A Puncture in the Euclidean Black Hole

Sunny Itzhaki

Based on work with R. Brustein, Amit Giveon

and Yoav Zigdon

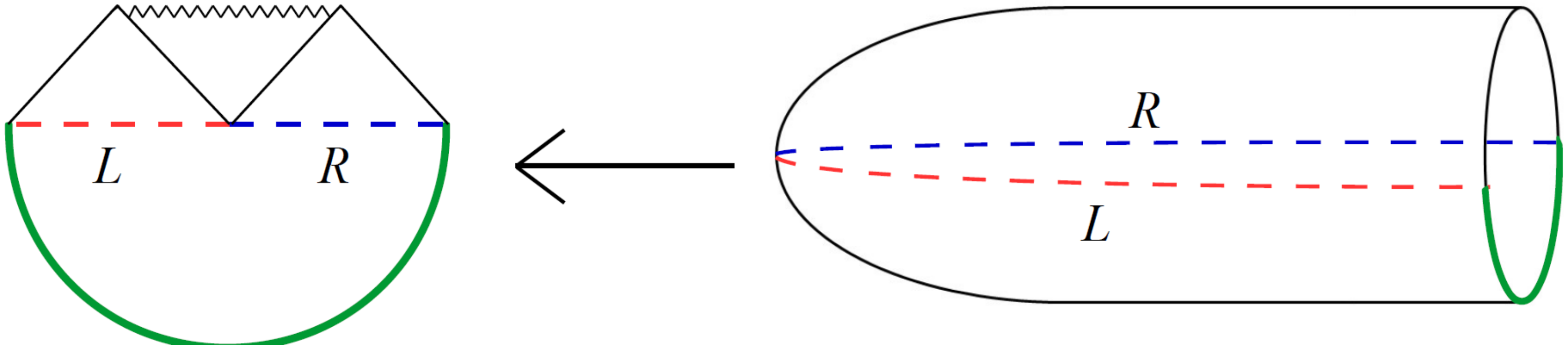
ITMP 8/12/21

Thinking about the cigar geometry was proven to be a useful thing to do:

Examples:

1. Gibbons-Hawking.
2. Hartle-Hawking
3. Recent derivation

   of the Page curve.



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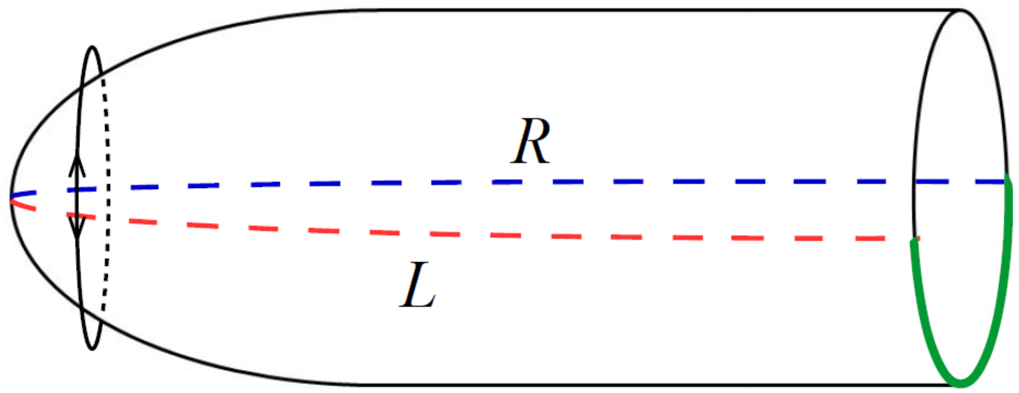

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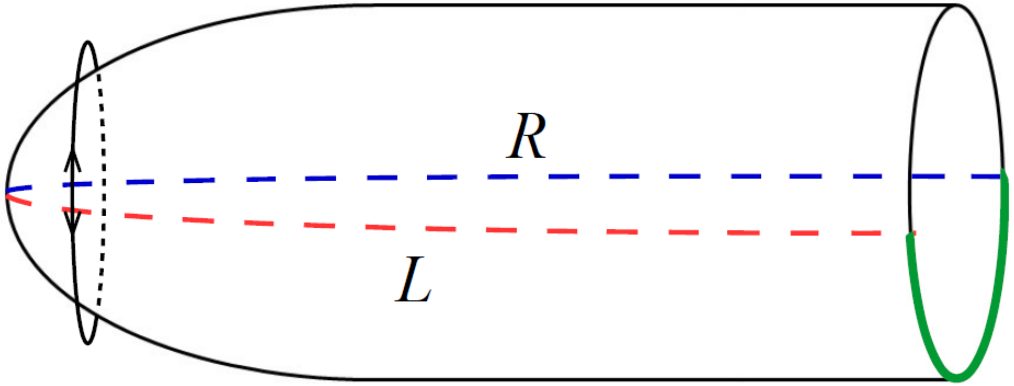

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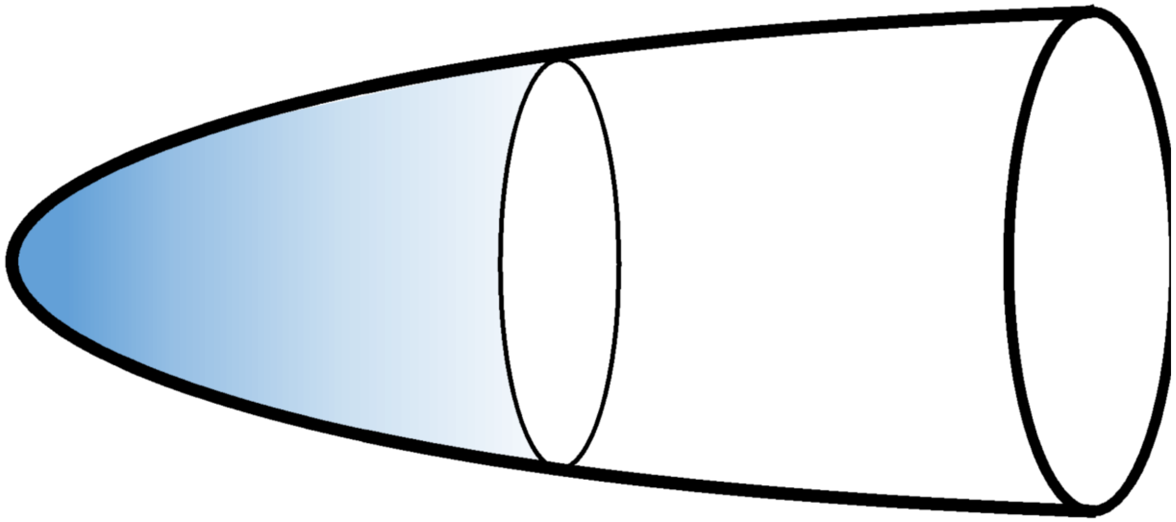
$$\chi(\rho) \propto e^{-S_{NG}(\rho)}$$



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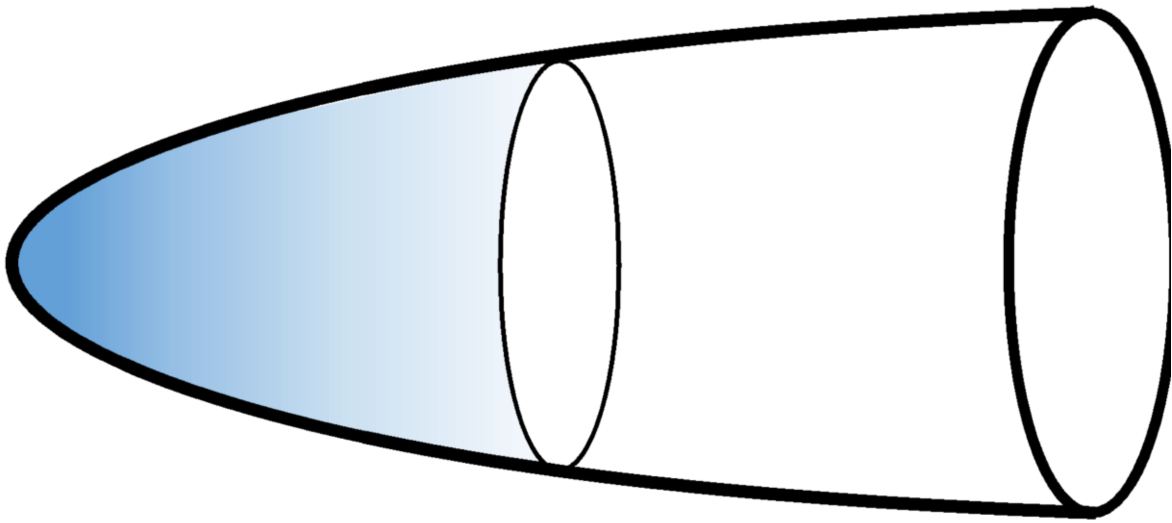

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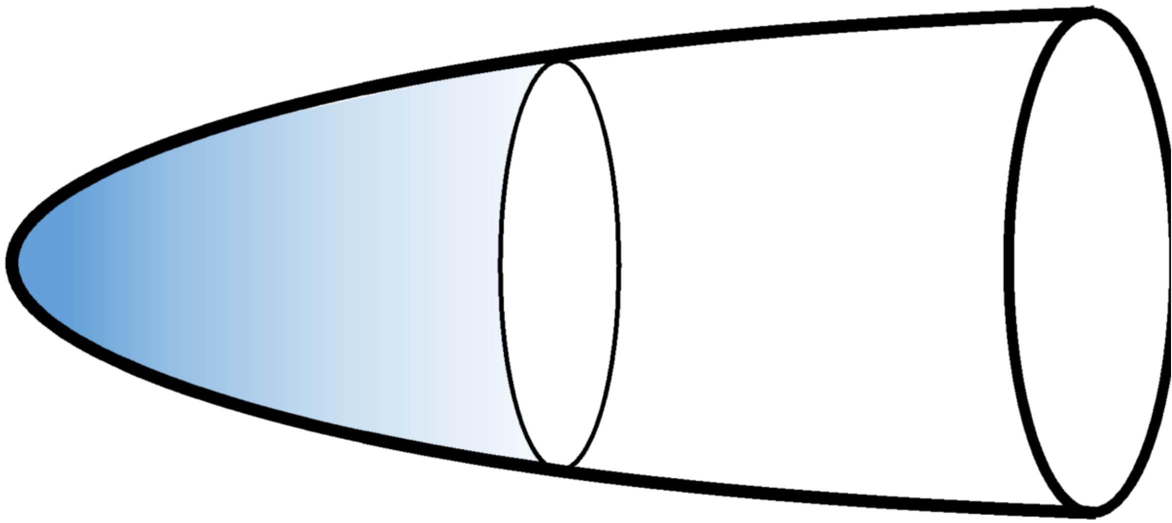

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Not ideal, but

$SL(2,R)_k/U(1)$ = near horizon limit of k near extremal NS5-branes.

(Maldacena and Strominger)

The starting point is the Horowitz-Polchinski action

$$I = \int d^2x \sqrt{g} e^{-2\Phi} \left(-\frac{1}{2\kappa^2}(R - 2\Lambda + 4\partial^\mu \Phi \partial_\mu \Phi) + \partial^\mu \chi \partial_\mu \chi^* + \frac{\beta^2 g_{\tau\tau} - \beta_H^2}{4\pi^2} \chi \chi^* \right)$$

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FZZ: χ and the zero mode of the dilaton are two semi-class tails of the same exact state in the CFT

HP →FZZ

The HP equations of motion take the form

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\begin{aligned}
h\left(\frac{\Phi'}{h}\right)' &= (\chi')^2 + h^2\chi^2 \ , \\
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Our approach:

Solve numerically and see if something

interesting happens at A_s

For technical reasons (related to the boundary condition) it is easier to solve numerically for $k \to \infty$.

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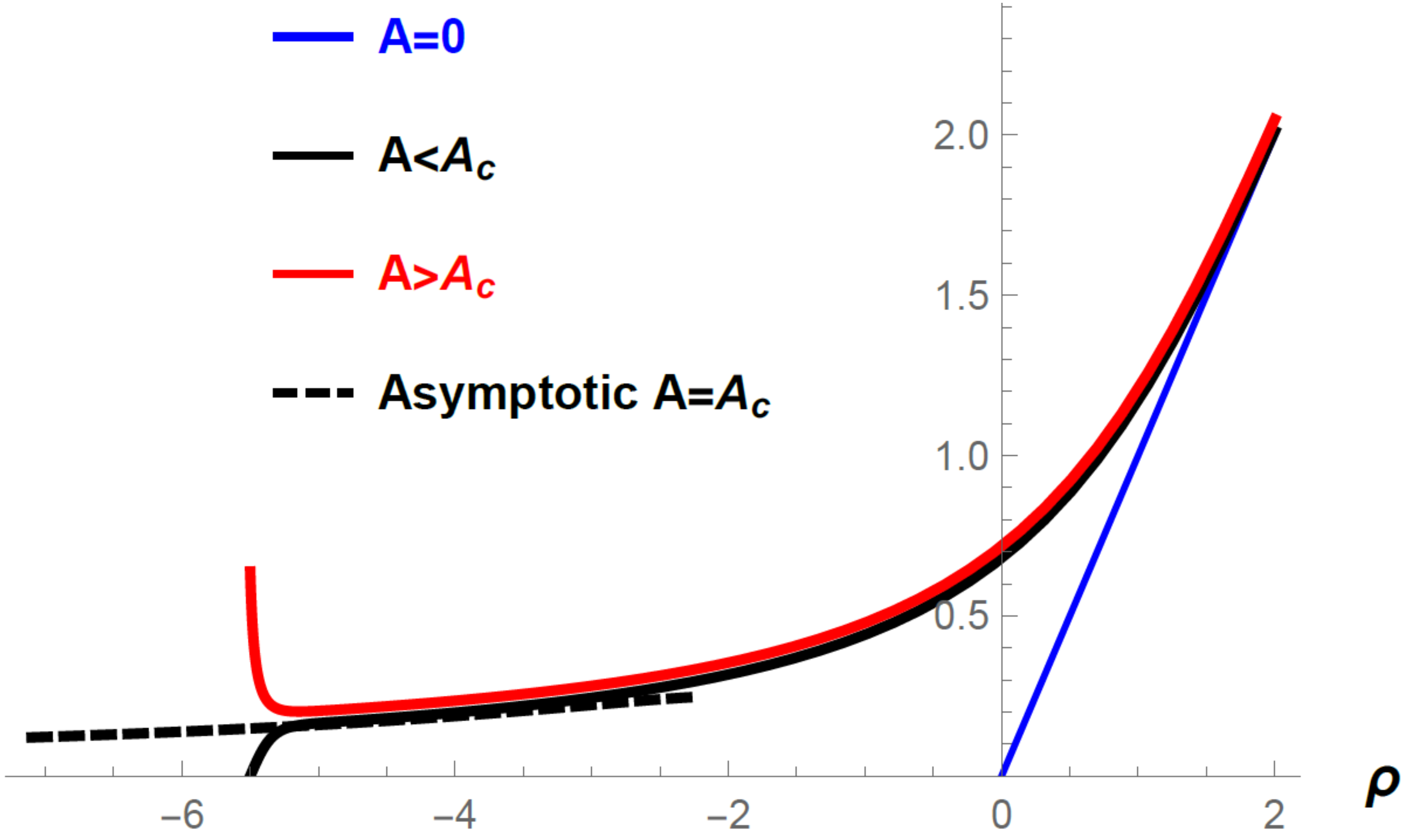

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There is a critical value of C

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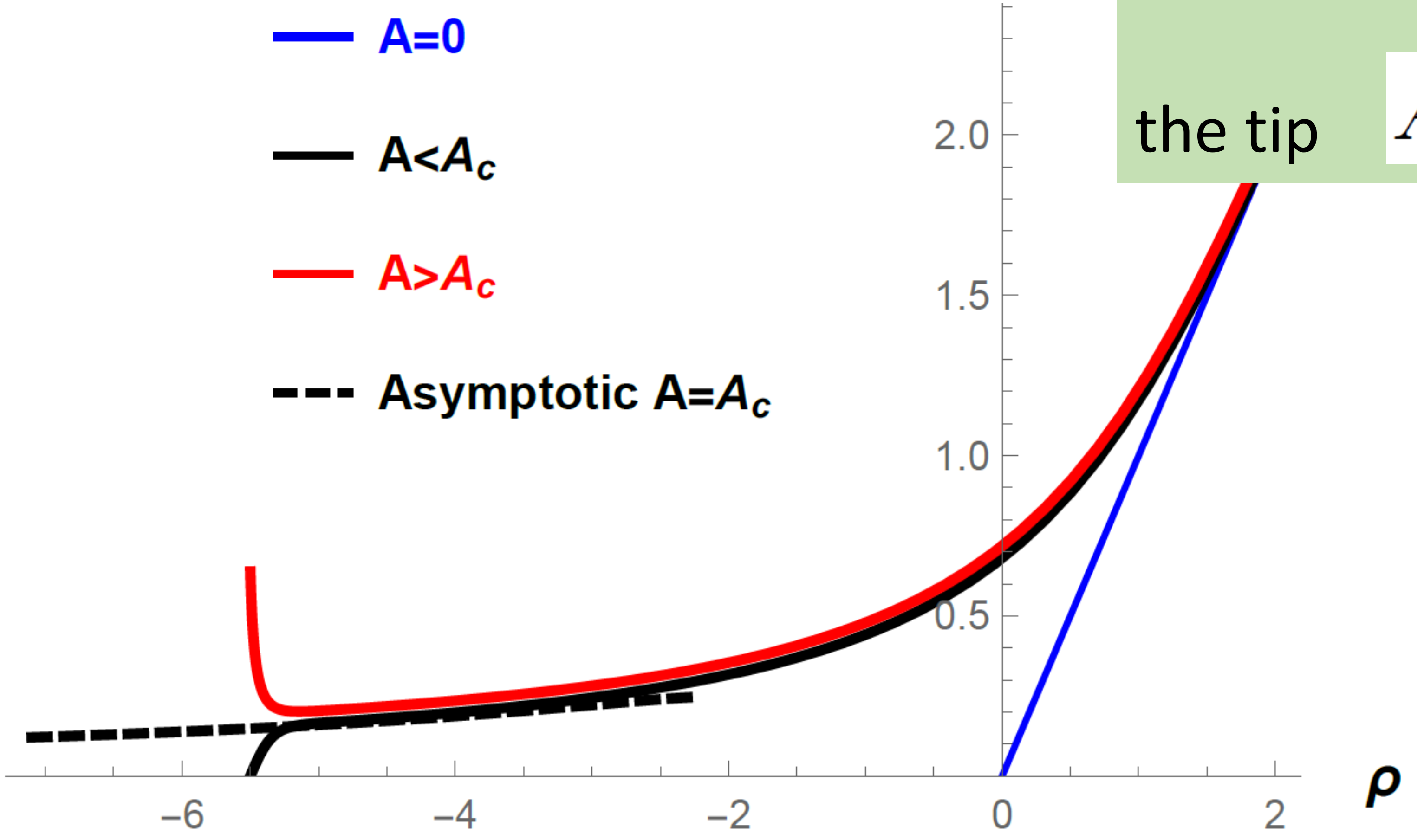

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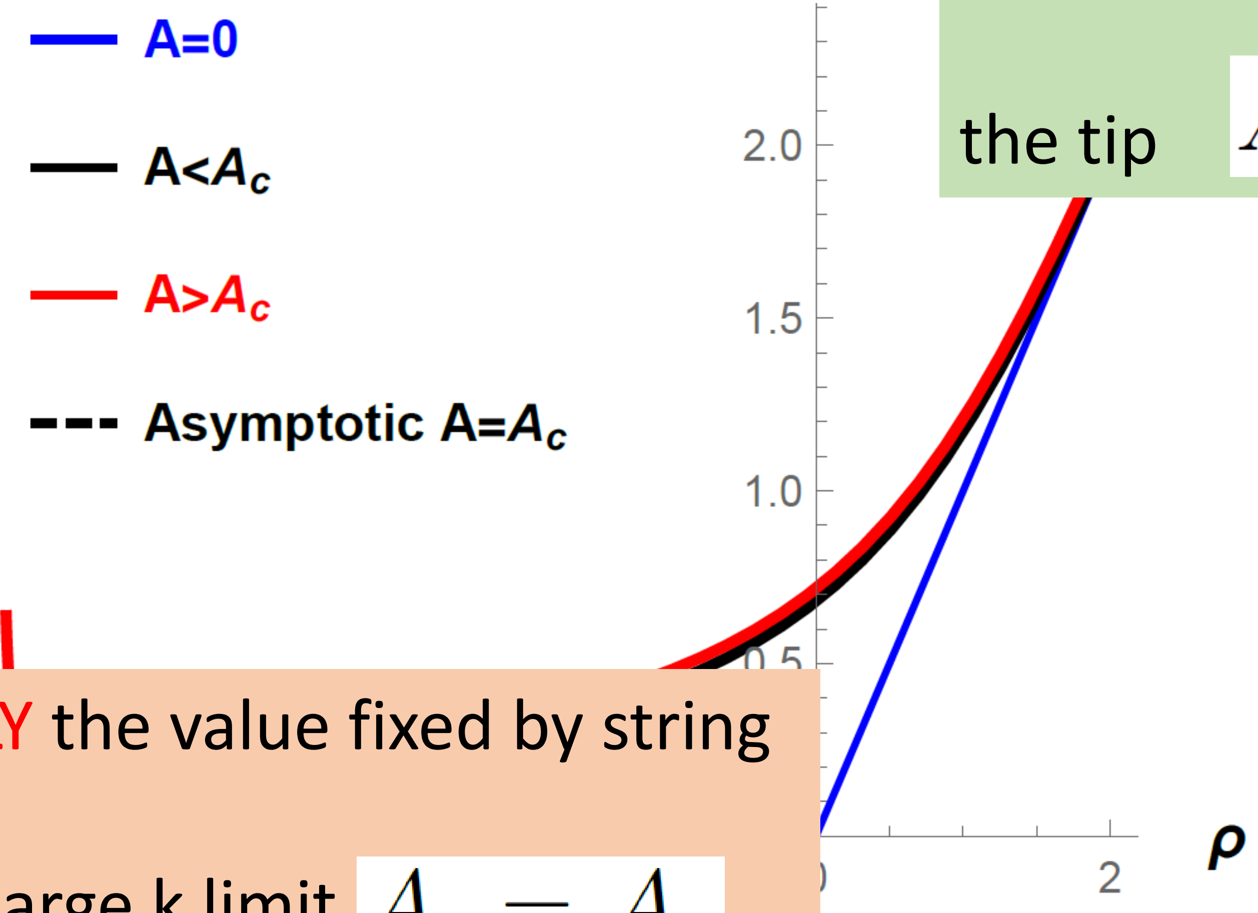

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$$\lim_{k\to\infty} (\Gamma(1+1/k))^k = \exp(-\gamma)$$

A comment:

This puncture removes the index obstruction in relating, in type II, HP with BH discussed recently by Chen Maldacena and Witten

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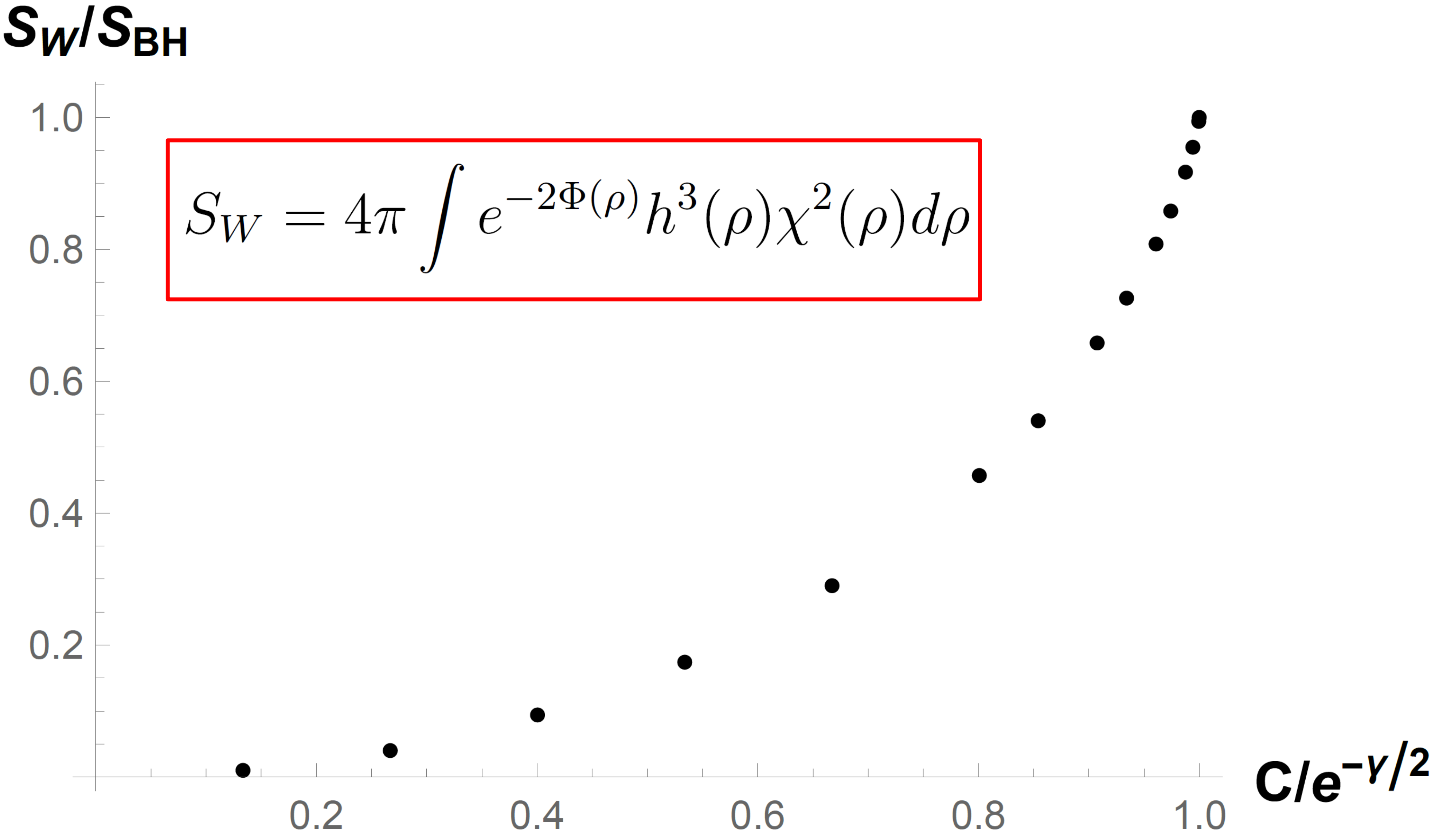
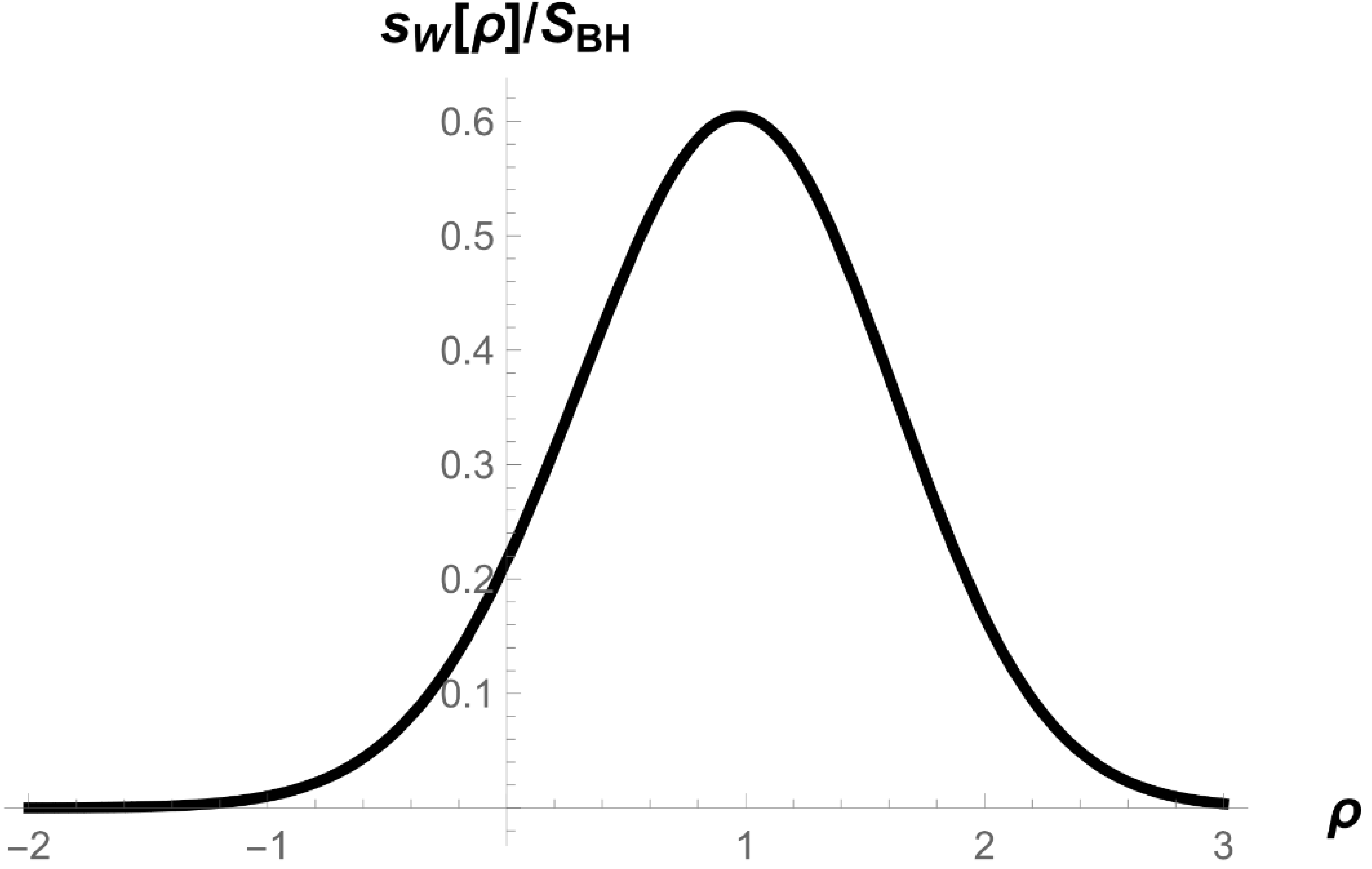


$$S_W = 4\pi \int e^{-2\Phi(\rho)} h^3(\rho) \chi^2(\rho) d\rho$$


$s_W[\rho]/S_{BH}$
0.6
0.5
0.4
0.3
0.2
0.1
−2
−1
1
2
3
ρ

Now we would like to argue that the information escapes the black hole through this puncture.

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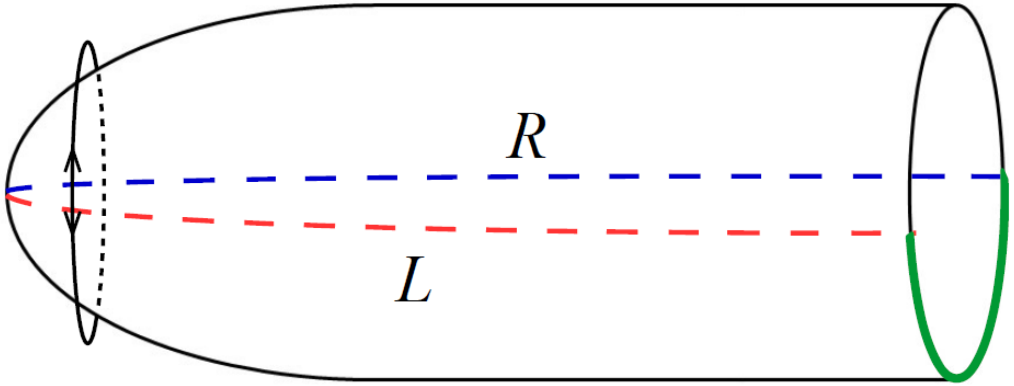

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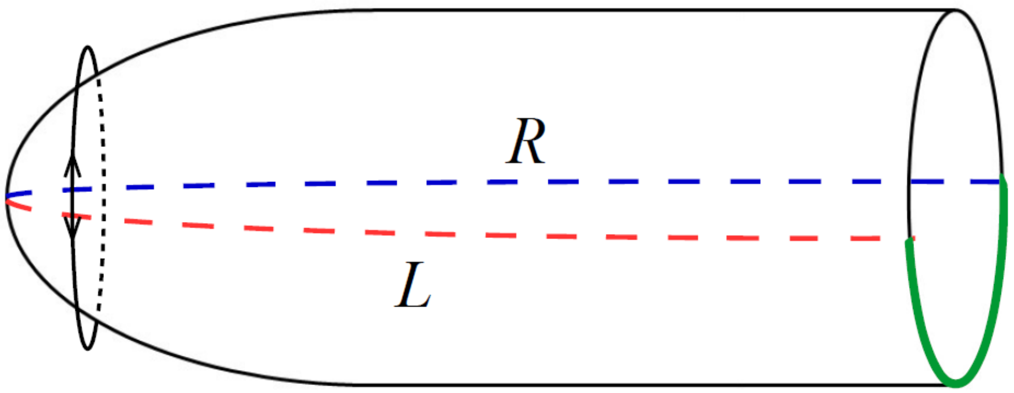

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A puncture at the tip means that we get an eternal BH without

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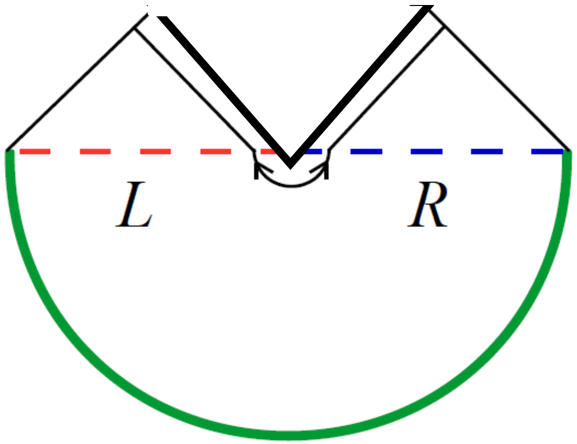

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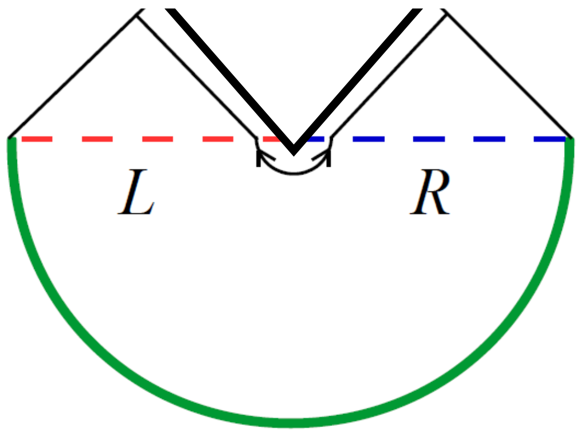

The wick rotation of the winding mode should have such a dramatic backreaction.

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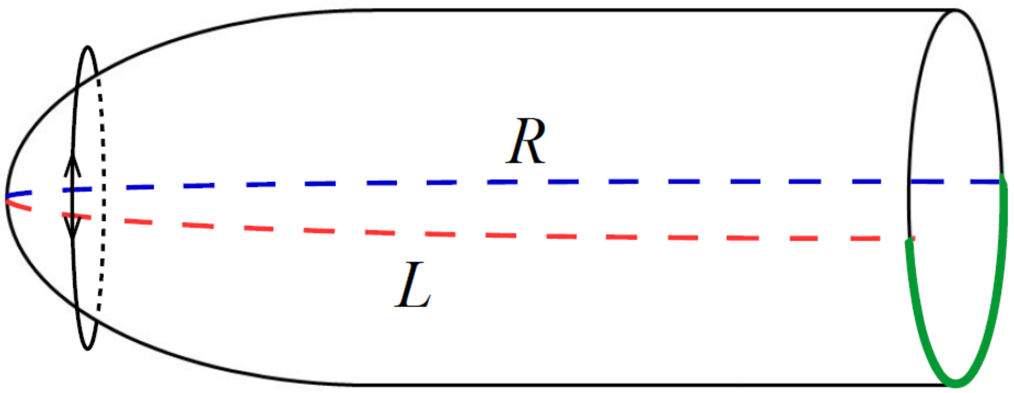

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2. What is the effect of the winding mode on the cigar?

In short the answer is that the winding mode becomes an

Instant Folded String (IFS).

Two arguments for IFS:

1. Uses the exact $SL(2,R)_k/U(1)$ CFT.

2. Effective description.

From the world sheet point of view a key observation was made by Giribet and Nunez :

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The non-perturbative effects can be described also by an operator that we refer to as F .

A rough way to think about is as $$F \sim W^{+} * W^{-}$$

Where $$\chi \sim W^{+} + W^{-}$$

The resolution is hidden in a observation made by Giribet and Nunez :

The non-perturbative effects can be described also by an operator that we refer to as F .

A rough way to think about is as $F \sim W^{+} * W^{-}$

Where $\chi \sim W^{+} + W^{-}$

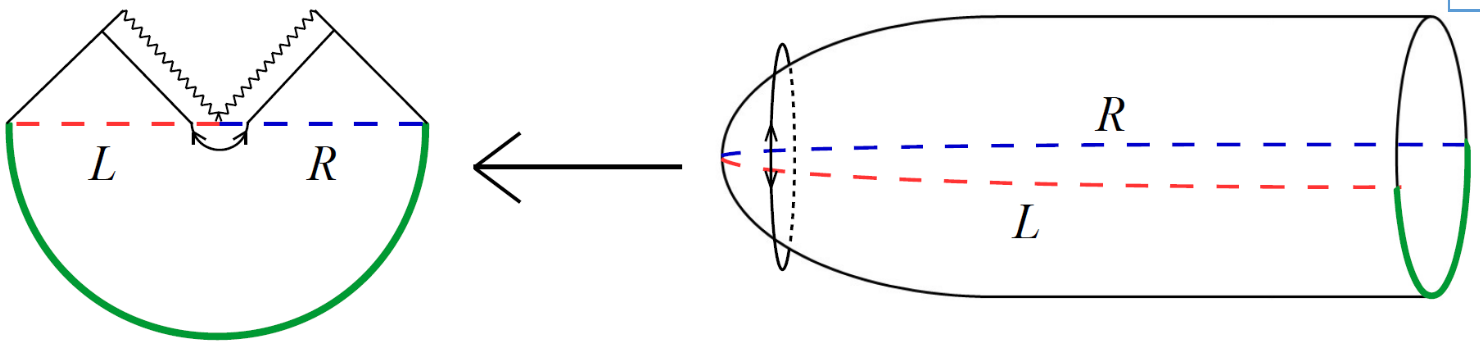
It is clear that F, unlike SL, is mutually local with ordinary exciations.

$F \sim W^{+} * W^{-}$ Target space (A. Giveon, NI)

Related work by

Jafferis and Schneider

Instead of



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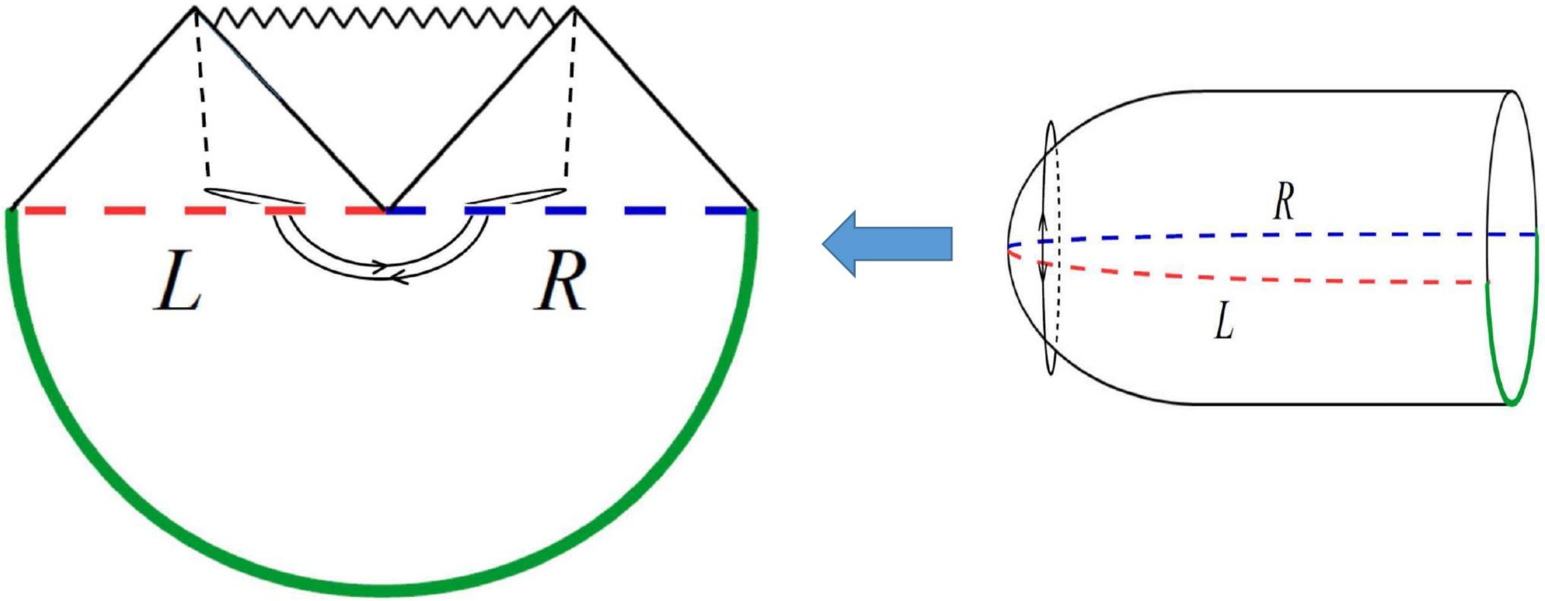

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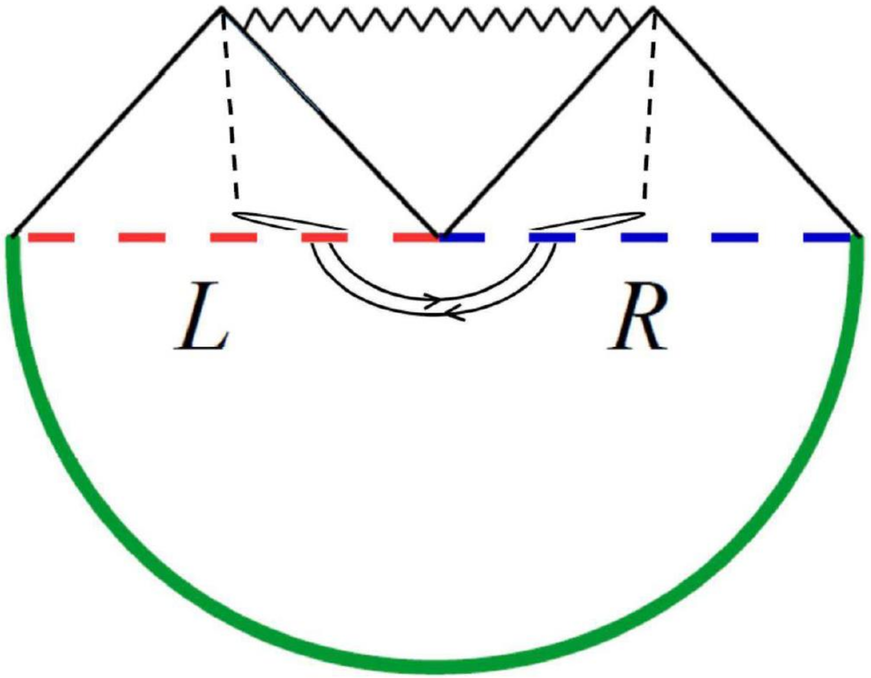

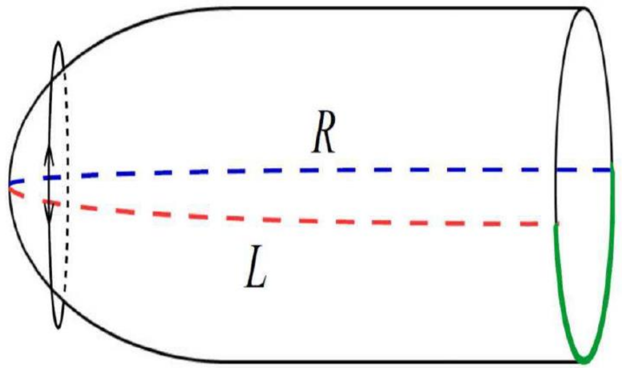

F is a folded string that fills the BH.

This sounds very strange: why are there folded strings inside the BH?

Can we understand this from low energy perspective?

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Can we understand this from low energy perspective?

How many folded strings are there?

What do they do to the black hole interior?

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3. It also leads to a non-perturbative solution (Maldacena)

New classical (non-perturbative) solutions that describe long strings

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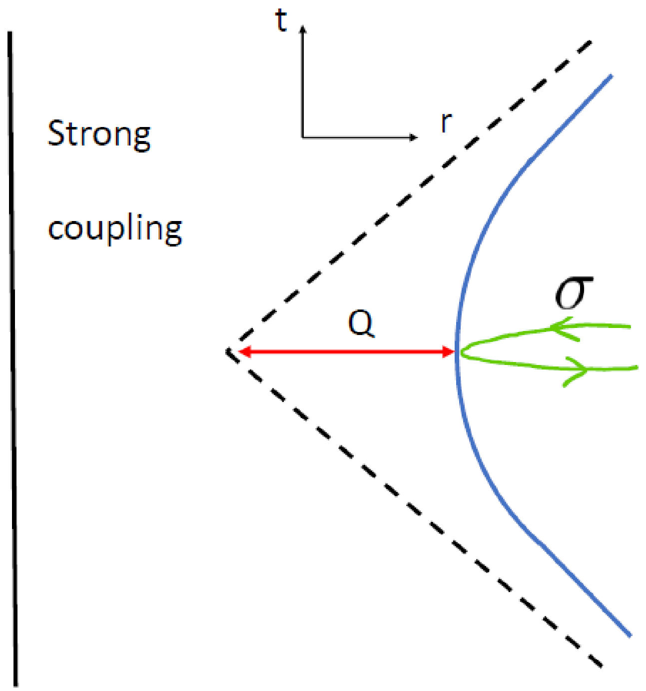

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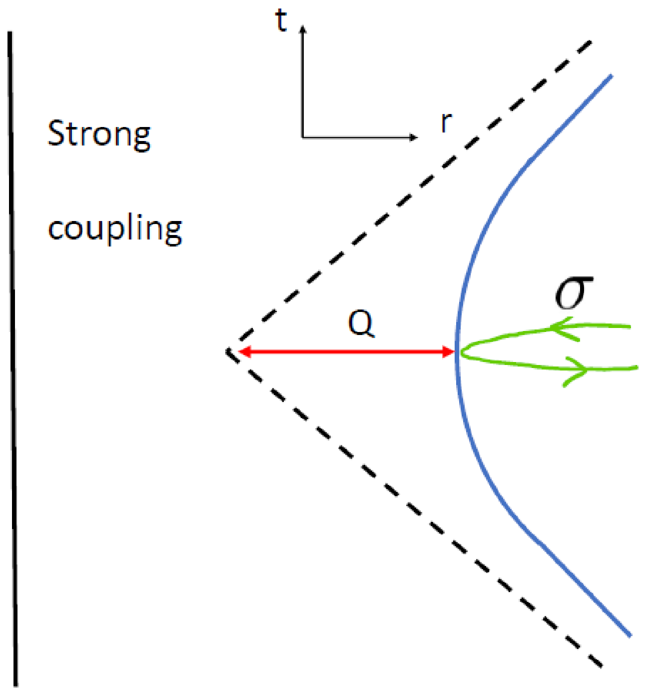

- The string can fold only away from the singularity.
- The length scale is tiny, of the order of Q .

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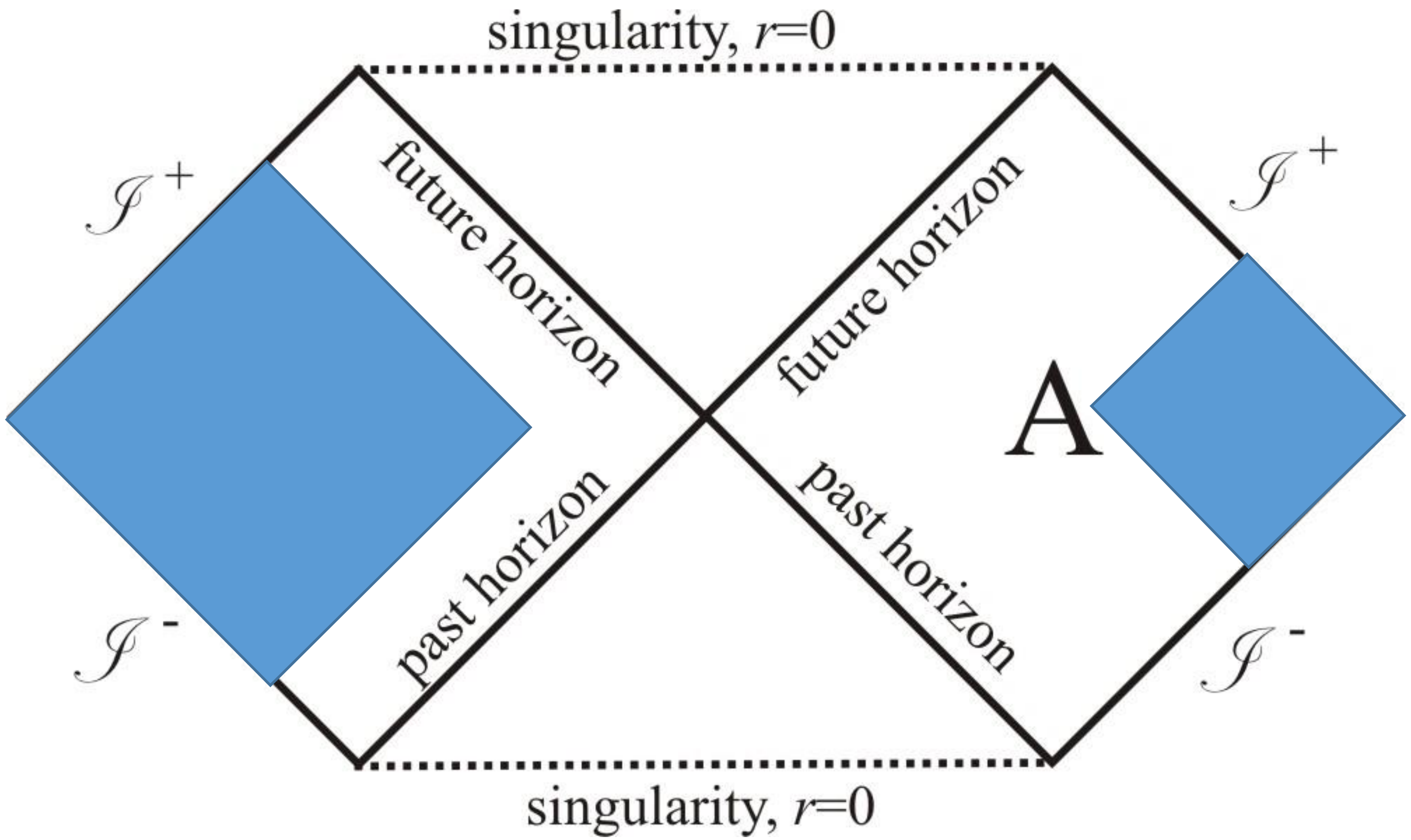

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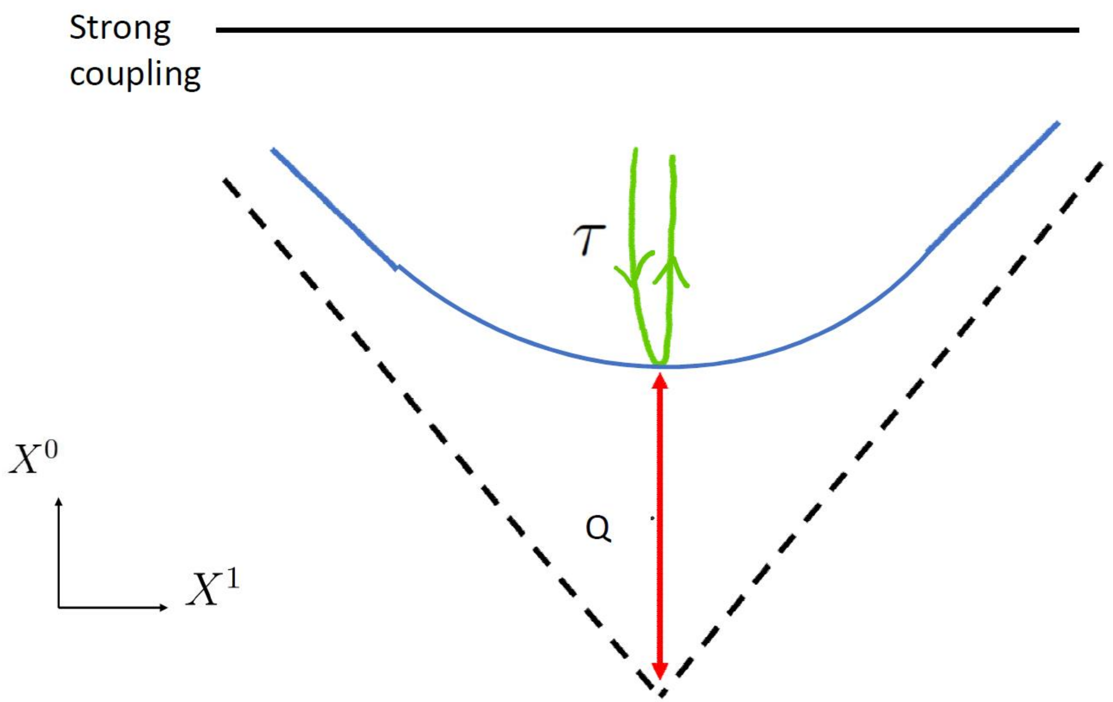

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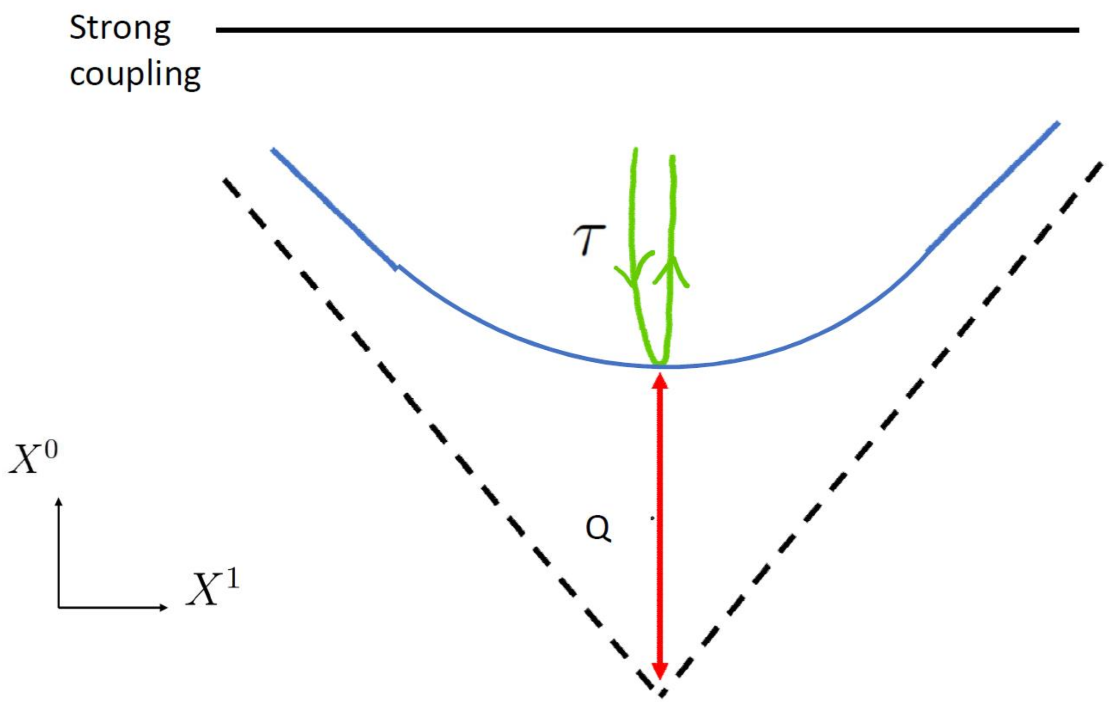
$\frac{dX^0}{d\tau}$ flips sign



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$\frac{dX^0}{d\tau}$ flips sign

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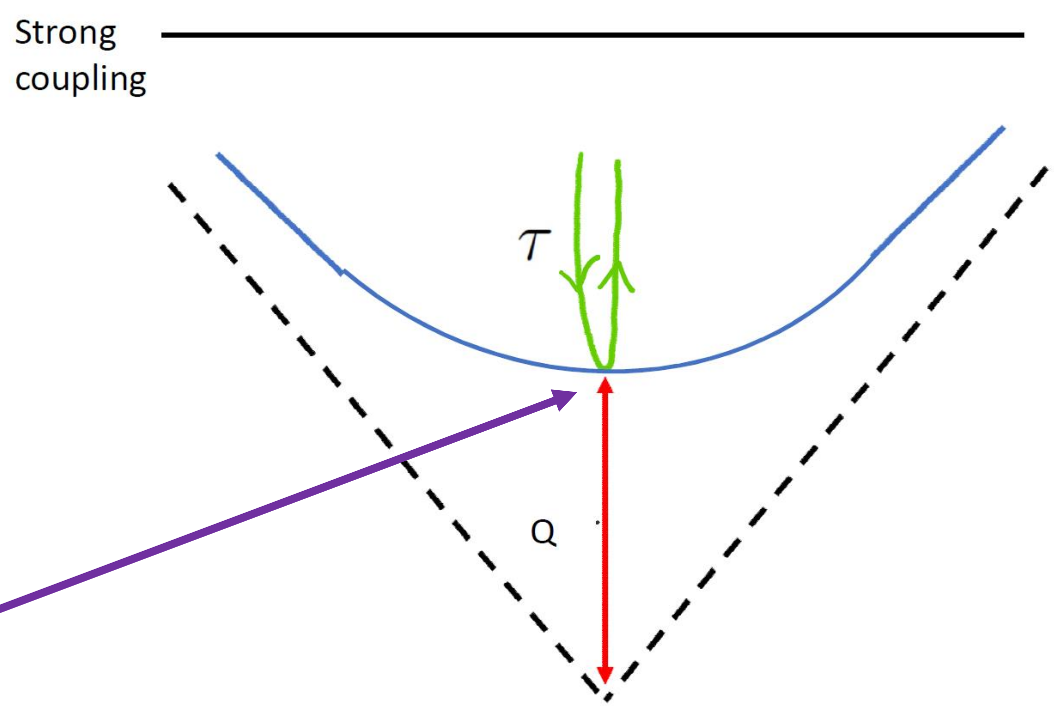
classically created at



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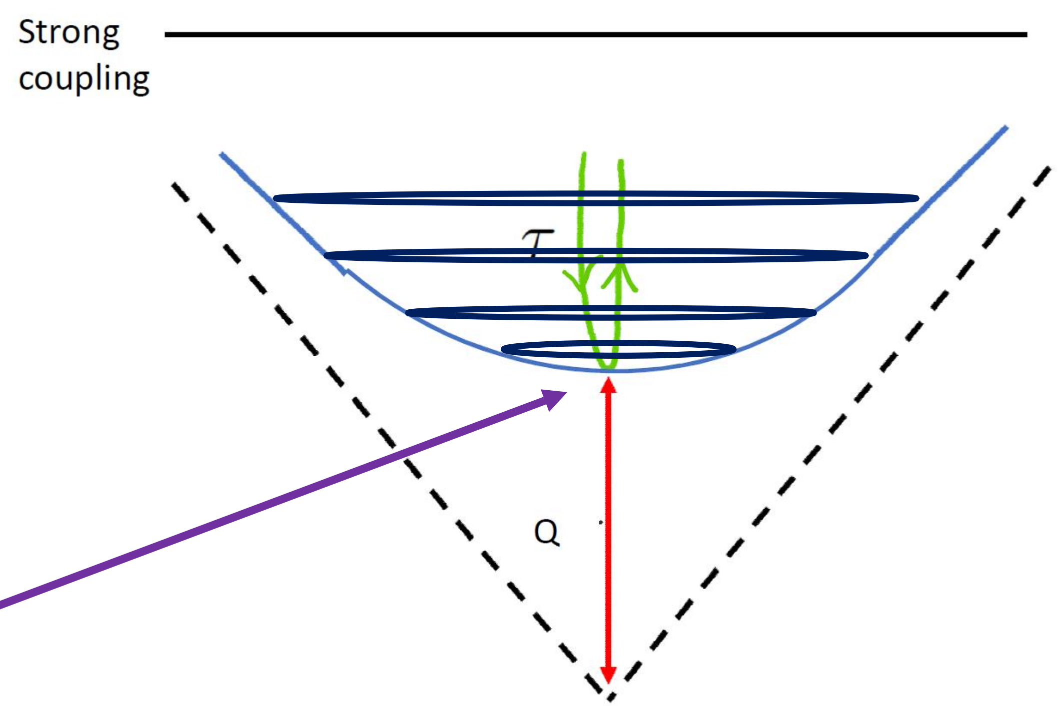

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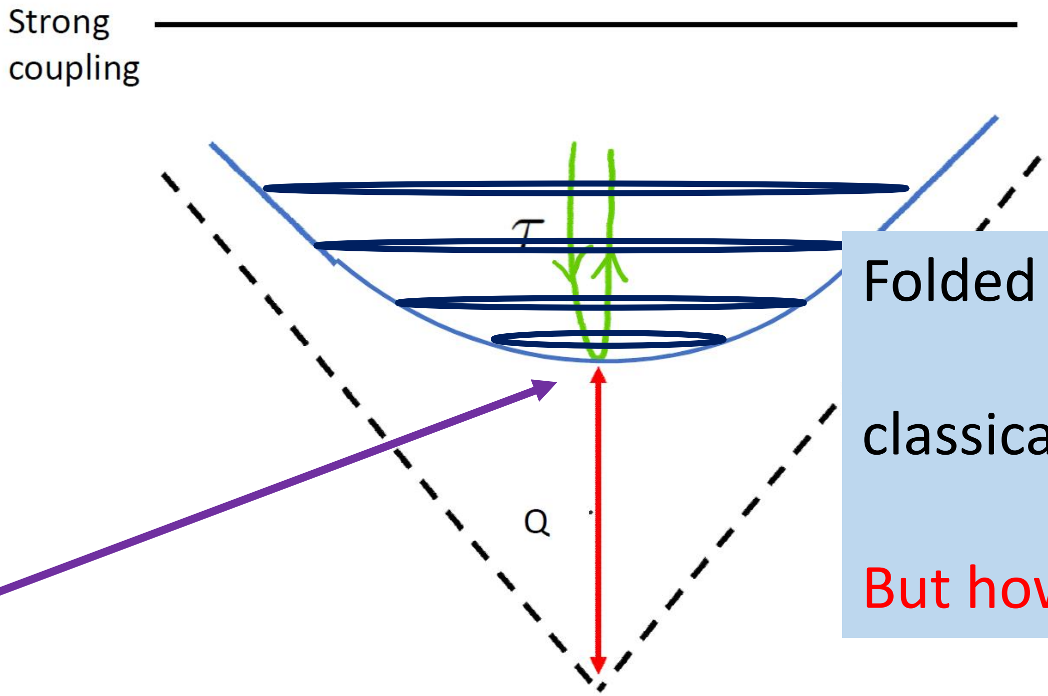
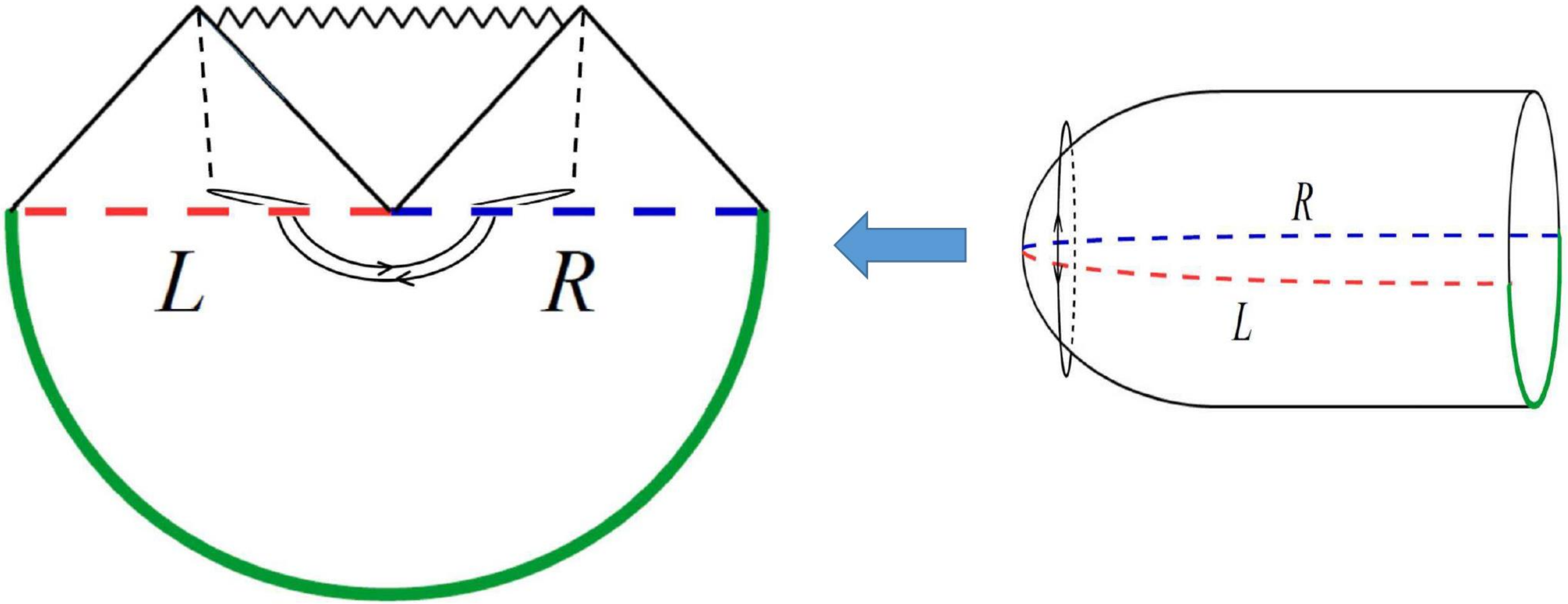
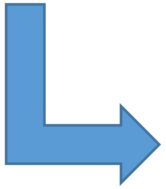


Folded strings are created classically behind the horizon.

But how many?

$F \sim W^{+} * W^{-}$ Target space





F is a folded string that fills the BH.

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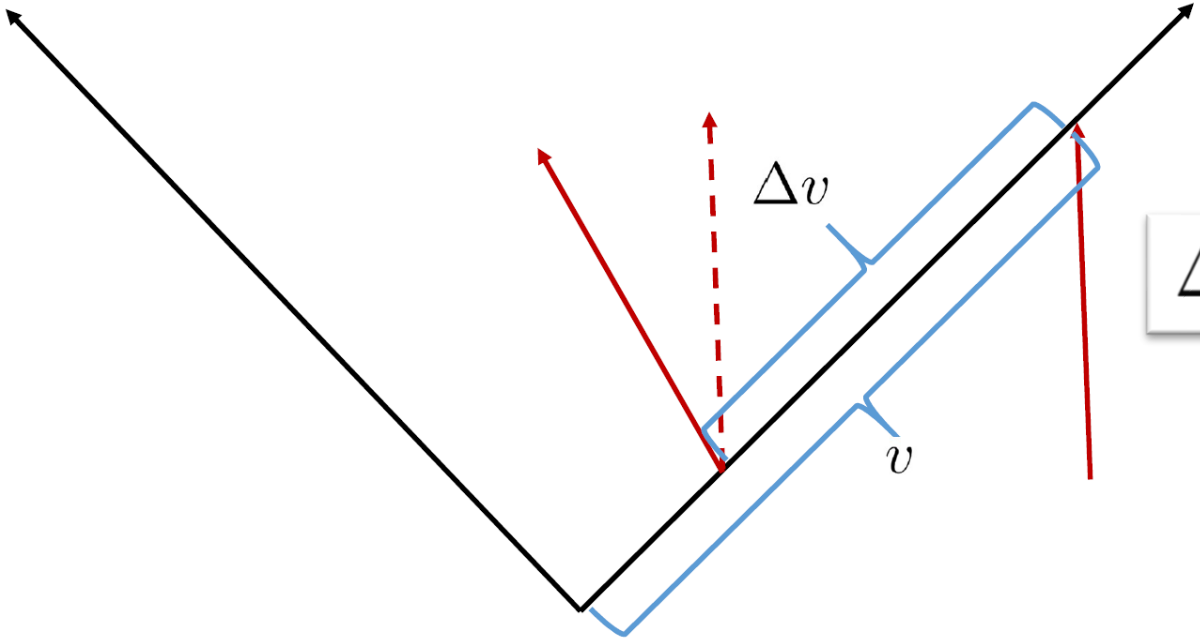

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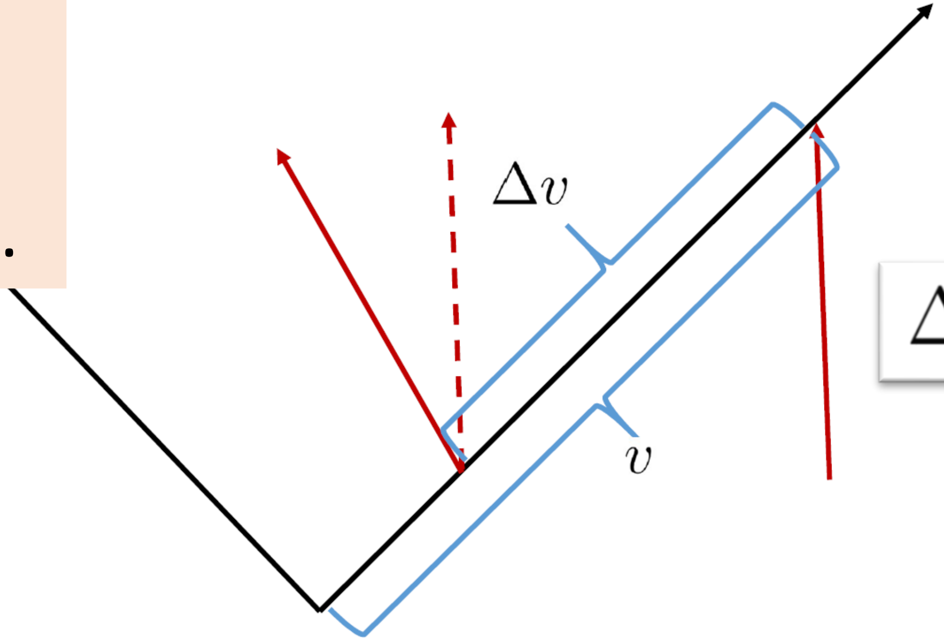

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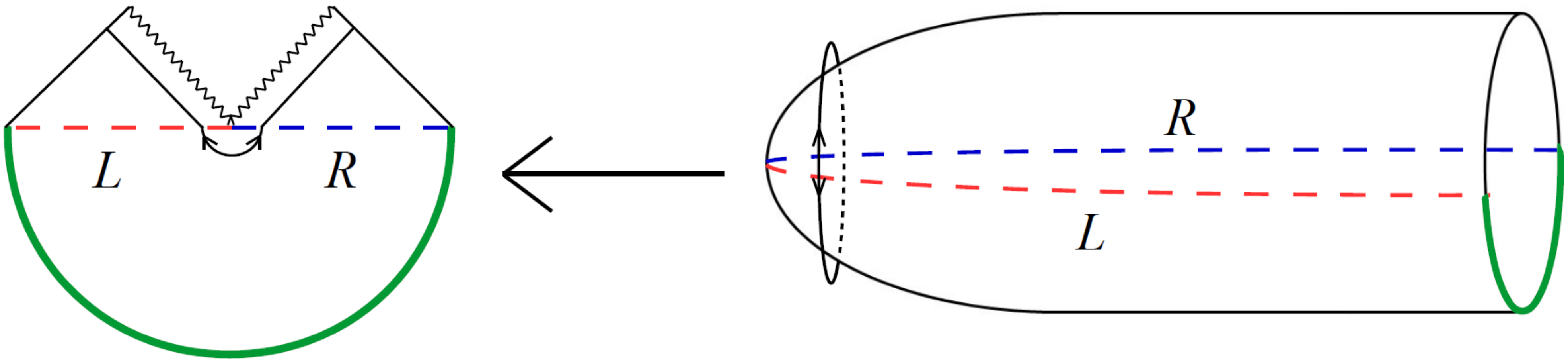


Three ways to calculate the number of IFS behind the BH: (Giveon, NI, Peleg):

1. Their backreaction renders the dilaton constant.
2. Their backreaction prevents falling to the BH.
3. "Entropy" considerations.

Low energy effective action (1, 2)

Exact CFT (3)

All 3 give the exact same answer:

$$N_{IFS} = \frac{2\pi}{kg_s^2}$$

Conclusions:

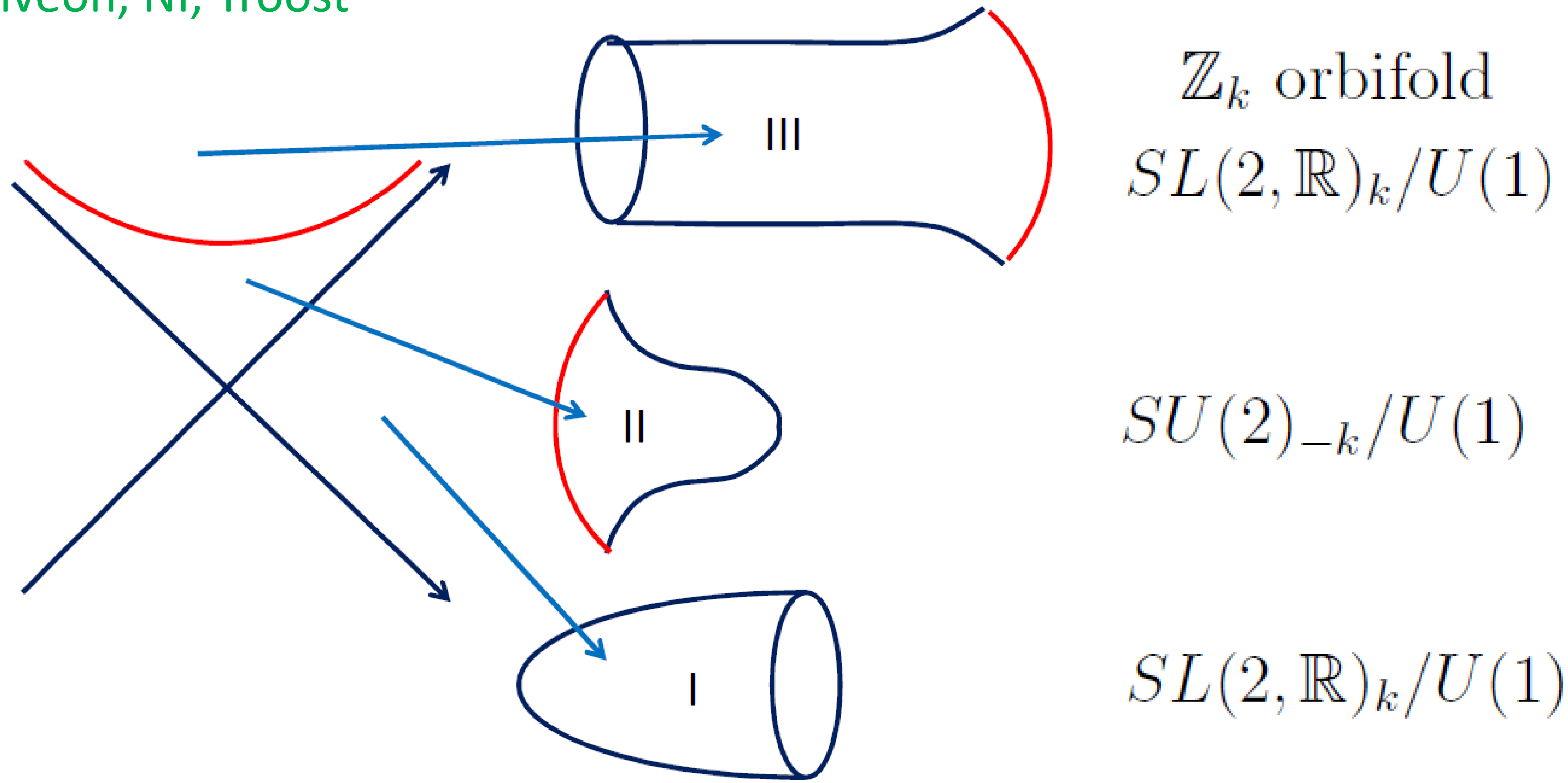
At least in the SL(2)/U(1) case string theory includes excitations that alter drastically GR physics:

Winding modes at the tip of a large BH.

IFS at the horizon of a large BH.

Thank you

Giveon, NI, Troost



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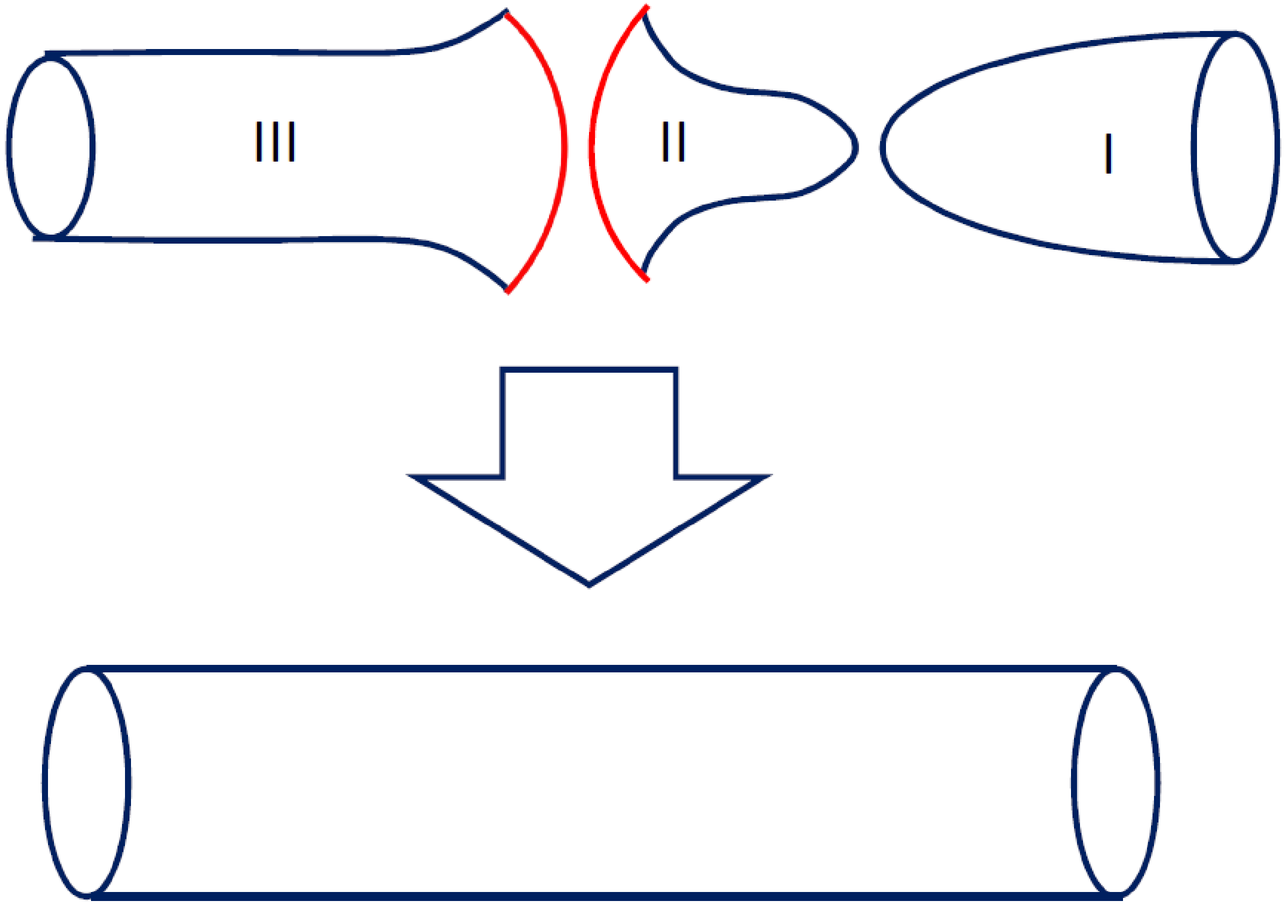

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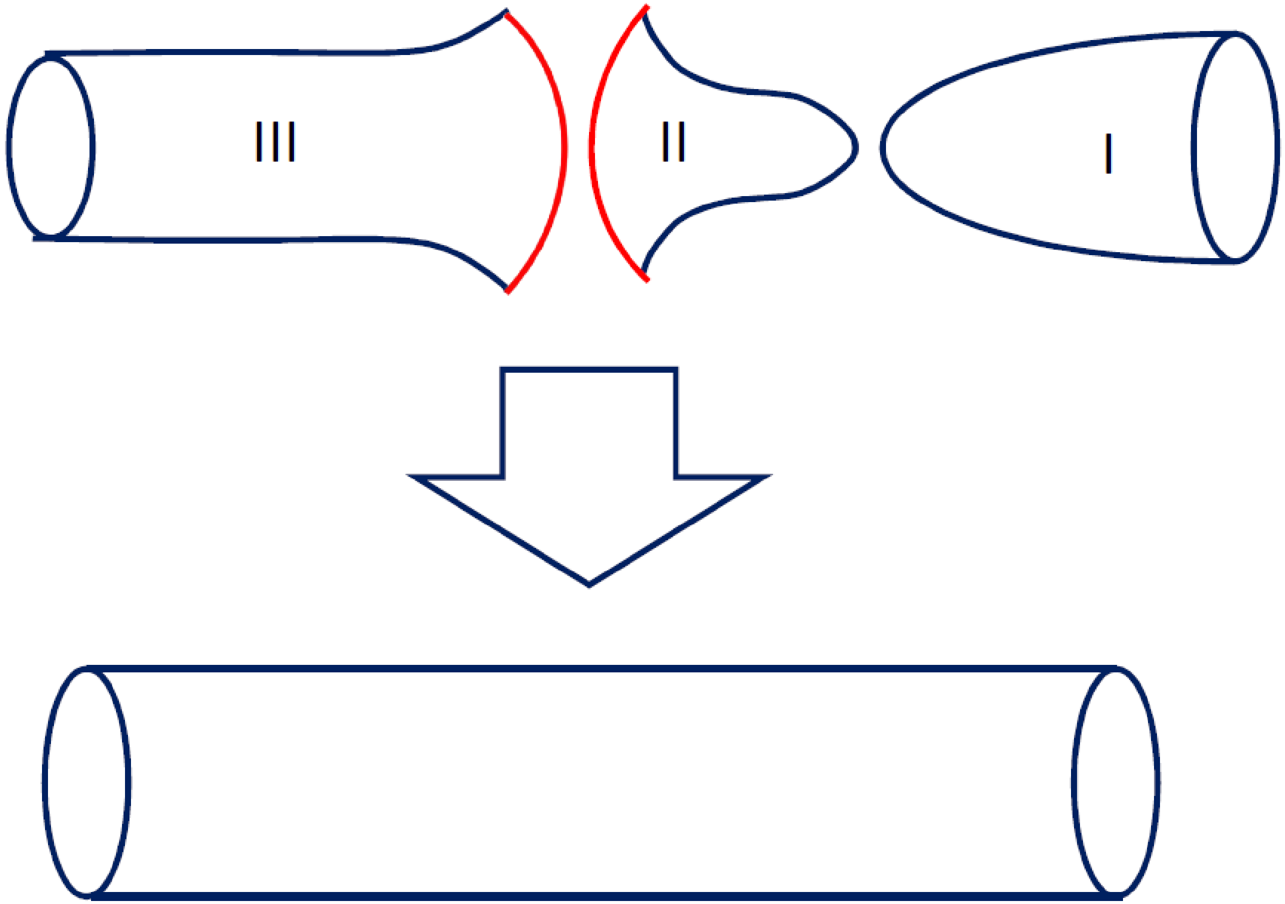
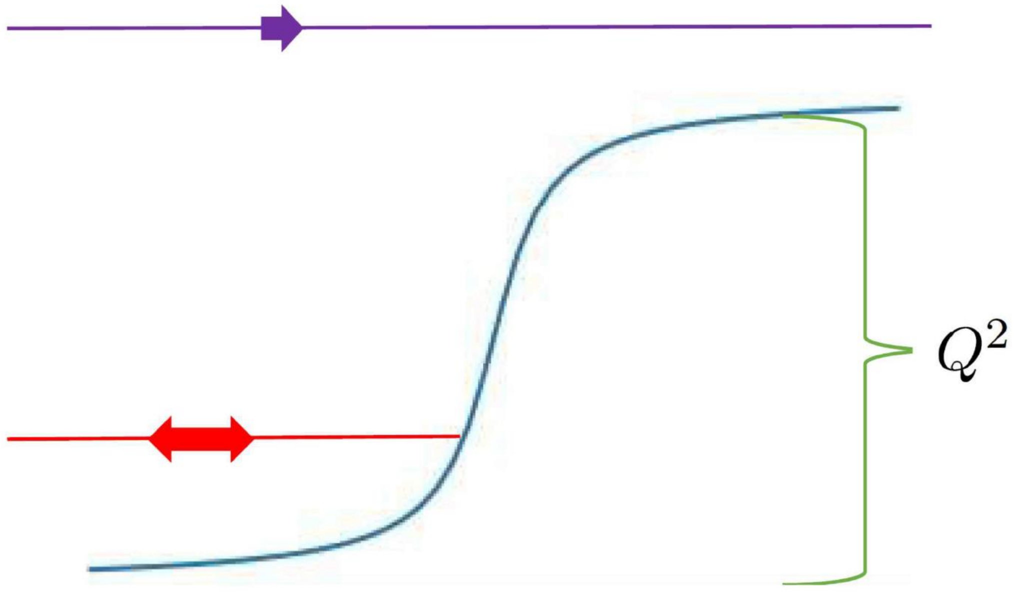


The glue is the

winding mode

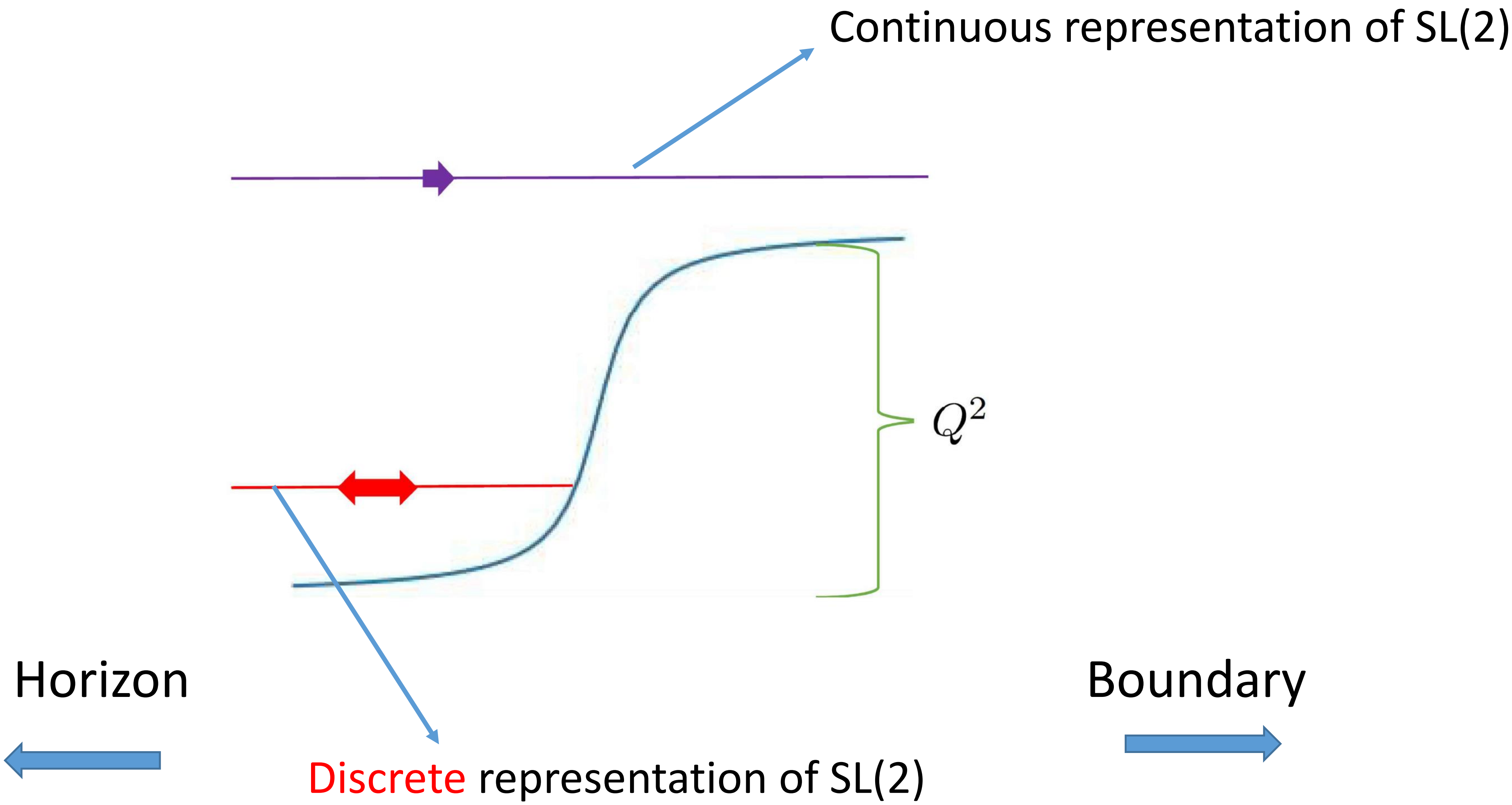
What about excitations of the BH?



Horizon

Boundary

What about excitations of the BH?



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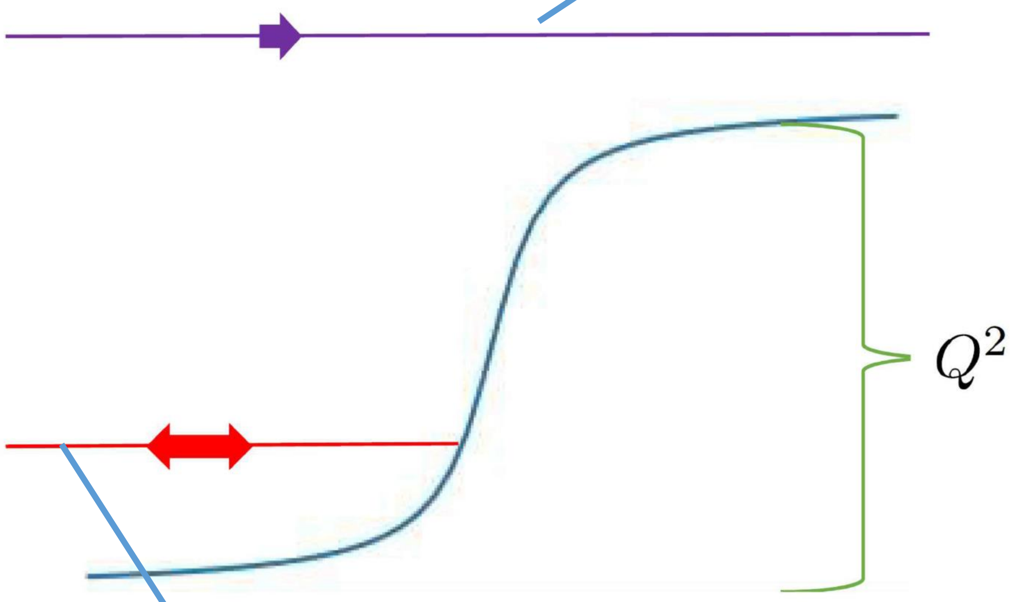
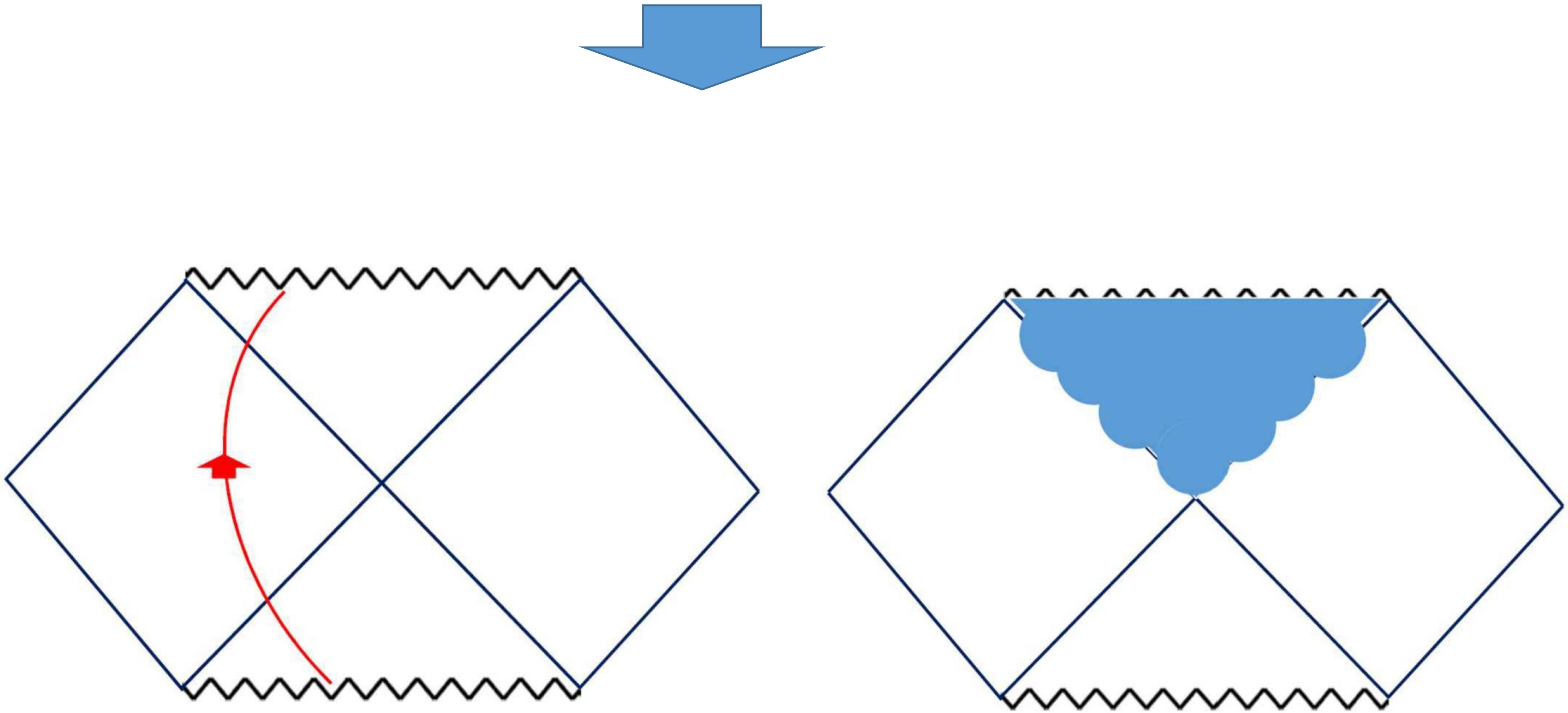
Continuous representation of SL(2)



Discrete representation of SL(2)

Reason: every state in the discrete rep. of AdS_3 comes with a partner determined by the spectral flow discussed by Maldacena and Ooguri .



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Can we understand this from low energy perspective?

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