Towards Virasoro-Shapiro on $AdS_5 \times S_5$.

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based on: 2012.2091 w/ Theresa Abl and Arthur Lipstein

Much progress!
holographic bootstrap.

- Focus on IIB on $AdS_5 \times S^5 \longleftrightarrow N=4$ SYM
- What can we learn from recent results on AdS side?
- Specifically: - Tree-level string corrections recently bootstrapped [Goncalves '14 ; Alday, Bissi, Perlmutter '18 Drummond, Nandan, Paul, Rigatos '19; Drummond, Paul, Santagata '20; Aprile, Viera '20]
  - Tree-level SUGRA found to have hidden 10d conformal symmetry. [Caron-Huot, Trinh '18]
- Find, 10d structure in CFT correlators; scalar superpotential on $AdS_5 \times S^5$.

Outline:

1.) Review basics of AdS/CFT. $\boxed{\text{scalars in AdS}}$, then → $\boxed{\text{10d IIB} \leftrightarrow \mathcal{N}=4 \text{ SYM}}$

2.) $\boxed{AdS \times S}$ /CFT : Tree-level string corrections from a scalar effective potential.

3.) Some techical details: - Covariant derivatives
- Mellin amplitude
- SUGRA

4.) Summary + conclusions

1.) Basics of AdS/CFT. generic.

$\boxed{\text{states in AdS} \longleftrightarrow \text{Operators in CFT}}$

eg scalar field — $\boxed{\varphi_\Delta(x) \longleftrightarrow O_\Delta(x)}$ Δ = dimension

mass $= \Delta(\Delta - d)$ scalar operator

$\boxed{\text{Amplitudes in AdS} \longleftrightarrow \text{Correlator in CFT.}}$

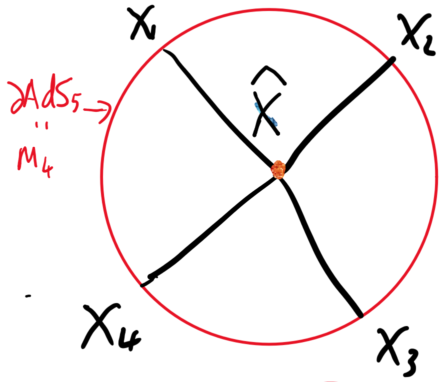
eg. φ^4 Term in effective action on AdS:

$$\boxed{\int d^5\hat{x}\; \varphi_{\Delta_1}(\hat{x})\, \varphi_{\Delta_2}(\hat{x})\, \varphi_{\Delta_3}(\hat{x})\, \varphi_{\Delta_4}(\hat{x})}$$

scalars mass $= \Delta_i(d - \Delta_i)$

correlator of ops

amplitude (Witten diagram)



$$= \int \frac{d^5\hat{X}}{(\hat{X}\cdot X_1)^{\Delta_1}\cdots(\hat{X}\cdot X_4)^{\Delta_4}} = \langle O_{\Delta_1}(x_1) O_{\Delta_2}(x_2) O_{\Delta_3}(x_3) O_{\Delta_4}(x_4)\rangle$$

correlator of ops corresponding to φ_{Δ_i}

$$\parallel$$

$$\boxed{D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1, x_2, x_3, x_4)}$$

"D-function"

Just replace $\boxed{\varphi_\Delta(\hat{X}) \to \frac{1}{(X_i\cdot\hat{X})^\Delta}}$

Bulk to boundary propagator

use embedding co-ords, $\hat{X}^{A=-1,0,\ldots,4}$, X_i^A

6d embedding space.

$\boxed{\hat{X}\cdot\hat{X} = -R^2}$ (AdS_5)

$\boxed{X_i \cdot X_i = 0}$ (Boundary Minkowski_4)

$$\eta^{AB} = \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

- Contact AdS Witten diagrams have Mellin transform:

$$D^{(d)}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i) = \mathcal{N}^{AdS_{d+1}}_{\Delta_i} \times \int \frac{d\delta_{ij}}{(2\pi i)^2} \prod_{i<j} \frac{\Gamma(\delta_{ij})}{(X_i.X_j)^{\delta_{ij}}}\, (M)$$

$\delta_{ij} = \delta_{ji}$, $\delta_{ii} = 0$

$\sum_i \delta_{ij} = \Delta_j$ — only 2 independent integration variables

Mellin amplitude: $\boxed{M=1}$

$\delta_{12} = -s$, $\delta_{14} = -t$

$\delta_{13} = -u = s + t + \Delta$, etc.

$\boxed{\text{Mellin amplitude}}$ from $\varphi^4 = 1$

(analogue of flat space amplitudes)

- Derivative interactions?

- Derivative interactions:

$$\boxed{\int \nabla\varphi_{\Delta_1}\cdot\nabla\varphi_{\Delta_2}\,\nabla\varphi_{\Delta_3}\cdot\nabla\varphi_{\Delta_4}\, d^5\hat{X}} \rightarrow \boxed{D_{\Delta_1+1,\Delta_2+1,\Delta_3+1,\Delta_4+1}\left(X_{12}^2X_{34}^2+X_{13}^2X_{24}^2+X_{14}^2X_{23}^2\right) + \cdots}$$

AdS$_5$ covariant derivative

$\downarrow$

Mellin amplitude quadratic in δ_{ij} (or s,t).

$$\boxed{M = s^2+t^2+u^2+\cdots}$$

$$\boxed{\begin{array}{lcl}\text{2k-derivative interaction} & = & \text{degree } k \text{ polynomial Mellin amplitude.}\end{array}}$$

Large s,t $\Rightarrow$ Flat space amplitude

Specific theory.

Most symmetric 4d QFT

IIB string theory on $AdS_5 \times S_5$ $\longleftrightarrow$ $N=4$ SYM $(d=4)$ $SU(N)$

Parameters.

$$g_s = g_{ym}^2$$
$$\alpha' = \frac{1}{\sqrt{\lambda}}, \quad \lambda = g_{ym}^2 N$$

gravity expansion parameter

$$G_N = g_s^2 \alpha'^4 = \frac{1}{N^2}$$

Small $G_N, \alpha' \Rightarrow$ large λ, N.

States of IIB:

supergravity: Dilaton, Axion, graviton, p-form field strengths (self-dual 5-form), fermions

linearized on flat superspace

$$D^4 \Phi = \bar{D}^4 \bar{\Phi} \Rightarrow D^5 \Phi = D^4 \bar{D}^4 \bar{\Phi} \sim \bar{D}^3 \partial \bar{\Phi}$$

$$\boxed{\Phi(x,\theta) = \varphi + \dots + \theta^4 R}$$

Dilaton Axion

Riemann

SUGRA = chiral superfield

[Howe, West]

also (massive) string states, $mass^2 \propto \alpha'$

States of $N=4$ SYM.

$\boxed{\varphi_I, F^{\mu\nu}, \text{fermions}}$ $I = 1 \dots 6$, $SO(6) \sim SU(4)$, $\mu,\nu = 0,\dots,3$, $\mathbb{R}^{1,3}$.

adjoint of $SU(N)$

Gauge invariant ops:

(multi-) traces of these over gauge group

eg. $\mathrm{Tr}((\varphi_I)^p)$ etc.

or $T_{\mu\nu} = \mathrm{Tr}(\varphi_J \partial_\mu \partial_\nu \varphi^J + \dots)$

stress-tensor

States on $AdS \times S$. $\longleftrightarrow$ Operators in $N=4$ SYM.

eg

$g_{\mu\nu}(\hat{x},\hat{y}) \rightarrow g_{\mu\nu}(\hat{x}) + g_{\mu\nu I}(\hat{x}) Y_I + \dots$ $\quad T_{\mu\nu}$

primaries

SUSY $\rightarrow \boxed{O_2 = \mathrm{Tr}(\varphi_{J_1} \varphi_{J_2})}$

eg graviton

eg. $g_{\mu\nu}(\hat{x},\hat{y}) \Rightarrow g_{\mu\nu}(\hat{x}) + g_{\mu\nu I}(\hat{x}) Y_I + \ldots$

$\hat{x}$: AdS_5, $\hat{y}$: S^5

$AdS \times S$ graviton, AdS graviton, K.K mode

$T_{\mu\nu}$

$T_{\mu\nu; I} = \mathrm{Tr}(\varphi_J \partial_\mu \partial_\nu \varphi^J \varphi_I \ldots)$

$\xrightarrow{SUSY}$ $O_2 = \mathrm{Tr}(\phi_{I_1} \phi_{I_2})$

$\xrightarrow{SUSY}$ $O_3 = \mathrm{Tr}(\phi_{I_1} \phi_{I_2} \phi_{I_3})$

SUGRA on $AdS \times S$

SUGRA K.K. modes $\longleftrightarrow$ $\boxed{O_p = \mathrm{Tr}(\underbrace{\varphi_{I_1} \varphi_{I_2} \cdots \varphi_{I_p}}_{\text{traceless rep.}}) + O(\tfrac{1}{N})}$

"single trace ops"

half BPS $\left[Q\, O_p = 0 \text{ for } 8/16\ Q\text{'s.} \right]$

- This is the usual story. Very complicated!
- Effective action for IIB string theory

$$\boxed{S = \underset{\uparrow\,\text{mess}}{S_{SUGRA}} + \alpha'^3 \underset{\uparrow\,\text{mess}}{(R^4 + \ldots)} + \alpha'^5 (D^4 R^4 + \ldots) + \ldots}$$

Can we factor out SUSY, use superspace in a nice way?

- On the CFT side can we re-combine everything in a 10d covariant manner?

Yes! At least for tree-level 4pnt string corrections.

Virasoro amplitude on $AdS_5 \times S_5$.

- 4-point functions of S.P.O's $\langle O_p O_q O_r O_s \rangle$ $p,q,r,s = 2,3,\ldots$
  dual to
  4-point graviton amplitude in $AdS_5 \times S_5$ + (SUSY partners)

- How does 10d structure appear in correlators?
  [answer for tree-level SUGRA $\alpha' \to 0$ limit: 10d conformal symmetry] S. Caron-Huot and A. K. Trinh.
  Also: L. Rastelli, K. Roumpedakis and X. Zhou
  S. Giusto, R. Russo, A. Tyukov and C. Wen

- We give a (slightly different) answer for α' corrections.

Here: $\boxed{\text{1st order in } \frac{1}{N^2} \;(\Leftrightarrow \text{classical limit}) \quad \text{all orders in } \frac{1}{\sqrt{\lambda}} = \alpha' \;(\Leftrightarrow \text{classical (tree-level string theory)})}$

Virasoro Shapiro Amplitude

4-point IIB string amplitude (flat space)

$S = 2k_1 \cdot k_2, \; T = 2k_1 \cdot k_4$
$U = 2k_1 \cdot k_3$

$$A_{VS}(S,T) = \frac{1}{STU} \frac{\Gamma(1-\frac{\alpha' S}{4})\Gamma(1-\frac{\alpha' T}{4})\Gamma(1-\frac{\alpha' U}{4})}{\Gamma(1+\frac{\alpha' S}{4})\Gamma(1+\frac{\alpha' T}{4})\Gamma(1+\frac{\alpha' U}{4})}, \qquad S+T+U=0$$

[Virasoro '69 (bosonic string - tachyons); Green, Schwarz, Witten (vol I) '87 (IIB string theory)]

More precisely this is $\hat{A}$ where →

$$A_n = \delta^{10}\left(\sum_{i=1}^{n} p_i\right) \delta^{16}(Q_n)\, \hat{A}_n \quad n=4$$

superamplitude

[Green, Wen 2019 - 10d spinor helicity]

superamplitude

[Green, Wen 2019 - 10d spinor helicity]

- Expand Virasoro-Shapiro:

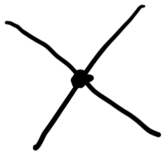
$$A_{VS}(S,T) = \frac{1}{STU}\exp\left(\sum_{n=1}^{\infty} 2\left(\frac{\alpha'}{4}\right)^{2n+1}\frac{\zeta_{2n+1}}{2n+1}(S^{2n+1}+T^{2n+1}+U^{2n+1})\right)$$
$$= \underbrace{\frac{1}{STU}}_{\text{SUGRA}} + 2\zeta_3(\tfrac{\alpha'}{4})^3 + (S^2+T^2+U^2)\zeta_5(\tfrac{\alpha'}{4})^5 + 2STU(\zeta_3)^2(\tfrac{\alpha'}{4})^6$$
$$+\tfrac{1}{2}(S^2+T^2+U^2)^2\zeta_7(\tfrac{\alpha'}{4})^7+\ldots$$

SUGRA

string corrections

- View the string corrections as arising from an effective φ^4 action. eg $\boxed{2\zeta_3\left(\frac{\alpha'}{4}\right)^3 \longleftrightarrow \frac{1}{4!}\,2\zeta_3\left(\frac{\alpha'}{4}\right)^3\int d^{10}x\,\varphi^4}$

(φ massless)



from φ^4

whereas $\boxed{S^2 = 4(k_1\cdot k_2)(k_3\cdot k_4) \longleftrightarrow \int d^{10}x\, \partial\varphi\cdot\partial\varphi\,\partial\varphi\cdot\partial\varphi}$

$$V^{\rm flat}_{VS}(\phi) = \frac{1}{2^3.4!}\Big(2\zeta_3(\tfrac{\alpha'}{2})^3\phi^4 + 3\zeta_5(\tfrac{\alpha'}{2})^5(\partial\phi.\partial\phi)^2 + 2(\zeta_3)^2(\tfrac{\alpha'}{2})^6(\partial\phi.\partial\phi)(\partial^2\phi.\partial^2\phi)$$
$$+3\zeta_7(\tfrac{\alpha'}{2})^7(\partial^2\phi.\partial^2\phi)^2+\ldots\Big)\,.$$

[This is really a pre-potential:
IIB SUGRA (linearized) on superspace = chiral superfield.

$$\Phi(x,\theta) = \Phi(x) + \cdots + \theta^4(R + F_5)$$

Dilaton/Axion

all other fields in IIB SUGRA

Riemann

self-dual 5-form field strength (source of D3-branes)

$$S_{\rm eff} \supset \int d^{10}x\, d^{16}\theta\; V^{\rm flat}_{VS}(\Phi)$$ ← chiral superpotential
]

Now

- Uplift effective action to $AdS_5 \times S_5$ background $\boxed{\partial \to \nabla}$
- a priori no guarantee this makes sense, but:

- SUGRA is still described by single chiral field on $AdS_5 \times S^5$ [Howe, PH. 2000]
- R^4 correction to SUGRA can not be written as a superpotential in the full non-linear theory [Berkovits, Howe; de Haro, Sinkovics, Skenderis, 2002] $\sqrt{g} \to \Delta(x,\theta)$ does not preserve SUSY.
- But: It does correctly give all terms containing $\boxed{R, F_5}$ [Rajaraman 2005]

Anyhow

- the fact that the following works strongly suggests $\boxed{\exists \text{ superpotential for IIB on } AdS_5 \times S_5.}$

∇_μ = AdS × S covariant derivative

$$V_{VS}^{\mathrm{AdS\times S}}(\phi) = \frac{1}{8.4!}\Big((\tfrac{\alpha'}{2})^3 A\phi^4 + (\tfrac{\alpha'}{2})^5\big(3B(\nabla\phi.\nabla\phi)^2 + 6C\nabla^2\nabla_\mu\phi\nabla^\mu\phi\phi^2\big)$$
$$+ (\tfrac{\alpha'}{2})^6\big(D(\nabla\phi.\nabla\phi)(\nabla^2\phi.\nabla^2\phi) + 6E\nabla_\mu\nabla^2\nabla_\rho\phi\nabla_\mu\nabla_\rho\phi\phi^2\big)$$
$$+ (\tfrac{\alpha'}{2})^7\big(6F(\nabla^2\phi.\nabla^2\phi)^2 + 6G_1\,(\nabla^\mu\nabla^\nu\nabla_\mu\nabla^\rho\nabla^\sigma\nabla_\rho\phi)\,(\nabla_\nu\nabla_\sigma\phi)\,\phi^2 + \dots\big) + \dots\Big)$$

$$A(\alpha') = 2\zeta_3 + A_1\frac{\alpha'}{2R^2} + A_2\left(\frac{\alpha'}{2R^2}\right)^2 + \dots$$
$$B(\alpha') = \zeta_5 + B_1\frac{\alpha'}{2R^2} + \dots$$
$$C(\alpha') = C_0 + C_1\frac{\alpha'}{2R^2} + \dots$$
$$D(\alpha') = 2(\zeta_3)^2 + D_1\frac{\alpha'}{2R^2} + \dots$$
$$E(\alpha') = E_0 + E_1\frac{\alpha'}{2R^2} + \dots$$
$$F(\alpha') = \tfrac{1}{2}\zeta_7 + F_1\frac{\alpha'}{2R^2} + \dots$$
$$G_i(\alpha') = G_{i;0} + G_{i;1}\frac{\alpha'}{2R^2} + \dots \qquad \text{for } i = 1,2,3,4,5$$

- Some terms not fixed by VS. (always true lifting from flat space)
  1.) Ambiguities due to $\boxed{[\nabla_\mu, \nabla_\nu] \neq 0}$
  2.) All coefficients can be expanded in $\boxed{\frac{\alpha'}{R^2}}$ radius of S_5

2.) All coefficients can be expanded in $\boxed{\frac{\alpha'}{R^2}}$ radius of S_5 = radius of AdS_5

dimensionless

How to extract correlators from the effective action?

How to extract correlators from this?

$$\boxed{\varphi_I(x) \to \varphi_M(X,Y) = \varphi_I Y^I}$$

$$O_p(X,Y) = \frac{1}{pN^{p/2}} \mathrm{Tr}(\phi^p_{YM}) \Rightarrow \boxed{\langle O_p O_q \rangle = \frac{1}{p} \delta_{pq} \left(\frac{Y_1 \cdot Y_2}{X_1 \cdot X_2}\right)^p}$$

normalization is the one that manifests 10d conformal symmetry

$\boxed{Y^I Y^I = 0}$, projects onto symmetric traceless rep.

$(I=1\cdots 6)$

Note: $\boxed{\underset{SO(6)}{Y^I}, \quad \underset{SO(2,4)}{X^A}}$ v. similar! $\longleftrightarrow$ $\boxed{\underset{S^5}{\hat{Y}^I}, \quad \underset{AdS_5}{\hat{X}^A}}$

- Superconformal Ward identities + non-renormalization theorem

$$\Rightarrow \boxed{\langle O_p O_q O_r O_s \rangle = \langle O_p O_q O_r O_s \rangle_{\text{free}} + I(X_i, Y_i)\, f(X,Y)}$$

$\therefore$ define

$$\langle O_p O_q O_r O_s \rangle_{\text{int}} = \frac{\langle O_p O_q O_r O_s \rangle - \langle O_p O_q O_r O_s \rangle_{\text{free}}}{I(X_i, Y_i)}$$

Universal factor from SUSY plays the role of $\delta^{16}(Q)$ or $\int d^{16}\theta$ for correlators (see Green, Wen)

$$\langle OOOO \rangle = \sum_{p,q,r,s=2}^{\infty} \langle O_p O_q O_r O_s \rangle_{\text{int}}$$

package all correlators into a single object.

For this combination, leading $\frac{1}{\sqrt{\lambda}} = \alpha'$ (SUGRA) has 10d conformal symmetry.

- N.B. $\langle O_2 O_2 O_2 O_2 \rangle$, $\boxed{O_2 \text{ has } \Delta = 2,\ su(4) \text{ rep } [020].}$ $O_{(IJ)}$

BUT. $\boxed{I(X_i, Y_i)}$ removes $su(4)$ charge + increases Δ

$$\Rightarrow \boxed{\langle OOOO \rangle, \ \boxed{\Delta = 4}\ su(4) \text{ singlets.}}$$

massless 10d scalar has $\Delta = 4$ ...

- Obtain $\langle OOOO \rangle$ directly from a 10d bulk computation

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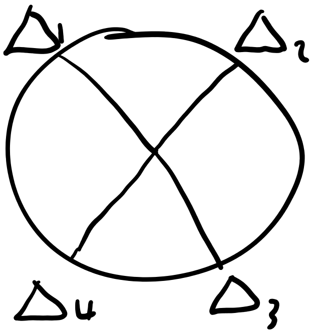
AdS x S Witten diagrams.

Recall AdS Witten diagrams: Bulk to boundary propagator:

$$G_\Delta(\hat{X}, X_i) = \frac{1}{(-2\hat{X}\cdot X_i)^\Delta}$$

AdS in 6d embedding space $\hat{X}\cdot\hat{X} = -1$

Minkowski in 6d embedding space $X_i \cdot X_i = 0$



$$= \quad D^{(d)}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i) = \frac{1}{(-2)^{2\Sigma_\Delta}}\int_{\mathrm{AdS}} \frac{d^{d+1}\hat{X}}{(\hat{X}.X_1)^{\Delta_1}(\hat{X}.X_2)^{\Delta_2}(\hat{X}.X_3)^{\Delta_3}(\hat{X}.X_4)^{\Delta_4}}$$

$\Sigma_\Delta = \frac{1}{2}(\Delta_1+\Delta_2+\Delta_3+\Delta_4)$

[Chen, Sakamoto '20]

- Write analogous expressions on S_5: $G_p(\hat{Y}, Y_i) = (-2\hat{Y}\cdot Y_i)^p$

S_5: $\hat{Y}\cdot\hat{Y} = 1$

$Y_i \cdot Y_i = 0$ (co-ords used to define 1/2 BPS ops)

$$B_{p_1p_2p_3p_4}(Y_1, Y_2, Y_3, Y_4) = (-2)^{2\Sigma_p}\int_S d^{d+1}\hat{Y}(\hat{Y}.Y_1)^{p_1}(\hat{Y}.Y_2)^{p_2}(\hat{Y}.Y_3)^{p_3}(\hat{Y}.Y_4)^{p_4}$$

. Completely explicit formula for this (later)

- Now define 10d $AdS_5 \times S_5$ bulk to boundary propagator.

$$G(\hat{X},\hat{Y};X,Y) = \left(-2\hat{X}.X - 2\hat{Y}.Y\right)^{-\Delta}$$

when $\Delta = d$.

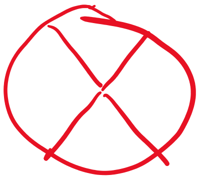
massless 10d field $AdS_5 \times S_5$ Bulk to boundary propagator

satisfies $\boxed{\nabla^2 G = \left(\nabla^2_{\hat{X}} + \nabla^2_{\hat{Y}}\right) G = 0}$

10d $\quad$ AdS_5 $\quad$ S_5

- AdS×S generalization of Witten diagram

$$D^{\mathrm{AdS}_{d+1}\times \mathrm{S}^{d+1}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i,Y_i) = \frac{1}{(-2)^{2\Sigma_\Delta}} \int_{\mathrm{AdS}\times\mathrm{S}} \frac{d^{d+1}\hat{X}\, d^{d+1}\hat{Y}}{(P_1+Q_1)^{\Delta_1}(P_2+Q_2)^{\Delta_2}(P_3+Q_3)^{\Delta_3}(P_4+Q_4)^{\Delta_4}}$$



$AdS_5 \times S_5$

$$P_i = \hat{X}.X_i\,, \qquad Q_i = \hat{Y}.Y_i$$

- Easy to expand in Y_i:

$$\frac{1}{(P+Q)^{\Delta}} = \sum_{p=0}^{\infty} (-1)^p \frac{(p+1)_{\Delta-1}}{\Gamma(\Delta)} \frac{Q^p}{P^{p+\Delta}}$$

$\Downarrow$

$$D^{\mathrm{AdS}\times\mathrm{S}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i,Y_i) = \sum_{p_i=0}^{\infty} \prod_{i=1}^{4} \frac{(p_i+1)_{\Delta_i-1}}{\Gamma(\Delta_i)}\, \underbrace{D^{(d)}_{p_1+\Delta_1,p_2+\Delta_2,p_3+\Delta_3,p_4+\Delta_4}(X_i)}_{\text{AdS}}\, \underbrace{B_{p_1p_2p_3p_4}(Y_i)}_{\text{sphere}}$$

- Also easy to convert to Mellin space. (later)

Relation to correlators.

Example.

$AdS_5 \times S_5$ effective action : $S_{\alpha'^3} = \frac{1}{8.4!}\left(\frac{\alpha'}{2}\right)^3 \times 2\zeta_3 \times \int_{\mathrm{AdS}\times\mathrm{S}} d^5\hat{X} d^5\hat{Y} \phi(\hat{X},\hat{Y})^4$

$\boxed{\phi \to G}$

$$\langle OOOO\rangle|_{\alpha'^3} = \frac{1}{8.4!}\left(\frac{\alpha'}{2}\right)^3 \times 2\zeta_3 \times \frac{(C_4)^4}{(-2)^{16}} \int_{\mathrm{AdS}\times\mathrm{S}} \frac{d^5\hat{X} d^5\hat{Y}}{(P_1+Q_1)^4(P_2+Q_2)^4(P_3+Q_3)^4(P_4+Q_4)^4}$$
$$= \frac{1}{8.4!}\left(\frac{\alpha'}{2}\right)^3 (C_4)^4 \times 2\zeta_3 \times D_{4444}^{\mathrm{AdS}_5\times\mathrm{S}^5}\,. \qquad (41)$$

All $\frac{1}{2}$BPS correlators at $O(\alpha'^3)$

- Above expansion in Y gives all individual correlators $\boxed{\langle O_p O_q O_r O_s\rangle}$ as explicit D-functions $\times$ B functions

- This turns out to give the correct result!!!

- Agrees with results from [Gonçalves '14; Alday, Bissi, Perlmutter '18 Drummond, Nandan, Paul, Rigatos '19] bootstrap method (self-consistency + flat space limit)

Derivative interactions? α'^5

$$S_{\alpha'^5} = \frac{1}{8}\left(\frac{\alpha'}{2}\right)^5 \left(\zeta_5 S^{\text{main}}_{\alpha'^5} + C_0 S^{\text{amb}}_{\alpha'^5} + A_2 S^{\text{main}}_{\alpha'^3}\right) ,$$

$$S^{\text{main}}_{\alpha'^5} = \frac{3}{4!}\int_{\text{AdS}\times\text{S}} d^5\hat{X} d^5\hat{Y} (\nabla\phi.\nabla\phi)(\nabla\phi.\nabla\phi),$$

$$S^{\text{amb}}_{\alpha'^5} = \frac{6}{4!}\int_{\text{AdS}\times\text{S}} d^5\hat{X} d^5\hat{Y} \nabla^2\nabla_\mu\phi\nabla^\mu\phi\phi^2 ,$$

$$S^{\text{main}}_{\alpha'^3} = \frac{1}{4!}\int_{\text{AdS}\times\text{S}} d^5\hat{X} d^5\hat{Y} \phi^4 .$$

$\phi \to G$

$\nabla_\mu = (\nabla_A, \nabla_I)$

$\nabla_A = \partial_A + \hat{X}_A \hat{X}\cdot\partial$

$\nabla_I = \partial_I - \hat{Y}_I \hat{Y}\cdot\partial$

$$\langle OOOO\rangle|_{\alpha'^5;\text{main}} = \frac{1}{4!}\frac{(C_4)^4}{(-2)^{16}}\int_{\text{AdS}\times\text{S}} d^5\hat{X}d^5\hat{Y} \frac{N_{12}N_{34}+N_{13}N_{24}+N_{14}N_{23}}{(P_1+Q_1)^5(P_2+Q_2)^5(P_3+Q_3)^5(P_4+Q_4)^5}\times 4^4$$

$$N_{ij} = X_i.X_j + Y_i.Y_j + P_iP_j - Q_iQ_j$$

- Straightforward to expand in Y's as before and obtain explicit expressions for $\boxed{\langle O_p O_q O_r O_s\rangle_{\alpha'^5}}$ in terms of D-functions, B-functions, with $X_i\cdot X_j$, $Y_i\cdot Y_j$ decorations.

- This also gives exactly the same result as obtained in:

[Alday, Bissi, Perlmutter '18; Drummond, Paul, Santagata '20; Aprile, Viera '20]

(with the same two ambiguities - ambiguities: fixed by Localization [Binder, Chester, Pufu, Wang '20] in N=4 SYM) checked in Mellin space.

- Going further is now straightforward in principle....

- Going further is not straight...

But. a.) Doing the covariant derivatives becomes tedious.

$\rightarrow$ $\boxed{\text{Computer algorithm}}$ for AdS, S, AdS×S covariant derivatives

b.) More convenient to display results in Mellin space "Mellin amplitudes".

AdS×S covariant derivatives.

- AdS (and S) covariant derivatives given in embedding space using projection tensors (projecting onto AdS or S surface)

$$\mathcal{P}_A^B = \delta_A^B + \hat{X}_A\hat{X}^B\,, \qquad \mathcal{P}_I^J = \delta_I^J - \hat{Y}_I\hat{Y}^J$$

satisfy: $\mathcal{P}_A^B\hat{X}^A = 0\,,$ $\quad \mathcal{P}_I^J\hat{Y}^J = 0\,,$

$\mathcal{P}_A^B\mathcal{P}_B^C = \mathcal{P}_A^C\,,$ $\quad \mathcal{P}_I^J\mathcal{P}_J^K = \mathcal{P}_I^K$

projector

orthogonal to normal vector

- Then the covariant derivative of any tensor is given by:

$$\nabla_A \mathrm{T}_{\mathrm{A_1\ldots A_N}} = \mathcal{P}_{\mathrm{A}}^{\mathrm{C}}\mathcal{P}_{\mathrm{A_1}}^{\mathrm{C_1}}\ldots\mathcal{P}_{\mathrm{A_N}}^{\mathrm{C_N}}\partial_{\mathrm{C}}\left(\mathcal{P}_{\mathrm{C_1}}^{\mathrm{E_1}}\ldots\mathcal{P}_{\mathrm{C_N}}^{\mathrm{E_N}}\mathrm{T}_{\mathrm{E_1\ldots E_N}}\right)$$ [Penedones; Sleight]

- Project the tensor
- Apply partial deriv.
- Project the resulting tensor

eg. Two AdS derivatives acting on a scalar:

$$\nabla_B\nabla_A\phi = \mathcal{P}_B^{B'}\mathcal{P}_A^{A'}\partial_{B'}\mathcal{P}_{A'}^{A''}\partial_{A''}\phi = \mathcal{P}_B^{B'}\mathcal{P}_A^{A'}\partial_{B'}\partial_{A'}\phi + \mathcal{P}_{BA}\hat{X}.\partial\phi$$

- Useful graphical notation:

· Useful graphical notation:

$$\nabla_B\nabla_A =$$

A • $= (\mathcal{P}\cdot\partial)_A$

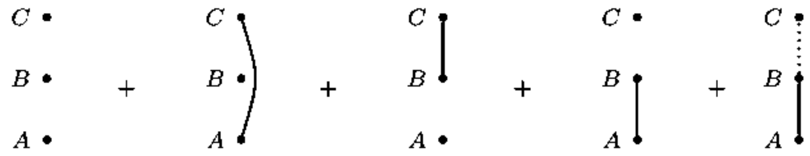
A — B $= \mathcal{P}_{AB}\,\hat{X}\cdot\partial$

all derivatives commuted through.

$$\nabla_C\nabla_B\nabla_A\phi = \mathcal{P}_C^{C'}\mathcal{P}_B^{B'}\mathcal{P}_A^{A'}\partial_C\left(\mathcal{P}_B^{B'}\mathcal{P}_A^{A'}\partial_{B'}\partial_{A'} + \mathcal{P}_{BA}\hat{X}.\partial\right)\phi$$

$$= \left(\mathcal{P}_C^{C'}\mathcal{P}_B^{B'}\mathcal{P}_A^{A'}\partial_{C'}\partial_{B'}\partial_{A'} + \mathcal{P}_{CA}\hat{X}.\partial\partial_B + \mathcal{P}_{CB}\hat{X}.\partial\partial_A + \mathcal{P}_{BA}\hat{X}.\partial\partial_C + \mathcal{P}_{BA}\partial_C\right)\phi$$

$=$



,

$= \mathcal{P}_{AB}$

Algorithm for $\nabla_{A_1}\nabla_{A_2}\dots\nabla_{A_n}\phi$

1. Draw n vertices vertically. Each corresponds to an embedding space index ordered so the bottom one corresponds to A_n and the top one to A_1.
2. Draw any number of solid edges between any two vertices such that each vertex is connected to at most one solid edge.
3. Draw any number of dotted edges from the upper vertex of a solid edge up to either a higher disconnected vertex or a higher vertex that is the lower vertex of a solid edge. No vertex can be attached to more than one dotted edge.
4. Sum over all the resulting graphs with the following interpretation:

A • $= \mathcal{P}_A^{A'}\partial_{A'}$

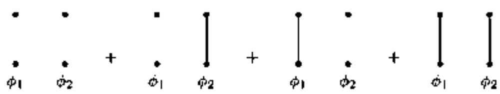
$= \mathcal{P}_{AB}\hat{X}.\partial$

$= \mathcal{P}_{AB}$ (49)

So solid edges come with a decoration $\hat{X}.\partial$ unless they have a dotted line attached to the top in which case the decoration is removed. (Otherwise the dotted lines can be ignored.)

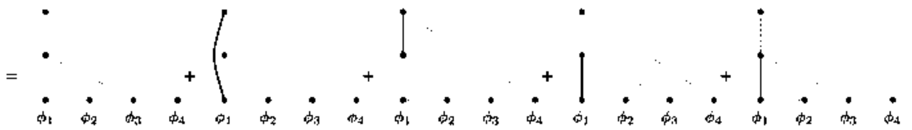
Tensoring these together gives results for derivative contact terms.
eg. Additional grey lines for contractions.

$\nabla_B\nabla_A\phi_1\,\nabla_B\nabla_A\phi_2 =$



$$= \mathcal{P}^{AB}\mathcal{P}^{CD}(\partial_A\partial_C\phi_1)(\partial_B\partial_D\phi_2) + \mathcal{P}^{AB}(\partial_A\partial_B\phi_1)\hat{X}.\partial\phi_2$$
$$+ \mathcal{P}^{AB}(\partial_A\partial_B\phi_2)\hat{X}.\partial\phi_1 + \mathcal{P}_A^A(\hat{X}.\partial\phi_1)(\hat{X}.\partial\phi_2)\,.$$

$$\nabla^C\nabla^B\nabla^A\phi_1\,\nabla_A\phi_2\,\nabla_B\phi_3\,\nabla_C\phi_4$$



$$=\mathcal{P}^{AA'}\mathcal{P}^{BB'}\mathcal{P}^{CC'}(\partial_A\partial_B\partial_C\phi_1)(\partial_{A'}\phi_2)(\partial_{B'}\phi_3)(\partial_{C'}\phi_4)+\mathcal{P}^{BB'}\mathcal{P}^{AC}(\hat{X}.\partial\partial_B\phi_1)(\partial_A\phi_2)(\partial_{B'}\phi_3)(\partial_C\phi_4)$$
$$+\mathcal{P}^{AA'}\mathcal{P}^{BC}(\hat{X}.\partial\partial_A\phi_1)(\partial_{A'}\phi_2)(\partial_B\phi_3)(\partial_C\phi_4)+\mathcal{P}^{CC'}\mathcal{P}^{AB}(\hat{X}.\partial\partial_C\phi_1)(\partial_A\phi_2)(\partial_B\phi_3)(\partial_{C'}\phi_4)$$
$$+\mathcal{P}^{CC'}\mathcal{P}^{AB}(\partial_C\phi_1)(\partial_A\phi_2)(\partial_B\phi_3)(\partial_{C'}\phi_4)\,. \tag{51}$$

- AdS×S use same graphs. (Subtlety for interpreting contractions + additional minus signs.)

- Finally interpret as acting on bulk to boundary propagators

  $\phi_i \to (\hat{X}\cdot X_i)^{-\Delta_i}$

  so $\boxed{\partial_{\mu_1}\cdots\partial_{\mu_n}\phi_i \to (-1)^n(\Delta_i)_n X_i^{\mu_1}\cdots X_i^{\mu_n}(\hat{X}\cdot X_i)^{-(\Delta_i+n+1)}}$

  acts on $\hat{X}$.

- easily implementable on mathematica.

Mellin amplitudes.

- Contact AdS Witten diagrams have Mellin transform:

$$D^{(d)}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i) = \mathcal{N}^{\mathrm{AdS}_{d+1}}_{\Delta_i} \times \int \frac{d\delta_{ij}}{(2\pi i)^2} \prod_{i<j} \frac{\Gamma(\delta_{ij})}{(X_i.X_j)^{\delta_{ij}}}\, \mathcal{M} \qquad \sum_i \delta_{ij} = \Delta_j \,,$$

Mellin amplitude: $\boxed{\mathcal{M}=1}$

- Expansion of spherical contact diagram:

$$B_{p_1p_2p_3p_4}(Y_i) = \mathcal{N}^{\mathrm{S}^{d+1}}_{p_i} \sum_{\{d_{ij}\}} \prod_{i<j} \frac{(Y_i.Y_j)^{d_{ij}}}{\Gamma(d_{ij}+1)} \qquad \sum_i d_{ij} = p_j$$

$$\left\{(d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}) : 0 \le d_{ij} = d_{ji}, \quad d_{ii} = 0, \quad \sum_{i=1}^{4} d_{ij} = p_j\right\}$$

Explicit expression straightforward to obtain.

Reduces to combinatorics (following 2- and 3-point cases obtained in [Chen, Sakamoto '20])

Note close similarity to the Mellin transform of contact diagrams. But straight sum instead of integral.

- Put these together into expansion of $AdS_5 \times S^5$ Witten diagram:

integral over δ_{ij} AND sum over d_{ij}

$$D^{\mathrm{AdS}_{d+1}\times \mathrm{S}^{d+1}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i, Y_i) = \frac{\pi^{d+1}}{(-2)^{\Sigma_\Delta} \prod_i \Gamma(\Delta_i)} \times \sum_{p_i=0}^{\infty} (-1)^{\Sigma_p} \int \frac{d\delta_{ij}}{(2\pi i)^2} \sum_{(d_{ij})} \left(\prod_{i<j} \frac{(Y_i.Y_j)^{d_{ij}}}{(X_i.X_j)^{\delta_{ij}}} \frac{\Gamma(\delta_{ij})}{\Gamma(d_{ij}+1)} \right) \times (\Sigma_p + d/2 + 1)_{\Sigma_\Delta - d - 1}$$

$$\text{where} \quad \sum_{i \neq j} \delta_{ij} = p_j + \Delta_j \qquad \sum_{i\neq j} d_{ij} = p_j \,.$$

$\therefore$ define the 10d $AdS_5 \times S_5$ $\boxed{\text{Mellin amplitude}}$ for any 4-point function

$$f(X_i, Y_i) = \frac{1}{4!}\frac{\pi^{d+1}}{(-2)^{\Sigma_\Delta}}\left(\prod_i \frac{C_{\Delta_i}}{\Gamma(\Delta_i)}\right) \times \sum_{p_i=0}^{\infty}(-1)^{\Sigma_p}\int \frac{d\delta_{ij}}{(2\pi i)^2}\sum_{(d_{ij})}\left(\prod_{i<j}\frac{(Y_i.Y_j)^{d_{ij}}}{(X_i.X_j)^{\delta_{ij}}}\frac{\Gamma(\delta_{ij})}{\Gamma(d_{ij}+1)}\right) \times \mathcal{M}_{\Delta_i}[f]\,,$$

$$\text{where} \qquad \sum_{i\neq j}\delta_{ij} = p_j + \Delta_j \qquad\qquad \sum_{i\neq j} d_{ij} = p_j\,.$$

[redacted]

Mellin amplitude

- So Mellin amplitude of $D^{AdS\times S}$ is $\boxed{\left(\Sigma_p + \frac{d}{2}+1\right)_{\Sigma_\Delta - d - 1}}$

eg. Mellin amplitude of α'^3 is $\boxed{(\Sigma_p+3)_3}$ $(d=4)$

$p = \frac{1}{2}\Sigma p_i$

- It was in this form that [Gonçalves '14; Alday, Bissi, Perlmutter '18 Drummond, Nandan, Paul, Rigatos '19] gave their results. "Close to an AdS contact diagram"

  – inspired this work.

[4 pnt.]

- More generally ANY (derivative) contact interaction will give an AdS×S integral as a sum of terms of the form:

$$\frac{1}{4!}\frac{\prod_i C_{\Delta_i}}{(-2)^{2\Sigma_\Delta}}\int_{\mathrm{AdS}\times \mathrm{S}} d^{d+1}\hat{X} d^{d+1}\hat{Y} \times \frac{(X_i.X_j)^{n^X_{ij}}(Y_i.Y_j)^{n^Y_{ij}} Q_i^{n_i^Q} P_i^{n_i^P} \times (\Delta_1)_{n_1}(\Delta_2)_{n_2}(\Delta_3)_{n_3}(\Delta_4)_{n_4}}{(P_1+Q_1)^{\Delta_1+n_1}(P_2+Q_2)^{\Delta_2+n_2}(P_3+Q_3)^{\Delta_3+n_3}(P_4+Q_4)^{\Delta_4+n_4}}$$

$\big\downarrow$ Mellin amplitude

$$\mathcal{M}_{\Delta_i}[\text{[redacted]}] = (-2)^{\Sigma_X}2^{\Sigma_Y}(-1)^{2\Sigma_Q}\times$$

$$\mathcal{M}_{\Delta_i}[\text{[redacted]}] = (-2)^{\Sigma_X} 2^{\Sigma_Y} (-1)^{2\Sigma_Q} \times$$

$$\left(\prod_{i<j} (\delta_{ij})_{n^X_{ij}} (d_{ij}-n^Y_{ij}+1)_{n^Y_{ij}}\right)\left(\prod_i (p_i+n_i^X+\Delta_i)_{n_i^P}(p_i-n_i^Q-n_i^Y+1)_{n_i^Q}\right)(\Sigma_p-\Sigma_Y+\tfrac{d}{2}+1)_{\Sigma_\Delta-d-1+\Sigma_X+\Sigma_Y}$$

$$\text{where} \qquad \sum_{i\neq j}\delta_{ij} = p_j+\Delta_j \qquad\qquad \sum_{i\neq j} d_{ij} = p_j \ . \qquad \text{[redacted]}$$

INPUT

- $\boxed{\varphi^4 \text{ interaction with covariant derivatives from } S_{\text{eff}}}$

$\downarrow$

Automated $\boxed{\text{Algorithm for applying } \nabla}$

$\downarrow$

$\boxed{\text{Sum of terms of above form}}$

$\downarrow$

$\boxed{\text{Mellin amplitude}}$.

- eg. α'^5: Mellin amplitude

$$\begin{aligned}\mathcal{M}^{\text{main}}_{\alpha'^5} = 4\Big[&(\Sigma_p-1)_5 \left(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2\right) \\ &+(\Sigma_p-1)_4\left(-10\left(\tilde{s}\,\mathbf{s}+\tilde{t}\,\mathbf{t}+\tilde{u}\,\mathbf{u}\right)-5\left(c_s\,\mathbf{s}+c_t\,\mathbf{t}+c_u\,\mathbf{u}\right)\right) \\ &+(\Sigma_p-1)_3\left(20\left(\tilde{s}^2+\tilde{t}^2+\tilde{u}^2\right)+4\left(c_s^2+c_t^2+c_u^2\right)+20\left(\tilde{s}\,c_s+\tilde{t}\,c_t+\tilde{u}\,c_u\right)\right) \\ &+(\Sigma_p-1)_3\left(-12\,\Sigma_p^2\right)\Big] \,.\end{aligned}$$

where we solved

$$\sum_i \delta_{ij} = p_j+2 \qquad\qquad \sum_i d_{ij} = p_j-2 \qquad \text{as}$$

$$\delta_{12} = -s + c_s\,,\qquad \delta_{14} = -t + c_t\,,\qquad \delta_{13} = -u\,,$$
$$\delta_{23} = -t\,,\qquad \delta_{24} = -u + c_u\,,\qquad \delta_{34} = -s\,,$$
$$d_{12} = \tilde{s} + c_s\,,\qquad d_{14} = \tilde{t} + c_t\,,\qquad d_{13} = \tilde{u}\,,$$
$$d_{23} = \tilde{t}\,,\qquad d_{24} = \tilde{u} + c_u\,,\qquad d_{34} = \tilde{s}\,,$$
$$\mathbf{s} = s + \tilde{s}\,,\qquad \mathbf{t} = t + \tilde{t}\,,\qquad \mathbf{u} = u + \tilde{u}\,,$$

$$c_s = \frac{p_1 + p_2 - p_3 - p_4}{2}\,,\quad c_t = \frac{p_1 + p_4 - p_2 - p_3}{2}\,,\quad c_u = \frac{p_2 + p_4 - p_3 - p_1}{2}$$

Agrees precisely with:

[Aprile, Viera '20] rewriting of [Drummond, Paul, Santagata '20]

(and two abiguities)

Obtained via an intricate bootstrap:
- Ansatz for the form of the Mellin amplitude
- Ansatz for the form of the anomalous dimensions in $O_p O_q$ OPE.
- Flat space limit ($s\to\infty, t\to\infty$ in Mellin $\Rightarrow$ flat space amplitude)

$$\mathcal{M}_{\alpha'^5} = \frac{1}{8}\left(\frac{\alpha'}{2}\right)^5 \left(\zeta_5 \mathcal{M}^{\rm main}_{\alpha'^5} + C_0 \mathcal{M}^{\rm amb}_{\alpha'^5} + A_2 \mathcal{M}^{\rm main}_{\alpha'^3}\right)$$

Localization [Binder, Chester, Pufu, Wang '19] fixes $C_0 = -\frac{3}{2}\zeta_5\,,\quad A_2 = -30\,\zeta_5$

- Used this procedure to obtain.

$\boxed{\alpha'^6 \text{ and } \alpha'^7 \text{ corrections}}$

- Interestingly: $\mathcal{M}_{\alpha'^6} = \frac{1}{8}\left(\frac{\alpha'}{2}\right)^6 \left(2(\zeta_3)^2 \mathcal{M}^{\rm main}_{\alpha'^6} + E_0 \mathcal{M}^{\rm amb}_{\alpha'^6} + B_1 \mathcal{M}^{\rm main}_{\alpha'^5} + C_1 \mathcal{M}^{\rm amb}_{\alpha'^5} + A_3 \mathcal{M}^{\rm main}_{\alpha'^3}\right)$

[Chester Pufu '20] computed $\langle 2222\rangle|_{\alpha'^6}$ fixed coeffs using localization:

$\leftarrow$ gives

$$E_0 = \frac{B_1}{8} + \frac{2(\zeta_3)^2}{3}\,,\qquad A_3 = 0\,,$$ $\leftarrow$ not observed

$$E_0 = \frac{B_1}{8} + \frac{2(\zeta_3)^2}{3} \;, \quad A_3 = 0 \;,$$ ← not observed before

. Suggests all odd terms in $\frac{\alpha'}{R^2}$ expansion

vanish ($A_1 = 0$ too)

- $R^2 \to -R^2$ symmetry

- R^2 arising from 10d curvature / F_5 where $R^2, -R^2$ always appear together?

$$\mathcal{M}^{2222}_{\alpha'^6} = \frac{1}{8}\left(\frac{\alpha'}{2}\right)^6 \Big(672(\zeta_3)^2\, s\,t\,u + 14\big(3\,B_1 + 4\big((\zeta_3)^2 - 6\,E_0\big)\big)\big(s^2+t^2+u^2\big) + A_3 - 96\,B_1 + 768\,E_0 - 3200(\zeta_3)^2\Big) \;,$$

Higher orders

$\alpha'^6 : \partial^6 R^4$

$$S_{\alpha'^6} = \frac{1}{8}\left(\frac{\alpha'}{2}\right)^6 \left(2(\zeta_3)^2 S^{\rm main}_{\alpha'^6} + E_0 S^{\rm amb}_{\alpha'^6} + B_1 S^{\rm main}_{\alpha'^5} + C_1 S^{\rm amb}_{\alpha'^5} + A_3 S^{\rm main}_{\alpha'^3}\right)$$

$$\begin{aligned}
S^{\rm main}_{\alpha'^6} &= \frac{1}{4!}\int_{\rm AdS\times S} d^5\hat{X} d^5\hat{Y} (\nabla\phi.\nabla\phi)(\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi)\ ,\\
S^{\rm amb}_{\alpha'^6} &= \frac{6}{4!}\int_{\rm AdS\times S} d^5\hat{X} d^5\hat{Y}\, \nabla_\mu\nabla^2\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\phi^2\ ,
\end{aligned}$$

$\Downarrow$ covariant deriv algorithm

$$\begin{aligned}
&\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle|_{\alpha'^6;\rm main}\\
&= \frac{1}{4!}\frac{1}{6}\frac{(\mathcal{C}_4)^4}{(-2)^{16}}\int_{\rm AdS\times S}\frac{d^5\hat{X}d^5\hat{Y}}{\prod_i (P_i+Q_i)^5}\left[\frac{N_{12}M_{34}}{(P_3+Q_3)(P_4+Q_4)} + \text{perms}\right]\times 4^4\times 5^2
\end{aligned}$$

$$M_{ij} = (X_i.X_j + P_iP_j + Y_i.Y_j - Q_iQ_j)^2 - \frac{1}{5}(P_iP_j - Q_iQ_j)^2$$

$\Downarrow$ convert to Mellin

$$\begin{aligned}
\hat{\mathcal{M}}^{\rm main}_{\alpha'^6} = \ &\frac{8}{3}\Big[(\Sigma_p{-}1)_6\left(\mathbf{s}^3+\mathbf{t}^3+\mathbf{u}^3\right)\\
&+(\Sigma_p{-}1)_5\left(6\,\Sigma_p\left(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2\right) - 18\left(\mathbf{s}^2\,\tilde{s}+\mathbf{t}^2\,\tilde{t}+\mathbf{u}^2\,\tilde{u}\right) - 9\left(\mathbf{s}^2\,c_s+\mathbf{t}^2\,c_t+\mathbf{u}^2\,c_u\right)\right)\\
&+(\Sigma_p{-}1)_4\Big(90\left(\mathbf{s}\,\tilde{s}^2+\mathbf{t}\,\tilde{t}^2+\mathbf{u}\,\tilde{u}^2\right)+\frac{39}{2}\left(\mathbf{s}\,c_s^2+\mathbf{t}\,c_t^2+\mathbf{u}\,c_u^2\right)+90\left(\mathbf{s}\,\tilde{s}\,c_s+\mathbf{t}\,\tilde{t}\,c_t+\mathbf{u}\,\tilde{u}\,c_u\right)\\
&\qquad - 60\,\Sigma_p\left(\mathbf{s}\,\tilde{s}+\mathbf{t}\,\tilde{t}+\mathbf{u}\,\tilde{u}\right) - 30\,\Sigma_p\left(\mathbf{s}\,c_s+\mathbf{t}\,c_t+\mathbf{u}\,c_u\right)\Big)\\
&+(\Sigma_p{-}1)_3\Big(-120\left(\tilde{s}^3+\tilde{t}^3+\tilde{u}^3\right)-9\left(c_s^3+c_t^3+c_u^3\right)-180\left(\tilde{s}^2\,c_s+\tilde{t}^2\,c_t+\tilde{u}^2\,c_u\right)\\
&\qquad -78\left(c_s^2\,\tilde{s}+c_t^2\,\tilde{t}+c_u^2\,\tilde{u}\right)+120\,\Sigma_p\left(\tilde{s}^2+\tilde{t}^2+\tilde{u}^2\right)+27\,\Sigma_p\left(c_s^2+c_t^2+c_u^2\right)\\
&\qquad +120\,\Sigma_p\left(\tilde{s}\,c_s+\tilde{t}\,c_t+\tilde{u}\,c_u\right)-50\,\Sigma_p^3-16\,\Sigma_p\Big)\Big]\ . \qquad (83)
\end{aligned}$$

$$\mathcal{M}_{\alpha'^6} = \frac{1}{8}\left(\frac{\alpha'}{2}\right)^6 \left(2\,(\zeta_3)^2\mathcal{M}^{\rm main}_{\alpha'^6} + E_0\mathcal{M}^{\rm amb}_{\alpha'^6} + B_1\mathcal{M}^{\rm main}_{\alpha'^5} + C_1\mathcal{M}^{\rm amb}_{\alpha'^5} + A_3\mathcal{M}^{\rm main}_{\alpha'^3}\right)$$

From $\langle 2222\rangle$ bootstrap results, including localization:

$$E_0 = \frac{B_1}{8} + \frac{2(\zeta_3)^2}{3}\ , \quad A_3 = 0$$

- assuming also: $A_1 = B_1 = C_1 = 0$

  then it is completely fixed.

- Obtain AdS5xS5 effective action from CFT (incl. localisation)

- Also α'^7. 1 new "main" contribution + 5 ambiguities

$$(\nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi)^2,$$

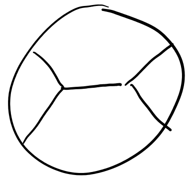
$$\begin{aligned}
&(\nabla^\mu\nabla^\nu\nabla_\mu\nabla^\rho\nabla^\sigma\nabla_\rho\phi)\,(\nabla_\nu\nabla_\sigma\phi)\,\phi^2\,, \quad \big(\nabla^2\nabla^\mu\nabla^\nu\nabla^\rho\nabla_\nu\phi\big)\,(\nabla_\mu\nabla_\rho\phi)\,\phi^2\,,\\
&\big(\nabla^2\nabla^\mu\nabla^\nu\nabla^\rho\nabla_\nu\phi\big)\,(\nabla_\rho\phi)\,(\nabla_\mu\phi)\,\phi\,, \quad (\nabla^\mu\nabla^\nu\nabla^\rho\nabla_\nu\nabla_\rho\phi)\,(\nabla^\sigma\nabla_\mu\phi)\,(\nabla_\sigma\phi)\,\phi\,,\\
&(\nabla^\mu\nabla^\nu\nabla^\rho\nabla^\sigma\nabla_\rho\phi)\,(\nabla_\mu\nabla_\sigma\phi)\,(\nabla_\nu\phi)\,\phi\,.
\end{aligned}$$

- Simultaneously computed by
  by bootstrap techniques. Precise agreement!

[Aprile, Drummond, Paul, Santagata]

Supergravity

- So far string corrections to effective action
- What about SUGRA itself?
- SUGRA includes 3-point interactions



- BUT - 4pnt correlators

  can be re-written in 10d form

D_{2422} for $\langle 2222\rangle$ from [Arutyunov, Frolov]

- $$\langle OOOO\rangle_{\text{sugra}} \propto \frac{1}{(X_1.X_3+Y_1.Y_3)}\,\frac{1}{(X_1.X_4+Y_1.Y_4)}\,\frac{1}{(X_3.X_4+Y_3.Y_4)}\,D_{2422}(X_i,Y_i)$$

[Caron-Huot, Trinh]

- Mellin amplitude

$$\mathcal{M}_{\text{sugra}} \propto \frac{1}{(\boldsymbol{\delta}_{13}-1)(\boldsymbol{\delta}_{14}-1)(\boldsymbol{\delta}_{34}-1)} = \frac{1}{(\delta_{13}-d_{13}-1)(\delta_{14}-d_{14}-1)(\delta_{34}-d_{34}-1)}$$

- Think of this as something like $\boxed{(\nabla_3\cdot\nabla_4)^{-1}(\nabla_1\cdot\nabla_4)^{-1}(\nabla_1\cdot\nabla_3)^{-1}\,\varphi^4}$ interaction in superpotential

<u>10d conformal symmetry?</u> [Caron-Huot, Trinh]

Here we have 10d Lorentz manifest, SUGRA has additional 10d conformal symmetry. Why?

has additional ...

- AdS×S contact terms $\longrightarrow$ Flat space contact terms: (rescale Y's and differentiate)

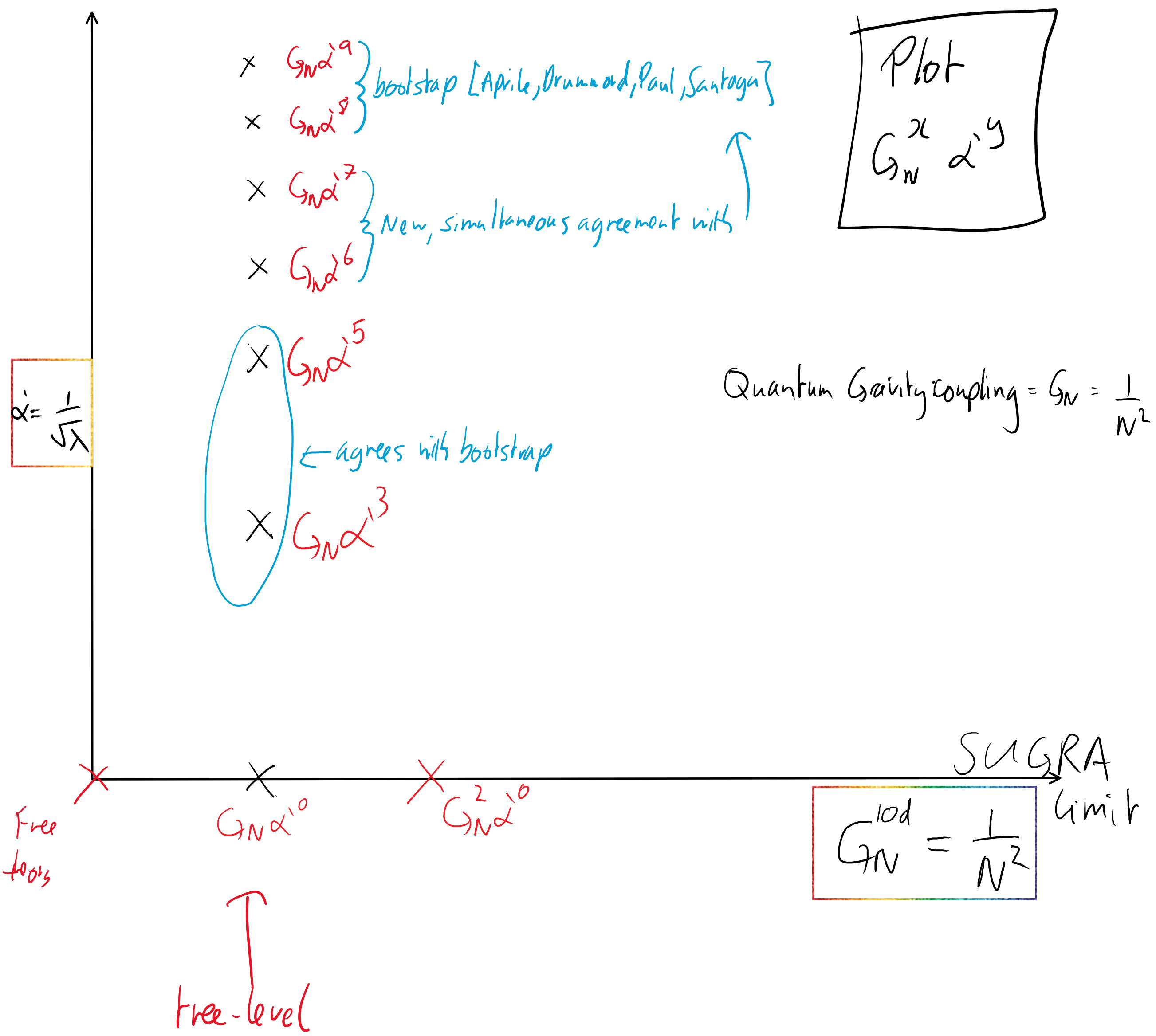
$$D^{\mathrm{AdS}_{d+1}\times\mathrm{S}^{d+1}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i,\sqrt{r}Y_i) = \frac{2}{\Gamma(\Sigma_\Delta - d - 1)}\frac{1}{r^{d/2}}\left(\frac{d}{dr}\right)^{\Sigma_\Delta - d - 1} r^{\Sigma_\Delta - d/2 - 1} D^{(2d+2)}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i,\sqrt{r}Y_i)$$

- In particular if $\boxed{\Sigma_\Delta = \frac{d}{\cancel{2}} + 1}$ they are equal!

$$D^{\mathrm{AdS}_{d+1}\times\mathrm{S}^{d+1}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i,Y_i) = \frac{\pi^{d+1}}{(-2)^{\Sigma_\Delta}\prod_i \Gamma(\Delta_i)} \times D_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i,Y_i) \qquad \Sigma_\Delta = d+1$$

- SUGRA $\boxed{D_{2422}}$ $\Rightarrow$ $\Sigma_\Delta = \frac{2+4+2+2}{2} = \boxed{5} = d+1$ ✓ $d=4$

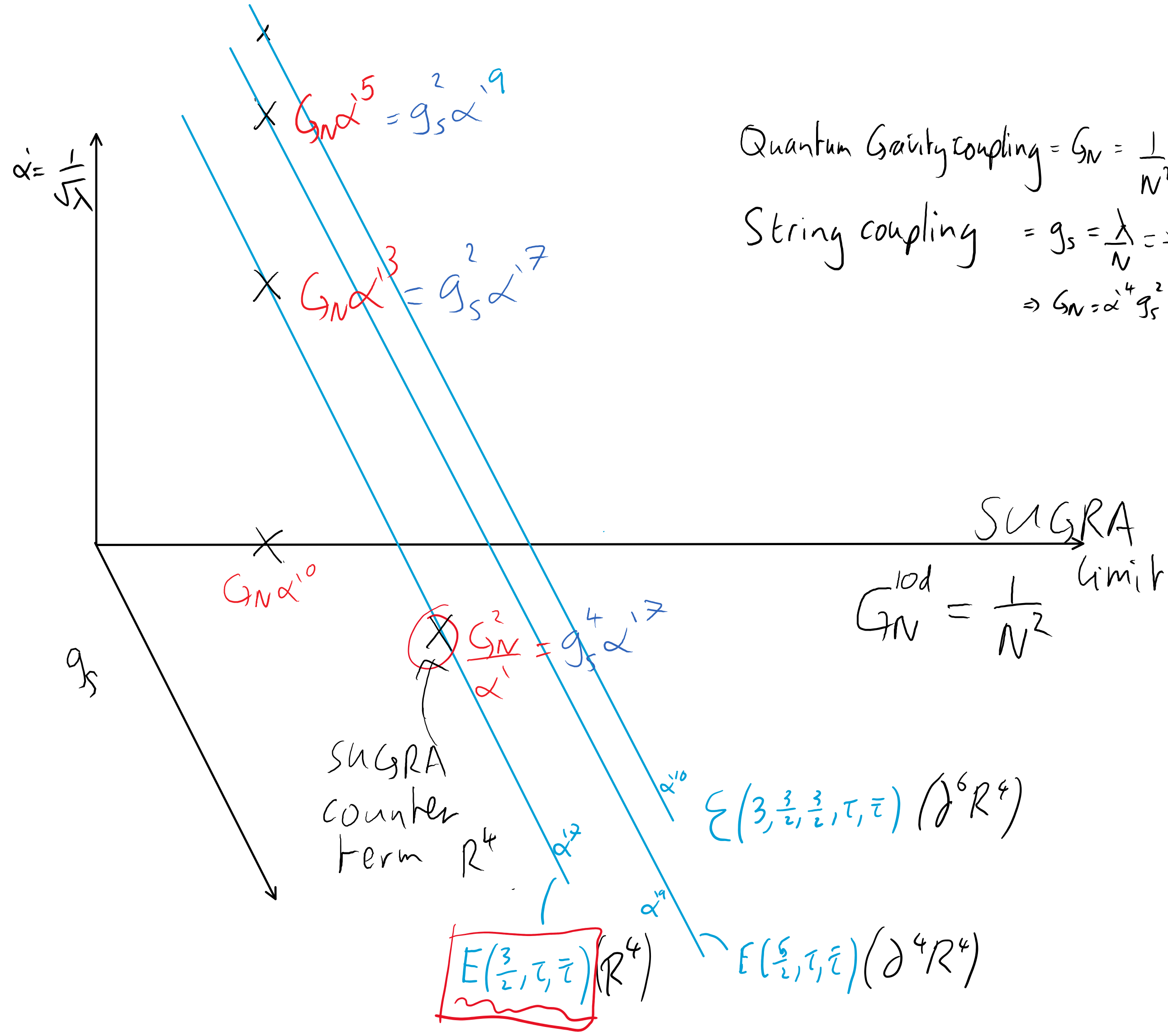
Summary of $\mathcal{N}=4$ SYM at strong coupling.



Quantum Gravity coupling $= G_N = \frac{1}{N^2}$

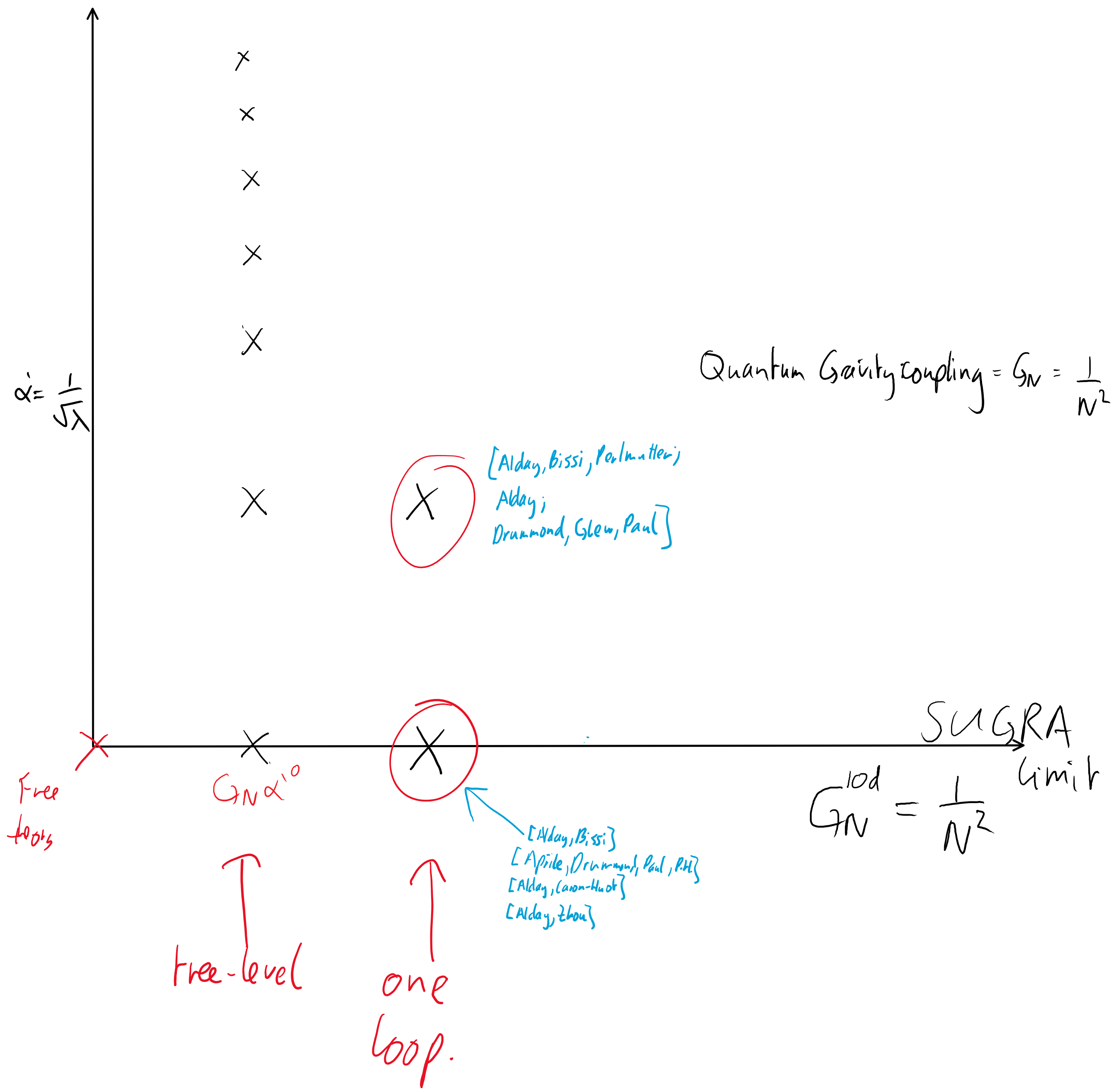
String coupling $= g_s = \frac{\lambda}{N} = \frac{\sqrt{G_N}}{\alpha'^2}$

$\Rightarrow G_N = \alpha'^4 g_s^2$



[Green, Gutperle, Vanhove; Green, Sethi; Green, Kwon, Vanhove; Chester, Green, Pufu, Wang, Wen]

Also progress in loop corrections



$\alpha' = \frac{1}{\sqrt{\lambda}}$
Quantum Gravity coupling = $G_N = \frac{1}{N^2}$
[Alday, Bissi, Perlmutter; Alday; Drummond, Glew, Paul]
SUGRA limit
$G_N^{10d} = \frac{1}{N^2}$
Free theory
$G_N \alpha'^0$
[Alday, Bissi]
[Aprile, Drummond, Paul, RH]
[Alday, Caron-Huot]
[Alday, Zhou]
tree-level
one loop.

Conclusions

Virasoro-Shapiro in AdS x S.

- $\boxed{\text{Clarify why this works}}$. Implies scalar superpotential on $AdS_5 \times S_5$. New way to compute beyond tree level?
- Obtain full $\overset{(4pnt)}{\text{effective action}}$ / VS amplitude on $AdS_5 \times S_5$?
- $\boxed{\text{Higher points?}}$ eg Maximally nilpotent correlators [Green, Wen]
- $\boxed{\text{Other backgrounds?}}$ $AdS_7 \times S_4$, $AdS_4 \times S^7$ Problem: factor out SUSY? Related? not conformally flat.

But $\left.\begin{matrix} AdS_3 \times S^3 \\ AdS_2 \times S^2 \end{matrix}\right\}$ should be OK.

Quick summary.

Flat IIB string theory 4-point amplitude: [precision AdS/CFT]

$$A_{VS}(S,T) = \frac{1}{STU}\frac{\Gamma(1-\frac{\alpha' S}{4})\Gamma(1-\frac{\alpha' T}{4})\Gamma(1-\frac{\alpha' U}{4})}{\Gamma(1+\frac{\alpha' S}{4})\Gamma(1+\frac{\alpha' T}{4})\Gamma(1+\frac{\alpha' U}{4})}, \quad = \frac{1}{STU} + 2\zeta_3(\tfrac{\alpha'}{4})^3 + (S^2+T^2+U^2)\zeta_5(\tfrac{\alpha'}{4})^5 + 2STU(\zeta_3)^2(\tfrac{\alpha'}{4})^6 + \tfrac{1}{2}(S^2+T^2+U^2)^2\zeta_7(\tfrac{\alpha'}{4})^7 + \dots$$

$\downarrow$

Effective 10d flat-space superpotential

$$V_{VS}^{\text{flat}}(\phi) = \frac{1}{2^3.4!}\Big(2\zeta_3(\tfrac{\alpha'}{2})^3\phi^4 + 3\zeta_5(\tfrac{\alpha'}{2})^5(\partial\phi.\partial\phi)^2 + 2(\zeta_3)^2(\tfrac{\alpha'}{2})^6(\partial\phi.\partial\phi)(\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi) + 3\zeta_7(\tfrac{\alpha'}{2})^7(\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi)^2 + \dots\Big)\,.$$

$\downarrow$

(conjectured) Effective $AdS_5 \times S_5$ super-potential mod ambiguities.

$$V_{VS}^{\text{AdS}\times\text{S}}(\phi) = \frac{1}{8.4!}\Bigg((\tfrac{\alpha'}{2})^3 A\phi^4 + (\tfrac{\alpha'}{2})^5\Big(3B(\nabla\phi.\nabla\phi)^2 + 6C\nabla^2\nabla_\mu\phi\nabla^\mu\phi\phi^2\Big) + (\tfrac{\alpha'}{2})^6\Big(D(\nabla\phi.\nabla\phi)(\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi) + 6E\nabla_\mu\nabla^2\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\phi^2\Big) + (\tfrac{\alpha'}{2})^7\Big(6F(\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi)^2 + 6G_1\,(\nabla^\mu\nabla^\nu\nabla_\mu\nabla^\rho\nabla^\sigma\nabla_\rho\phi)\,(\nabla_\nu\nabla_\sigma\phi)\,\phi^2 + \dots\Big) + \dots\Bigg)$$

$$\begin{aligned}
A(\alpha') &= 2\zeta_3 + A_1\tfrac{\alpha'}{2R^2} + A_2\left(\tfrac{\alpha'}{2R^2}\right)^2 + \dots\\
B(\alpha') &= \zeta_5 + B_1\tfrac{\alpha'}{2R^2} + \dots\\
C(\alpha') &= C_0 + C_1\tfrac{\alpha'}{2R^2} + \dots\\
D(\alpha') &= 2(\zeta_3)^2 + D_1\tfrac{\alpha'}{2R^2} + \dots\\
E(\alpha') &= E_0 + E_1\tfrac{\alpha'}{2R^2} + \dots\\
F(\alpha') &= \tfrac{1}{2}\zeta_7 + F_1\tfrac{\alpha'}{2R^2} + \dots\\
G_i(\alpha') &= G_{i;0} + G_{i;1}\tfrac{\alpha'}{2R^2} + \dots
\end{aligned}$$

$\downarrow$

All $\frac{1}{2}$BPS correlators using generalized AdS × S Witten diagrams.