**
- In general relativity, rotating black holes are described by the Kerr metric
- It is interesting to construct deformations of the Kerr spacetime, in order to test general relativity and find signatures of modified gravity
- Usually, *ad hoc* deformations of the Kerr spacetime are constructed **[Psaltis+, 2011; Johannsen, 2013; Papadopoulos+, 2018; ...]**
- Using the disformal map, one can construct deformed versions of the Kerr spacetime which are solutions to higher-order scalar-tensor theories

Scalar-tensor theories

- Lovelock's theorem: GR is unique in 4 dimensions for a theory constructed with 1 metric tensor and up to its 2nd derivatives
- The simplest extension is to consider an additional scalar field ϕ
- Scalar-tensor theories have been generalized over the years: Jordan-Brans-Dicke **[1959, 1961]**, Horndeski **[1974]**, Degenerate Higher Order Scalar-Tensor (DHOST) theories **[Langlois, Noui, 2015; Crisostomi+, 2016]**
- Higher derivative theories generically lead to unphysical degree of freedom (Ostrogradsky ghost). The DHOST class is free of ghosts thanks to degeneracy conditions
- The quadratic DHOST Lagrangian density is of the form (with $X = (\partial\phi)^2$, $\phi_\mu = \partial_\mu \phi$, $\phi_{\mu\nu} = \nabla_\mu \nabla_\nu \phi$)

$$\mathcal{L} = f(\phi, X)R + K(\phi, X) - G_3(\phi, X)\Box\phi + A_1(\phi, X)\,\phi_{\mu\nu}\phi^{\mu\nu} + A_2(\phi, X)\,(\Box\phi)^2 + A_3(\phi, X)\,\phi_{\mu\nu}\phi^\mu\phi^\nu\Box\phi + A_4(\phi, X)\,\phi_{\mu\alpha}\phi^{\alpha\nu}\phi^\mu\phi_\nu + A_5(\phi, X)\,(\phi_{\mu\nu}\phi^\mu\phi^\nu)^2$$

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Rotating black holes in GR

Kerr solution

- Vacuum solution of GR describing a rotating black hole **[Kerr, 1963]**. The metric g verifies $R_{\mu\nu} = 0$.
- In Boyer-Lindquist coordinates, the metric tensor is:

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4aMr\sin^2\theta}{\rho^2} dt d\varphi + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] d\varphi^2$$
$$+ \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

where M is the mass, a is the angular momentum per unit mass, and

$$\rho^2 = r^2 + a^2\cos^2\theta\ ,$$
$$\Delta = r^2 + a^2 - 2Mr\ .$$

- $R_{\mu\nu\alpha\sigma}R^{\mu\nu\alpha\sigma}$ is singular at $\rho = \sqrt{r^2 + a^2\cos^2\theta} = 0$, so there is a ring singularity at

$$r = 0 \qquad \text{and} \qquad \theta = \frac{\pi}{2}$$

Symmetries and circularity

- The metric is stationary and axisymmetric, which corresponds to 2 Killing directions

$$\xi_{(t)} = \partial_t \qquad \text{and} \qquad \xi_{(\varphi)} = \partial_\varphi$$

- The spacetime is circular, i.e. symmetric under the reflection $(t, \varphi) \to (-t, -\varphi)$, because the Killing fields verify the condition

$$\xi_{(t)} \wedge \xi_{(\varphi)} \wedge \mathrm{d}\xi_{(t)} = \xi_{(t)} \wedge \xi_{(\varphi)} \wedge \mathrm{d}\xi_{(\varphi)} = 0 \ .$$

- The Kerr spacetime also admits a nontrivial Killing 2-tensor K verifying the equation

$$\nabla_{(\mu} K_{\nu\sigma)} = 0 \ .$$

- This defines a third nontrivial constant of motion along geodesics (Carter's constant). The geodesic equations thus reduce to a first order system.

Stationary observers

- Consider constant (r, θ) observers, with a 4-velocity

$$u = \partial_t + \omega \partial_\varphi$$

- The condition $u^2 \leq 0$ implies $\omega \in [\omega_-, \omega_+]$, where

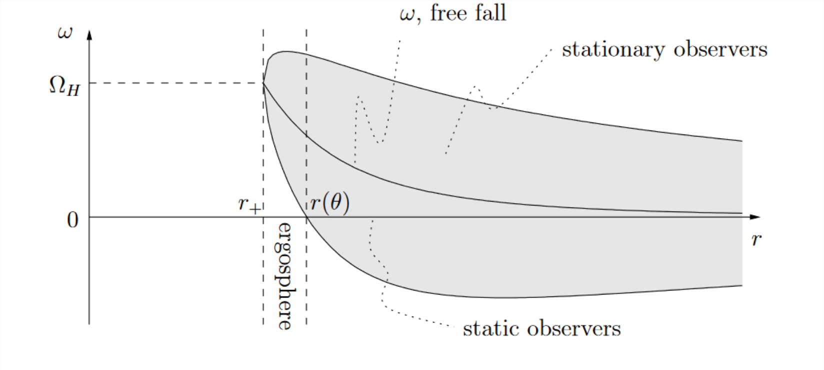
$$\omega_\pm = \frac{|g_{t\varphi}|}{g_{\varphi\varphi}} \left(1 \pm \sqrt{1 - \frac{g_{tt} g_{\varphi\varphi}}{g_{t\varphi}^2}} \right)$$

- Inside the ergosphere, where $g_{tt} > 0$, one necessarily has $\omega_- > 0$
- This surface is defined by $g_{tt} = 0$, which implies

$$r_E = M + \sqrt{M^2 - a^2 \cos^2 \theta}$$

- These observers stop to exist at the outer event horizon when $g_{tt} g_{\varphi\varphi} - g_{t\varphi}^2 = 0$, at the radius

$$r_+ = M + \sqrt{M^2 - a^2}$$



[Graf, GR lecture notes]

Killing horizon

- Rigidity theoreom (Hawking): The event horizon $\mathcal{H}$ of a real analytic, stationary, regular, vacuum spacetime is a Killing horizon: $\exists$ a Killing field k normal to $\mathcal{H}$ which verifies $k^2 = 0$ on $\mathcal{H}$.
- For the outer horizon of the Kerr spacetime, this Killing vector is

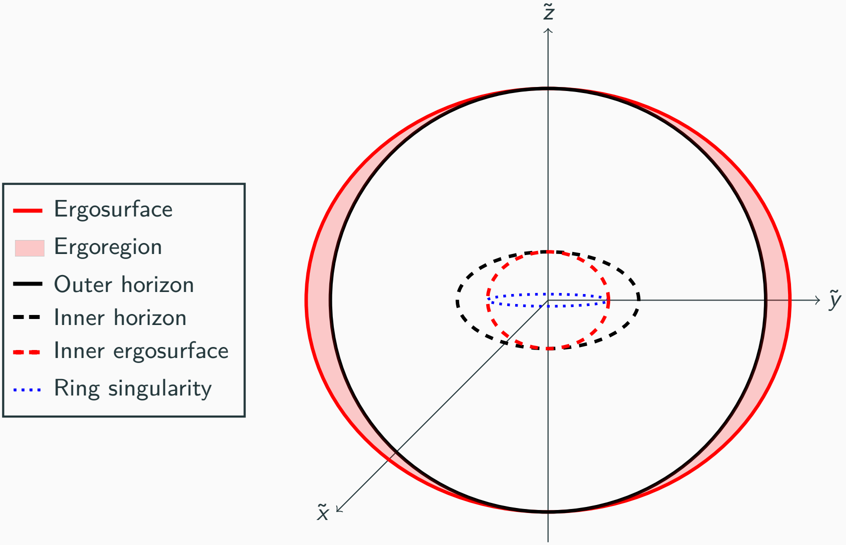
$$k = \partial_t + \frac{a}{2Mr_+}\partial_\varphi$$

- One can define the surface gravity κ_+ of $\mathcal{H}$ as

$$k^\mu \nabla_\mu k^\nu = \kappa_+ k^\nu$$

- The surface gravity is constant on $\mathcal{H}$ and is related to the Hawking temperature $T_H = \kappa_+/2\pi$

Important hypersurfaces in the Kerr spacetime



Construction and properties of the disformed Kerr metric

Stealth-Kerr solution in higher-order gravity

$$\mathcal{L} = f(\phi, X)R + K(\phi, X) - G_3(\phi, X)\Box\phi + A_1(\phi, X)\,\phi_{\mu\nu}\phi^{\mu\nu} + A_2(\phi, X)\,(\Box\phi)^2$$
$$+ A_3(\phi, X)\,\phi_{\mu\nu}\phi^{\mu}\phi^{\nu}\Box\phi + A_4(\phi, X)\,\phi_{\mu\alpha}\phi^{\alpha\nu}\phi^{\mu}\phi_{\nu} + A_5(\phi, X)\,\left(\phi_{\mu\nu}\phi^{\mu}\phi^{\nu}\right)^2$$

$+$ degeneracy conditions on the A_i

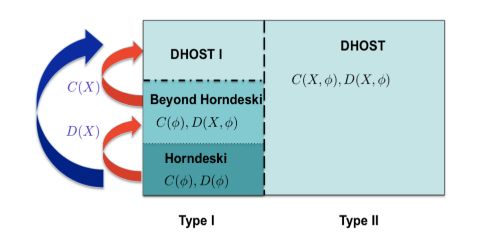
- A stealth-Kerr solution was constructed **[Charmousis+, 2019]**, where the scalar field is the Hamilton-Jacobi potential of the Kerr spacetime

$$g = g_{\text{Kerr}}$$
$$\phi = -Et + L_z\varphi \pm \int \frac{\sqrt{\mathcal{R}(r)}}{\Delta}\mathrm{d}r \pm \int \sqrt{\Theta(\theta)}\mathrm{d}\theta$$

- The scalar defines a geodesic direction because

$$\nabla_\nu\left(\nabla^\mu\phi\nabla_\mu\phi\right) = 0 \Rightarrow \nabla^\mu\phi\nabla_\mu\nabla_\nu\phi = 0$$

Stability of the DHOST Ia class under the disformal map



[Langlois, 2018]

- The Ia subclass can be obtained from Horndeski theories by a disformal transformation of the metric **[Ben Achour+; Crisostomi+, 2016; ...]**:

$$\tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + D(\phi, X) \partial_\mu \phi \partial_\nu \phi$$

- The theories are different because of the matter coupling:

$$\tilde{S}[\tilde{g}_{\mu\nu}, \phi] + S_{\mathrm{m}}[\tilde{g}_{\mu\nu}, \Psi_{\mathrm{m}}] \xrightarrow{\text{DISFORMAL}} S[g_{\mu\nu}, \phi] + S_{\mathrm{m}}[g_{\mu\nu}, \Psi_{\mathrm{m}}]$$

Disformed Kerr metric

$$\tilde{g}_{\mu\nu} = g^{\mathrm{K}}_{\mu\nu} - \frac{D}{q^2}\,\partial_\mu\phi\,\partial_\nu\phi\,,$$

$$\phi = q\left[t + \int \frac{\sqrt{2Mr(a^2+r^2)}}{\Delta}\mathrm{d}r\right]\,.$$

- We start from the Kerr solution g^{K}, and perform a disformal transformation

$$\mathrm{d}\tilde{s}^2 = -\left(1 - \frac{2\tilde{M}r}{\rho^2}\right)\mathrm{d}t^2 + \frac{\rho^2\Delta - 2\tilde{M}rD(1+D)(a^2+r^2)}{\Delta^2}\mathrm{d}r^2 - \frac{4\sqrt{1+D}\tilde{M}ar\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\varphi$$
$$+\frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right]\mathrm{d}\varphi^2 + \rho^2\mathrm{d}\theta^2 - 2D\frac{\sqrt{2\tilde{M}r(a^2+r^2)}}{\Delta}\mathrm{d}t\mathrm{d}r$$

with $\tilde{M} = M/(1+D)$ and the rescaling $t \to \sqrt{1+D}t$

- a is the black hole spin, $\Delta = r^2 + a^2 - 2Mr$, $\rho^2 = r^2 + a^2\cos^2\theta$
- If $a = 0$, we recover the Schwarzschild metric with mass $\tilde{M}$ **[Babichev+, 2017]**

Regular solution

- The disformed metric has the following curvature scalars

$$\tilde{R} = -\frac{Da^2 Mr[1 + 3\cos(2\theta)]}{(1+D)\rho^6}, \quad \tilde{R}_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta} = \frac{M^2 Q_2(r,\theta)}{\rho^{12}(r^2+a^2)(1+D)^2}, \ldots$$

- The solution is not Ricci-flat, but the only singularity is at $\rho = 0$, like Kerr. To verify this, one changes coordinates to

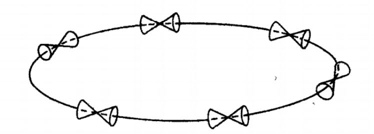
$$t \to v - r - \int \frac{2Mr}{\Delta}\mathrm{d}r\,, \qquad \varphi \to -\Phi - a\int \frac{\mathrm{d}r}{\Delta}$$

- The metric components are regular in these coordinates, and the scalar field reads

$$\phi = q_0\left(v - r + \int \frac{\mathrm{d}r}{1+\sqrt{\frac{r^2+a^2}{2Mr}}}\right)$$

Stably causal spacetime

- There can exist closed timelike curves in a spacetime, even in GR (Kerr with $a > M$ for example). We want to avoid such pathologies.



[Wald's book]

Theorem: A spacetime $(M_0, g_{\mu\nu})$ is stably causal if and only if there exists a differentiable function f on M_0 such that $\nabla^\mu f$ is a future (past) directed timelike vector field

- We have such a function by construction, the scalar field ϕ itself. It serves as a global time.
- The spacetime is globally causal if the region $r > 0$ is causally disconnected from the region $r < 0$ (where CTCs are present even for Kerr)
- Some of the *ad hoc* deformations of Kerr proposed in the past contain such pathologies **[Johannsen, 2013]**

Properties of the disformed metric

- We still have axisymmetry (two commuting Killing vectors ∂_t and ∂_φ)
- However, defining $\xi_{(t)} = \tilde{g}_{t\mu}\mathrm{d}x^\mu$ and $\xi_{(\varphi)} = \tilde{g}_{\varphi\mu}\mathrm{d}x^\mu$, we now have

$$\xi_{(t)} \wedge \xi_{(\varphi)} \wedge \mathrm{d}\xi_{(t)} = -D\frac{4a^2\tilde{M}r\sqrt{2\tilde{M}r(a^2+r^2)}\cos\theta\sin^3\theta}{\rho^4}\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\varphi$$

- This noncircularity means we cannot write the metric in a form that is invariant under the reflection $(t,\varphi) \to (-t,-\varphi)$
- It also has an impact on the separability structure of the spacetime, and we no longer have a nontrivial Killing tensor **[Benenti+, 1979,1980]**
- Interesting because noncircular spacetimes can exist even in GR (for instance in the presence of toroidal magnetic fields), and the circular ansatz fails in certain situations **[Van Aelst+, 2019]**

Asymptotically similar to Kerr

- Asymptotically, the Kerr metric can be written

$$\mathrm{d}s^2_{\mathrm{Kerr}} = -\left[1 - \frac{2\tilde{M}}{r} + \mathcal{O}\left(\frac{1}{r^3}\right)\right]\mathrm{d}T^2 - \left[\frac{4\tilde{a}\tilde{M}}{r^3} + \mathcal{O}\left(\frac{1}{r^5}\right)\right][x\mathrm{d}y - y\mathrm{d}x]\,\mathrm{d}T + \left[1 + \mathcal{O}\left(\frac{1}{r}\right)\right]\left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right]$$

- The physical parameters determined from the asymptotic expansion are

$$\tilde{M} = \frac{M}{1+D}\,, \qquad \tilde{a} = a\sqrt{1+D}$$

- After a coordinate transformation, one can write the disformal metric as

$$\mathrm{d}\tilde{s}^2 = \mathrm{d}s^2_{\mathrm{Kerr}} + \frac{D}{1+D}\left[\mathcal{O}\left(\frac{\tilde{a}^2\tilde{M}}{r^3}\right)\mathrm{d}T^2 + \mathcal{O}\left(\frac{\tilde{a}^2\tilde{M}^{3/2}}{r^{7/2}}\right)\alpha_i\mathrm{d}T\mathrm{d}x^i + \mathcal{O}\left(\frac{\tilde{a}^2}{r^2}\right)\beta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]$$

with $\alpha_i, \beta_{ij} \sim \mathcal{O}(1)$.

Stationary observers

- Consider constant (r, θ) observers, with a 4-velocity

$$u = \partial_t + \omega \partial_\varphi$$

- The condition $u^2 \leq 0$ implies $\omega \in [\omega_-, \omega_+]$, where

$$\omega_\pm = \frac{1}{\tilde{g}_{\varphi\varphi}} \left(-\tilde{g}_{t\varphi} \pm \sqrt{\tilde{g}_{t\varphi}^2 - \tilde{g}_{tt}\tilde{g}_{\varphi\varphi}} \right)$$

- Inside the static limit defined by $\tilde{g}_{tt} = 0$, one necessarily has $\omega_- > 0$
- These observers no longer exist when $\tilde{g}_{t\varphi}^2 - \tilde{g}_{tt}\tilde{g}_{\varphi\varphi} = 0$, which happens when

$$P(r, \theta) \equiv r^2 + a^2 - 2\tilde{M}r + \frac{2\tilde{M}Da^2 r \sin^2\theta}{\rho^2(r, \theta)} = 0$$

- The outermost surface $r = R_0(\theta)$ which satisfies $P(R_0(\theta), \theta) = 0$ is called the stationary limit

Nature of the stationary limit

$$P(r,\theta) \equiv r^2 + a^2 - 2\tilde{M}r + \frac{2\tilde{M}Da^2 r \sin^2\theta}{\rho^2(r,\theta)} = 0$$

- When $D = 0$, the stationary limit coincides with the event horizon
- In the general case, the normal vector N to this surface is

$$N_\mu = \left(0, 1, -R_0'(\theta), 0\right)$$

- One can check that

$$N^2|_{r=R_0} = \tilde{g}^{rr} + \tilde{g}^{\theta\theta} R_0'^2 > 0$$

- Hence the surface is timelike and cannot be the event horizon in the general case
- All Killing vectors of the form $\partial_t + \omega\partial_\varphi$ are spacelike inside this surface, so if there is an event horizon, it cannot be a Killing horizon

Event horizon ?

- For Kerr, the horizons are found by solving $g^{rr} = 0 \implies \Delta = 0$ which admits constant r solutions. In our case, we have $\tilde{g}^{rr} = 0 \implies P = 0$, which doesn't admit constant r solutions when $D \neq 0$
- We look for more general null hypersurface of the form $r = R(\theta)$. The normal has components

$$n_\mu = \big(0, 1, -R'(\theta), 0\big)$$

- The condition $n^2 = 0$ yields

$$R'(\theta)^2 + P(R,\theta) = R'(\theta)^2 + R^2 + a^2 - 2\tilde{M}R + \frac{2\tilde{M}Da^2 R\sin^2\theta}{\rho^2(R,\theta)} = 0$$

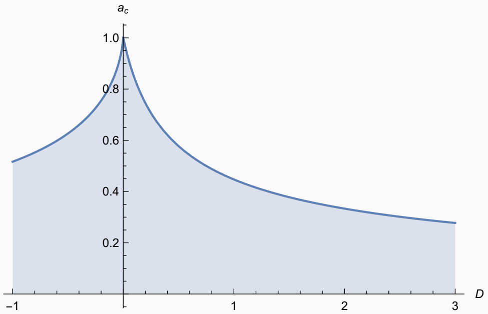
- To have a smooth solution, we must have

$$R'(0) = R'(\frac{\pi}{2}) = 0$$

Bounds on the rotation parameter

- After imposing $R'(\frac{\pi}{2}) = 0$, an expansion around $\theta = \frac{\pi}{2}$ yields a necessary condition to have $R''(\frac{\pi}{2}) \in \mathbb{R}$ (and similar arguments at $\theta = 0$).
- In units where $\tilde{M} = 1$, one must have $a < a_c$, where

$$Q_4(a_c^2) = 0, \qquad D < 0$$
$$a_c = \frac{1}{\sqrt{1+4D}}, \qquad D > 0$$



Event horizon ?

- In the Kerr spacetime, the event horizon is located at $r = R_+$. By considering the hypersurfaces $r = R_+ + \zeta$, one can show that these surfaces are timelike outside the event horizon, and become spacelike between the horizons
- Similarly, we introduce the family of surfaces

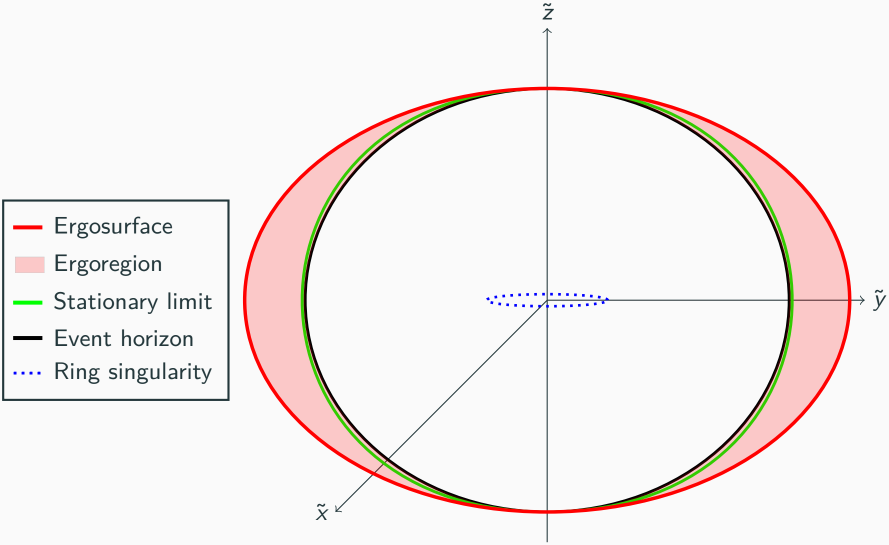
$$R_\zeta(\theta) = R(\theta) + \zeta$$

- Under the assumption $R(\theta) \geq \tilde{M}$, one can show that $\exists\, \zeta_0$ such that the surfaces $r = R_\zeta(\theta)$ are

$$\begin{aligned} &\text{timelike for} \qquad \zeta > 0 \\ &\text{spacelike for} \qquad \zeta_0 < \zeta < 0 \end{aligned}$$

- These correspond to coordinates adapted to the horizon, in which the horizon is located at $\zeta = 0$

Hypersurfaces in the disformed Kerr spacetime



Limit $D \to \infty$: noncircular Schwarzschild metric

- In the limit $D \to \infty$, one can obtain a simpler line element. After a coordinate transformation, it reads with $\tilde{\chi} = \tilde{a}/\tilde{M}$

$$\mathrm{d}\tilde{s}^2_{\mathrm{NCS}} = -\left(1 - \frac{2\tilde{M}}{r}\right)\left(\mathrm{d}T + \frac{2\tilde{\chi}\tilde{M}^2 \sin^2\theta}{r - 2\tilde{M}}\mathrm{d}\varphi\right)^2 + \left(1 - \frac{2\tilde{M}}{r}\right)^{-1}\left(\mathrm{d}r - \sqrt{\frac{2\tilde{M}^3}{r}}\tilde{\chi}\sin^2\theta\mathrm{d}\varphi\right)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)$$

- In the Newman-Penrose formalism, one can construct quantities that are independent of the tetrad choice with the Weyl scalars ψ_i

$$\mathcal{I} = \psi_0\psi_4 - 4\psi_1\psi_3 + 3\psi_2^2\,, \qquad \mathcal{J} = \psi_0\psi_2\psi_4 - \psi_1^2\psi_4 - \psi_0\psi_3^2 + 2\psi_1\psi_2\psi_3 - \psi_2^3\,.$$

- The metric is of general Petrov type I (type D when $\tilde{\chi} = 0$)

$$\mathcal{S} = \frac{27\mathcal{J}^2}{\mathcal{I}^3} = 1 - \frac{3\tilde{M}^4\tilde{\chi}^4\sin^4\theta}{4r^4} + \mathcal{O}\left(\tilde{\chi}^5\right)$$

Limit $D \to -1$: quasi-Weyl metric

- In the limit $D \to -1$, we have

$$\mathrm{d}\tilde{s}^2_{\mathrm{QW}} = -\left(1 - \frac{2\tilde{M}r}{r^2 + a^2\cos^2\theta}\right)\mathrm{d}t^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\mathrm{d}r^2 + 2\sqrt{\frac{2\tilde{M}r}{r^2 + a^2}}\mathrm{d}t\mathrm{d}r + \left(r^2 + a^2\cos^2\theta\right)\mathrm{d}\theta^2 + \left(r^2 + a^2\right)\sin^2\theta\mathrm{d}\varphi^2$$

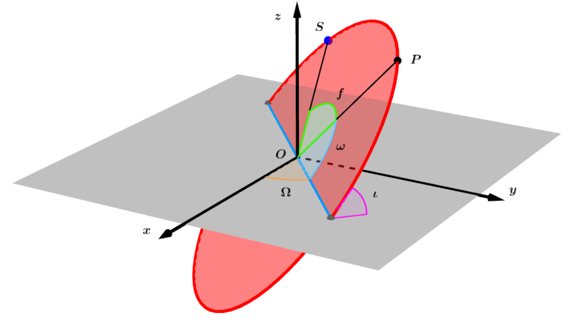
- It is *a priori* a singular limit, since after the redefinition $t \to t/\sqrt{1+D}$ we have

$$\phi = \frac{q_0}{\sqrt{1+D}}\left[t + (1+D)\int \frac{\sqrt{2\tilde{M}r(a^2+r^2)}}{\Delta}\mathrm{d}r\right] .$$

- After the field redefinition $\psi = \sqrt{1+D}\phi/q_0$, the scalar is simply $\psi = t$
- In both limits, one can isolate the corresponding DHOST theories
- These 2 simpler examples could be useful in understanding the properties of noncircular spacetimes

Stars orbiting a disformed black hole

Orbit of stars around Sgr A*



- We study the motion of stars around a deformed Kerr black hole using the osculating orbit method (the orbital elements $\{\Omega, \iota, \omega, p, e\}$ evolve in time)
- We calculate the secular variation of orbital elements up to 2PN order
- The pericenter precession ($\Delta\omega$) for the star S2 around Sagittarius A* has been measured **[GRAVITY, 2020]**
- In the future, smaller effects ($\Delta\Omega, \Delta\iota$) will be probed and will provide a measurement of J and Q, the spin and quadrupole moment of the black hole, providing a test of the no-hair theorem in GR: $Q = -Ma^2$

- Using the osculating orbit method, we calculate the secular variation of orbital elements up to the second post-Newtonian order
- For the disformed Kerr metrics in the generic case, we obtain the following expressions at 2PN order, with $\alpha = e\cos\omega$ and $\beta = e\sin\omega$

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}u} = 0\,,$$

$$\frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}u} = -\frac{3\tilde{M}\bar{\beta}}{\bar{p}} + 6\tilde{\chi}\bar{\beta}\cos\bar{\iota}\left(\frac{\tilde{M}}{\bar{p}}\right)^{3/2} + \frac{3\tilde{M}^2\bar{\beta}}{4\bar{p}^2}\left(10-\bar{\alpha}^2-\bar{\beta}^2\right) - \frac{3\tilde{M}^2\bar{\beta}\tilde{\chi}^2(5\cos^2\bar{\iota}-1)}{4\bar{p}^2(1+D)}\,,$$

$$\frac{\mathrm{d}\bar{\beta}}{\mathrm{d}u} = \frac{3\tilde{M}\bar{\alpha}}{\bar{p}} - 6\tilde{\chi}\bar{\alpha}\cos\bar{\iota}\left(\frac{\tilde{M}}{\bar{p}}\right)^{3/2} - \frac{3\tilde{M}^2\bar{\alpha}}{4\bar{p}^2}\left(10-\bar{\alpha}^2-\bar{\beta}^2\right) + \frac{3\tilde{M}^2\bar{\alpha}\tilde{\chi}^2(5\cos^2\bar{\iota}-1)}{4\bar{p}^2(1+D)}\,,$$

$$\frac{\mathrm{d}\bar{\iota}}{\mathrm{d}u} = 0\,,$$

$$\frac{\mathrm{d}\bar{\Omega}}{\mathrm{d}u} = 2\tilde{\chi}\left(\frac{\tilde{M}}{\bar{p}}\right)^{3/2} - \frac{3\tilde{M}^2\tilde{\chi}^2\cos\bar{\iota}}{2\bar{p}^2(1+D)}\,.$$

Secular variation of orbital elements

- The dimensionless quadrupole in the disformal case reads ($\tilde{\chi} = \tilde{a}/\tilde{M}$)

$$q^{(D)} = -\frac{\tilde{\chi}^2}{1+D}$$

- The no-hair theorem is violated for $D \neq 0$
- To maximize the effects of disformality, we define $\varepsilon = \tilde{M}/A$, where A is the semimajor axis, and assume the disformal parameter reads

$$D = -1 + \frac{\tilde{\chi}^2}{\lambda}\varepsilon\,, \qquad \{\lambda, \tilde{\chi}\} \sim \mathcal{O}(1)$$

- We expect 1PN terms to be modified in this case, and indeed we obtain

$$\Delta\bar{\varpi} \equiv \Delta\bar{\omega} + \cos\bar{\iota}\,\Delta\bar{\Omega} = \frac{6\pi\tilde{M}}{\bar{p}}\left[1 + \frac{\lambda}{4(1-\bar{e}^2)}\left(3\cos^2\bar{\iota} - 1\right)\right] + \mathcal{O}\left(\varepsilon^{3/2}\right)$$

- We obtain a signature of modified gravity, and a way to constrain the disformal parameter D using future observations

Lower bound on D with maximal disformality

- Using the previous expression for the orbit of S2 around Sgr A*, with $\varepsilon_0 = \tilde{M}/A_0$, the GRAVITY constraint implies

$$\left| \frac{\lambda \left(3\cos^2\bar{\iota} - 1\right)}{4(1-\bar{e}^2)} \right| \lesssim 0.2$$

- If we replace the eccentricity of $S2$ and assume $|3\cos^2\bar{\iota} - 1| \sim 1$, the inequality is saturated for $\lambda_0 \sim 0.2$. To maximize the effects of disformality, we take

$$D_0 = -1 + \frac{\tilde{\chi}^2 \varepsilon_0}{\lambda_0}$$

- For another star with $\varepsilon \neq \varepsilon_0$, we have

$$\Delta\bar{\varpi} = \frac{6\pi\tilde{M}}{\bar{p}}\left[1 + \frac{\varepsilon\lambda_0}{\varepsilon_0}\frac{\left(3\cos^2\bar{\iota} - 1\right)}{4(1-\bar{e}^2)}\right].$$

- This is valid for $\varepsilon^2 \lesssim 10^{-3} \lesssim \sqrt{\varepsilon}$

Conclusion

- We constructed deformations of the Kerr spacetime using the disformal transformation. While asymptotically very similar to Kerr, the solution presents many interesting properties: noncircularity, horizon not located at constant r and not a Killing horizon, the stationary limit is distinct from the event horizon
- We have calculated the secular variation of orbital parameters for stars around a deformed black hole, and shown that the no-hair theorem of GR is violated in general for these spacetimes. The 1PN corrections are modified if $D \sim -1$
- Other papers have studied some aspects of these solutions: the particular DHOST theories that these objects are a solution of **[Achour+, 2020]**; shadows of this black hole **[Long+, 2020]**; testing noncircularity with pulsars orbiting Sgr A* **[Takamori+, 2021]**; ...

Thank you for your attention.