Colour–kinematic duality, double copy and homotopy algebras

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Poincaré doesn't meddle with Lie

- We consider Yang–Mills (YM) theory

$$S = -\frac{1}{4}\int \mathrm{d}^d x\, F^{a\mu\nu} F^a_{\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{bc}{}^a A^b_\mu A^c_\nu$$

- A^a_μ has two indeces with a very different meaning: gauge index a (internal symmetry), Lorentz index μ (spacetime symmetry)

- Coleman–Mandula theorem says that is impossible to combine internal symmetries and spacetime symmetries in any but a trivial way

Poincaré doesn't meddle with Lie

What is the most general symmetry algebra $\mathfrak{S}$ of a QFT that leaves its S matrix invariant?

Coleman–Mandula theorem (1967)

If we consider a theory with:

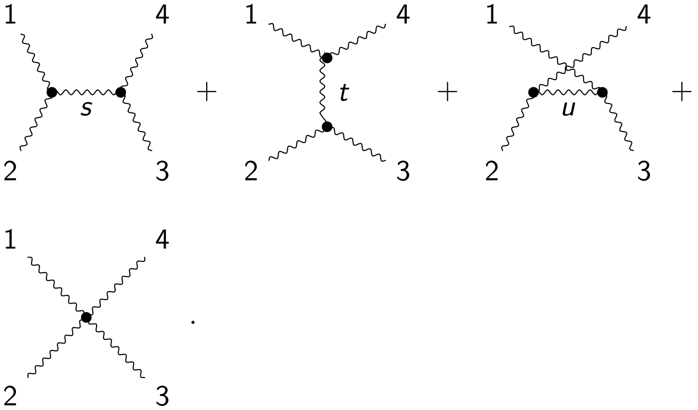
- $\mathfrak{S}$ containing Poincaré algebra $\mathfrak{p}$
- finite number of particles with mass less than M, for every $M > 0$
- nontrivial S matrix that is an analytic function of s and t
- *other technical assumptions*

then $\mathfrak{S} = \mathfrak{p} \oplus \mathfrak{g}$, with $\mathfrak{g}$ a Lie algebra

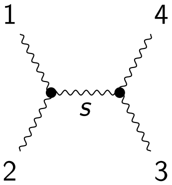
Flash review of CK duality and double copy

In YM theory, the tree-level scattering amplitude for four gluons is given by

$\mathcal{A}_{4,0} =$

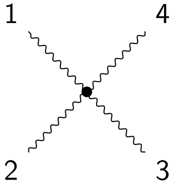


Flash review of CK duality and double copy



$$= -ig^2 \frac{f^{abe} f_e{}^{cd} n_s}{s} = -ig^2 \frac{c_s n_s}{s} \,,$$

and analogous expressions for t-channel ($-ig^2 \frac{f^{aed} f_e{}^{bc} n_t}{t} = -ig^2 \frac{c_t n_t}{t}$) and u-channel ($-ig^2 \frac{f^{aec} f_e{}^{db} n_u}{u} = -ig^2 \frac{c_u n_u}{u}$). We can blow up the quartic vertex into trivalent contributions, and distribute them into s-, t-, u-channel



$$= -ig^2 \left(c_s n_s^{(4)} - c_t n_t^{(4)} - c_u n_u^{(4)} \right)$$

Flash review of CK duality and double copy

$$\mathcal{A}_{4,0} = -ig^2 \frac{c_s n'_s}{s} - ig^2 \frac{c_t n'_t}{t} - ig^2 \frac{c_u n'_u}{u}$$

$$n'_s = n_s + s n_s^{(4)} , \quad n'_t = n_t - t n_t^{(4)} , \quad n'_u = n_u - u n_u^{(4)}$$

- Colour Jacobi identity

$$c_s - c_t - c_u = 0$$

implies that $\mathcal{A}_{4,0}$ is invariant under

$$n'_s \mapsto n'_s - s\alpha , \quad n'_t \mapsto n'_t + t\alpha , \quad n'_u \mapsto n'_u + u\alpha$$

- Kinematic Jacobi identity

$$n'_s - n'_t - n'_u = 0$$

Flash review of CK duality and double copy

- In general, we can write n-points L-loops YM amplitude as sums of trivalent graphs

$$\mathcal{A}^{\text{YM}}_{n,L} = \sum_i \int \prod_{l=1}^{L} \mathrm{d}^d p_l \frac{1}{S_i} \frac{C_i N_i}{D_i}$$

- i ranges over all trivalent L-loops graphs
- C_i: colour factor, composed of gauge group structure constants
- N_i: kinematic factor, composed of Lorentz-invariant contractions of polarisations and momenta

Flash review of CK duality and double copy

- Generalised gauge transformation

$$N_i \;\mapsto\; N_i + \Delta_i, \qquad \sum_i \int \prod_{l=1}^{L} \mathrm{d}^d p_l \frac{1}{S_i} \frac{C_i \Delta_i}{D_i} \;=\; 0$$

Bern–Carrasco–Johansson colour–kinematic duality (2008)

There is a choice of kinematic factors such that N_is obey the same algebraic relations (e.g., Jacobi identity) of the correspondent C_i

- True at tree-level, conjectured for loop-level
- If true, it would allow us to compute gravity amplitudes from YM ones

Flash review of CK duality and double copy

Yang–Mills double copy

If BCJ duality holds true, replacing the colour factor with a copy of the kinematic factor in $\mathcal{A}_{n,L}^{\mathrm{YM}}$ produces a $\mathcal{N} = 0$ supergravity amplitude

$$\mathcal{A}_{n,L}^{\mathrm{YM}} = \sum_i \int \prod_{l=1}^{L} \mathrm{d}^d p_l \frac{1}{S_i} \frac{C_i N_i}{D_i} \rightarrow \mathcal{A}_{n,L}^{\mathcal{N}=0} = \sum_i \int \prod_{l=1}^{L} \mathrm{d}^d p_l \frac{1}{S_i} \frac{\tilde{N}_i N_i}{D_i}$$

- All-loop statement, the problem is then to validate BCJ duality at loop level
- Until now, on-shell scattering amplitude approach: an off-shell Lagrangian realisation of colour-kinematic duality and double copy could solve the all-loop conundrum

First steps toward Lagrangian double copy

There exist a non-local YM Lagrangian with manifest tree-level BCJ duality for on-shell physical gluons (Bern–Dennen–Huang–Kreimeier '10, Tolotti–Weinzierl, '13)

$$\mathscr{L}^{\mathsf{YM}} = \mathscr{L}_2 + \mathscr{L}_3 + \cdots$$

with

$$\mathscr{L}_n = \sum_{\Gamma \in \mathrm{Tree}_{3,n}} O_{n,\Gamma}^{\mu_1 \cdots \mu_n} \frac{\mathrm{tr}\left\{ [A_{\mu_{\sigma(1)}}, A_{\mu_{\sigma(2)}}] [\ldots [A_{\mu_{\sigma(3)}}, A_{\mu_{\sigma(4)}}] \ldots, A_{\mu_{\sigma(n)}}] \right\}}{\Box_{j_{n,\Gamma,1}} \cdots \Box_{j_{n,\Gamma,n-3}}}$$

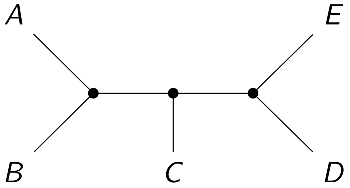
For $n \geq 5$, $\mathscr{L}_n = 0$ upon imposing Jacobi identity. For example

$$\mathscr{L}_5 \sim \mathrm{tr}\left\{ [A^\nu, A^\rho] \frac{1}{\Box} \left(\left[[\partial_\mu A_\nu, A_\rho], \frac{\Box}{\Box} A^\mu \right] + \right.\right.$$
$$\left.\left. + \left[[A_\rho, A^\mu], \frac{\Box}{\Box} \partial_\mu A_\nu \right] + \left[[A^\mu, \partial_\mu A_\nu], \frac{\Box}{\Box} A_\rho \right] \right) \right\}$$

First steps toward Lagrangian double copy

- Every perturbative Lagrangian field theory is equivalent to a theory with only cubic interactions (*strictification*)

$$\mathrm{f}_{ab}{}^{f}\mathrm{f}_{fcg}\mathrm{f}_{de}{}^{g}A^aB^bC^cD^dE^e \quad \Longleftrightarrow$$



- A quintic interaction term $\mathrm{f}_{ab}{}^{f}\mathrm{f}_{fcg}\mathrm{f}_{de}{}^{g}A^aB^bC^cD^dE^e$ can be strictified inserting auxiliary fields, and it is equivalent to

$$\bar{G}_{1a}G_1^a + \bar{G}_{2a}G_2^a + \mathrm{f}_{ab}{}^{f}A^aB^b\bar{G}_{1f} + \mathrm{f}_{fcg}G_1^fC^cG_2^g + \mathrm{f}_{de}{}^{g}\bar{G}_{2g}D^dE^e$$

First steps toward Lagrangian double copy

- We want to strictify non-local terms of the form

$$E_1^M \frac{1}{\Box} E_M^2$$

where E_1 and E_2 are polynomials in fields and their derivatives and M is a multi-index

- Inserting auxiliary fields, this is equivalent to

$$-G^M \Box \bar{G}_M + G^M E_M^2 + E_1^M \bar{G}_M$$

First steps toward Lagrangian double copy

- We can insert auxiliary fields to make TW action local, and strictify to a Lagrangian with only cubic interactions

$$\mathscr{L} = \frac{1}{2}\Phi^{\alpha i} g_{\alpha\beta} G_{ij} \Box \Phi^{\beta j} + \frac{1}{3!}\Phi^{\alpha i} f_{\alpha\beta\gamma} F_{ijk} \Phi^{\beta j} \Phi^{\gamma k}$$

- We plan to double copy the BRST-extended field space, but CK duality is satisfied only on-shell for physical gluons: eventual CK violations due to unphysical gluons and ghosts!

- We can compensate for these eventual CK violations with suitable field redefinitions and gauge choice

- Batalin–Vilkovisky (BV) formalism allows us to work with very general gauge choices, and homotopy algebras provide a natural setting for colour-kinematic factorisation and Lagrangian double copy

Homotopy algebras

- Informally, homotopy algebras are generalizations of classical algebras (e.g., associative, Lie) where the respective structural identities (e.g., associativity, Jacobi identity) hold up to homotopies

Classical algebra	Homotopy algebra
Associative algebra	A_∞-algebra
Associative commutative algebra	C_∞-algebra
Lie algebra	L_∞-algebra

- Homotopy structures are ubiquitous in Physics: while homotopy algebras emerged in the context of string field theory, they were later recognized as underlying structures of every Lagrangian field theory

BV formalism: motivations

- Batalin–Vilkovisky (BV) formalism is the bridge between (quantum) field theories and homotopy algebras
- To quantize a classical theory means to make sense of the path integral

$$\int_{\mathfrak{F}} \mu_{\mathfrak{F}}(\Phi)\ \mathrm{e}^{\frac{i}{\hbar}S[\Phi]}$$

- Standard approach: BRST formalism
- If the symmetries close off-shell, then BRST formalism is enough for quantization

BV formalism

- In the case of open symmetries, BRST complex is a complex only up to e.o.m.
- The BV quantisation is a sophisticated machinery, that allows us to gauge-fix and quantize these complicated field theories
- We extend the BRST complex, doubling the field content of the theory

$$\mathfrak{F}_{\mathrm{BV}} = T^*[1]\mathfrak{F}_{\mathrm{BRST}}$$

- Fields Φ^A are local coordinates on $\mathfrak{F}_{\mathrm{BRST}}$, antifields Φ^+_A are fibre coordinates. As a cotangent bundle, $\mathfrak{F}_{\mathrm{BV}}$ comes with a natural symplectic structure and Poisson brackets $\{-,-\}$

BV formalism

- We extend Q_{BRST} and S_{BRST} to Q_{BV} and S_{BV}, requiring

$$Q_{\text{BV}}|_{\mathfrak{F}_{\text{BRST}}} = Q_{\text{BRST}}, \quad Q_{\text{BV}} = \{S_{\text{BV}}, -\}$$

$$Q_{\text{BV}} S_{\text{BV}} = \{S_{\text{BV}}, S_{\text{BV}}\} = 0$$

  the latter is known as *BV master equation*

- The differential algebra of the BV formalism dualizes to a codifferential coalgebra, that can be equivalently described as an L_∞-algebra

BV formalism: gauge-fixing and quantization

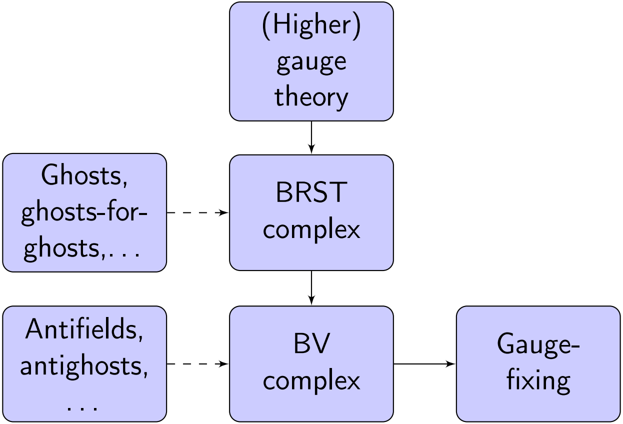
- Before quantization: imposing gauge-fixing in the BV formalism
- Gauge-fixing S_{BV} means evaluating it on an appropriate Lagrangian submanifold of $\mathfrak{F}_{\text{BV}}$
- We eliminate the antifields by introducing a gauge-fixing fermion Ψ:

$$\Phi_A^+ = \frac{\delta}{\delta \Phi^A}\Psi$$

- Gauge-independence of the expectation values for observables: BV quantum master equation

$$\{S_{\text{BV}}^{\hbar}, S_{\text{BV}}^{\hbar}\} - 2i\hbar\Delta_{\text{BV}}S_{\text{BV}}^{\hbar} = 0$$

BV formalism: summary



The dual picture: L_∞-algebras

- BRST and BV formalism introduce a differential Q on the graded commutative algebra of polynomial functions of fields, $\mathscr{C}^\infty(\mathfrak{F}[1])$
- This is an instance of an abstract geometrical construction, Q-vector spaces
- In the simplest case, we have an ordinary vector space $\mathfrak{g}$ with basis e^a, and the most general degree 1 differential acting on $\mathscr{C}^\infty(\mathfrak{g}[1])$ is

$$Q\xi^a \;=\; -\frac{1}{2}\mathrm{f}^a_{bc}\xi^b\xi^c,$$

  where the coordinate functions ξ^a are basis for $\mathfrak{g}^*$
- Requiring $Q^2 = 0$ is equivalent to require Jacobi identity for f^a_{bc}, i.e. that f^a_{bc} are the structure constant of a Lie algebra with bracket $[\mathrm{e}_b, \mathrm{e}_c] = \mathrm{f}^a_{bc}\mathrm{e}_a$

The dual picture: L_∞-algebras

- The differential algebra picture and the Lie algebra picture are easy to relate, introducing contracted coordinate functions

$$\mathsf{a} \;=\; \xi^a \otimes \mathsf{e}_a \;\in\; (\mathfrak{g}[1])^* \otimes \mathfrak{g}$$

$$Q\mathsf{a} \;=\; (Q\xi^a)\otimes\mathsf{e}_a \;=\; -\frac{1}{2}\mathsf{f}^a_{bc}\xi^b\xi^c\otimes\mathsf{e}_a \;=\; -\frac{1}{2}\xi^b\xi^c\otimes[\mathsf{e}_b,\mathsf{e}_c] \;=\; -\frac{1}{2}[\mathsf{a},\mathsf{a}]$$

- More general vector fields: we consider now the graded vector space $\mathfrak{F}_{\mathrm{BV}}$

$$\mathsf{a} \;=\; \Phi^I \otimes \mathsf{e}_I + \Phi^+_I \otimes \mathsf{e}^I \;\in\; (\mathfrak{F}_{\mathrm{BV}}[1])^* \otimes \mathfrak{F}_{\mathrm{BV}}$$

$$Q\mathsf{a} \;=\; -\sum_i \frac{1}{i!}\mu_i(\mathsf{a},\ldots,\mathsf{a})$$

- The multibrackets are multilinear, graded antisymmetric maps, called *higher products*

L_∞-algebras

$$Qa = -\sum_i \frac{1}{i!}\mu_i(a,\dots,a)$$

- Requiring $Q^2 = 0$ is equivalent to require that $(\mathfrak{F}_{\mathrm{BV}}, \mu_i)$ is an L_∞-algebra
- An L_∞-algebra is a graded vector space equipped with higher products, that satisfy a generalization of Jacobi identity
- Underlying every Lagrangian field theory is an L_∞-algebra that encodes the whole classical theory (symmetries, fields, equations of motion, Noether identities...)

Yang–Mills theory in the BV formalism

- Extend YM action with antifields (A^+, c^+) and trivial pairs (b, $\bar{c}^+$, b^+, $\bar{c}$)

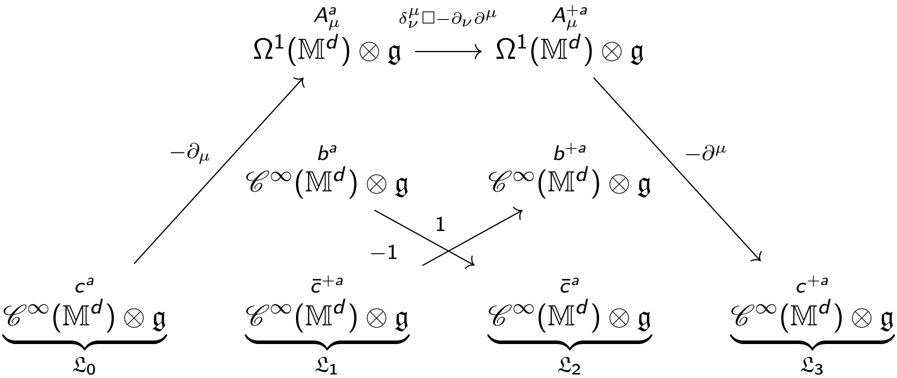
$$S_{\text{BV}}^{\text{YM}} = \int_{\mathbb{M}^d} \mathrm{d}^d x \left\{ -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + A^{+a}_{\mu} (\nabla^\mu c)^a + \tfrac{g}{2} f^a_{bc} c^{+a} c^b c^c + b^a \bar{c}^{+a} \right\}$$

- We can formulate YM theory as the Maurer–Cartan homotopy theory associated to a cyclic L_∞-algebra $(\mathfrak{L}, \mu_i, \langle -, - \rangle)$

$$S_{\text{MC}}[a] = \sum_{i \geq 1} \frac{1}{(i+1)!} \langle a, \mu_i(a, \dots, a) \rangle$$

Yang–Mills theory in the BV formalism

Chain complex (μ_1)



Yang–Mills theory in the BV formalism

- Other non-vanishing higher products

$$
\begin{aligned}
[\mu_2(A,c)]^a &= g f^a_{bc} c^b c^c \ , \quad [\mu_2(A,c)]^a_\mu = -g f^a_{bc} A^b_\mu c^c \\
[\mu_2(A^+,c)]^a_\mu &= -g f^a_{bc} A^{+b}_\mu c^c \ , \quad [\mu_2(c,c^+)]^a = g f^a_{bc} c^b c^{+c} \\
[\mu_2(A,A)]^a_\mu &= -3!\kappa f^a_{bc} \partial^\nu (A^b_\nu A^c_\mu) \\
[\mu_2(A,A^+)]^a &= 2g f_{bc}{}^a \Big(\partial^\nu (A^b_\nu A^c_\mu) + 2A^{b\nu} \partial_{[\nu} A^c_{\mu]} \Big) \\
[\mu_3(A,A,A)]^a_\mu &= 3! g^2 f_{ed}{}^b f_{bc}{}^a A^{\nu c} A^d_\nu A^e_\mu
\end{aligned}
$$

- Cyclic structure

$$
\begin{aligned}
\langle A, A^+ \rangle &= \int_{\mathbb{M}^d} \mathrm{d}^d x\, A^a_\mu A^{+a\mu} \ , \quad \langle b, b^+ \rangle = \int_{\mathbb{M}^d} \mathrm{d}^d x\, b^a b^{+a} \ , \\
\langle c, c^+ \rangle &= \int_{\mathbb{M}^d} \mathrm{d}^d x\, c^a c^{+a} \ , \quad \langle \bar{c}, \bar{c}^+ \rangle = -\int_{\mathbb{M}^d} \mathrm{d}^d x\, \bar{c}^a \bar{c}^{+a}
\end{aligned}
$$

Yang–Mills theory in the BV formalism

- Gauge-fixing: gauge-fixing fermion

$$\Psi \;=\; -\int \mathrm{d}^d x\, \bar{c}_a\left(\partial^\mu A^a_\mu + \tfrac{\xi}{2} b^a\right) .$$

  with ξ real parameter

- Gauge-fixed action

$$S^{\mathrm{gf}}_{\mathrm{YM}} \;=\; \int \mathrm{d}^d x \left\{ -\tfrac{1}{4} F_{a\mu\nu} F^{a\mu\nu} - \bar{c}_a \partial^\mu (\nabla_\mu c)^a + \tfrac{\xi}{2} b_a b^a + b_a \partial^\mu A^a_\mu \right\}$$

$$S^{\mathrm{gf}}_{\mathrm{YM}} \;=\; \int \mathrm{d}^d x \left\{ \tfrac{1}{2} A_{a\mu} \Box A^{a\mu} + \tfrac{1}{2} (\partial^\mu A^a_\mu)^2 - \bar{c}_a \Box c^a + \tfrac{\xi}{2} b_a b^a + b_a \partial^\mu A^a_\mu \right\} + S^{\mathrm{int}}_{\mathrm{YM}}$$

- Canonical field redefinition

$$
\begin{aligned}
\tilde{c}^a &= c^a & \tilde{c}^{+a} &= c^{+a} \\
\tilde{A}^a_\mu &= A^a_\mu & \tilde{A}^{+a}_\mu &= A^{+a}_\mu + \frac{1-\sqrt{1-\xi}}{\xi}\,\partial_\mu b^{+a} \\
\tilde{b}^a &= \sqrt{\frac{\xi}{\Box}}\left(b^a + \frac{1-\sqrt{1-\xi}}{\xi}\,\partial^\mu A^a_\mu\right) & \tilde{b}^{+a} &= \sqrt{\frac{\Box}{\xi}}\,b^{+a} \\
\tilde{\bar{c}}^a &= \bar{c}^a & \tilde{\bar{c}}^{+a} &= \bar{c}^{+a}
\end{aligned}
$$

- New action

$$
\tilde{S}_{\mathrm{YM}} = \int \mathrm{d}^d x \left\{ \tfrac{1}{2}\tilde{A}_{a\mu}\Box\tilde{A}^{a\mu} - \tilde{\bar{c}}_a\Box\tilde{c}^a + \tfrac{1}{2}\tilde{b}_a\Box\tilde{b}^a + \tilde{\xi}\,\tilde{b}_a\sqrt{\Box}\,\partial^\mu\tilde{A}^a_\mu \right\} + \tilde{S}^{\mathrm{int}}_{\mathrm{YM}}
$$

with $\tilde{\xi} = \sqrt{\frac{1-\xi}{\xi}}$

Yang–Mills theory in the BV formalism

- New chain complex

$$
\begin{array}{ccc}
\overset{\tilde{A}^a_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} & \xrightarrow{\ \Box\ } & \overset{\tilde{A}^{+a}_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} \\
 & -\tilde{\xi}\sqrt{\Box}\,\partial^\mu \searrow \quad \nearrow \tilde{\xi}\sqrt{\Box}\,\partial_\mu & \\
\overset{\tilde{b}^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}} & \xrightarrow[\ \Box\ ]{} & \overset{\tilde{b}^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}
\end{array}
$$

$$
\underbrace{\overset{\tilde{c}^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=\tilde{\mathfrak{L}}^{\mathrm{YM}}_0} \xrightarrow{-\Box} \underbrace{\overset{\tilde{\bar{c}}^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=\tilde{\mathfrak{L}}^{\mathrm{YM}}_1} \qquad \underbrace{\overset{\tilde{\bar{c}}^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=\tilde{\mathfrak{L}}^{\mathrm{YM}}_2} \xrightarrow{-\Box} \underbrace{\overset{\tilde{c}^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{=\tilde{\mathfrak{L}}^{\mathrm{YM}}_3}
$$

- This chain complex is readily interpreted as a (twisted) tensor product

Tensor products

- Tensor product between associative, commutative and Lie algebras

$\otimes$	Ass	Com	Lie
Ass	Ass	Ass	–
Com	Ass	Com	Lie
Lie	–	Lie	–

For two compatible algebras $\mathfrak{A}, \mathfrak{B}$

$$m_2^{\mathfrak{A}\otimes\mathfrak{B}}(a_1 \otimes b_1, a_2 \otimes b_2) \;=\; m_2^{\mathfrak{A}}(a_1, a_2) \otimes m_2^{\mathfrak{B}}(b_1, b_2)$$

- Tensor product between chain complexes $\mathfrak{A}, \mathfrak{B}$

$$\mathfrak{A} \otimes \mathfrak{B} \;=\; \bigoplus_i (\mathfrak{A} \otimes \mathfrak{B})_i, \quad (\mathfrak{A} \otimes \mathfrak{B})_i \;=\; \bigoplus_{k+l=i} \mathfrak{A}_k \otimes \mathfrak{B}_l$$

$$m_1^{\mathfrak{A}\otimes\mathfrak{B}}(a \otimes b) \;=\; m_1^{\mathfrak{A}}(a) \otimes b \pm a \otimes m_2^{\mathfrak{B}}(b)$$

- Strict homotopy algebras are nothing but differential graded algebras

Factorisation of YM theory

$$\mathfrak{L}_{\mathsf{YM}} \equiv \mathfrak{colour} \otimes \mathfrak{kinematic} \otimes_\tau \mathfrak{scalar}$$

- $\mathfrak{scalar}$ is the A_∞-algebra of a cubic scalar theory, with basis s_x
- $\mathfrak{colour}$ is the gauge Lie algebra, with basis e_a
- $\mathfrak{kinematic}$ is the following graded vector space

$$\overset{\mathsf{g}}{\mathbb{R}[1]} \oplus (\overset{\mathsf{v}^\mu}{\mathbb{R}^d} \otimes \overset{\mathsf{n}}{\mathbb{R}}) \oplus \overset{\mathsf{a}}{\mathbb{R}[-1]} \oplus \dots$$

where basis $(\mathsf{g}, \mathsf{v}^\mu, \mathsf{b}, \mathsf{a}, \dots)$ correspond to fields $(c, A, b, \bar{c}, \dots)$

$$c = \mathsf{e}_a \mathsf{g} \mathsf{s}_x c^a(x), \quad A = \mathsf{e}_a \mathsf{v}^\mu \mathsf{s}_x A^a_\mu(x), \quad \dots$$

Factorisation of YM theory

- $\otimes_\tau$ is a *twisted* tensor product, where the twist is controlled by a twist datum τ

$$\tau : \mathfrak{kinematic}^{\otimes n} \to \mathfrak{kinematic} \otimes \mathrm{End}(\mathfrak{scalar})^{\otimes n}$$

$$(\mathrm{k}_1 \cdots \mathrm{k}_\mathrm{n}) \mapsto \tau(\mathrm{k}_1, \ldots, \mathrm{k}_\mathrm{n}) \otimes \bigotimes_{i=1}^{n} \tau^i(\mathrm{k}_1, \ldots, \mathrm{k}_\mathrm{n})$$

- Thanks to the twist, $\mathfrak{kinematic}$ becomes a kinematic operator algebra, acting on $\mathfrak{scalar}$ with $\tau^i(\mathrm{k}_1, \ldots, \mathrm{k}_\mathrm{n})$

$$\begin{aligned} \mathrm{m}_1(k \otimes a) &= \pm\tau(k) \otimes \mathrm{m}_1(\tau^1(k)(a)) \,, \\ \mathrm{m}_2(k_1 \otimes a_1, k_2 \otimes a_2) &= \\ &= \pm\tau(k_1, k_2) \otimes \mathrm{m}_2(\tau^2(k_1, k_2)(a_1), \tau^2(k_1, k_2)(a_2)) \,. \end{aligned}$$

- τ is dictated by the field theory we consider

Factorisation of YM theory: chain complex level

- Scalar theory chain complex

$$* \rightarrow \underbrace{\overset{\mathsf{s}_x}{\mathfrak{F}[-1]}}_{\mathfrak{scalar}_1} \xrightarrow{\ \Box\ } \underbrace{\overset{\mathsf{s}_x^+}{\mathfrak{F}[-2]}}_{\mathfrak{scalar}_2} \rightarrow *$$

- Twist datum

$$\tau_1(\mathsf{g}) = \mathsf{g} \otimes \mathsf{id} \ , \quad \begin{aligned} \tau_1(\mathsf{v}^\mu) &= \mathsf{v}^\mu \otimes \mathsf{id} + \tilde{\xi}\mathsf{n} \otimes \frac{1}{\sqrt{\Box}}\partial^\mu \ , \\ \tau_1(\mathsf{n}) &= \mathsf{n} \otimes \mathsf{id} - \tilde{\xi}\mathsf{v}^\mu \otimes \frac{1}{\sqrt{\Box}}\partial_\mu \ , \end{aligned} \quad \tau_1(\mathsf{a}) = \mathsf{a} \otimes \mathsf{id}$$

To factorise the full interacting theory as

$$\mathfrak{L}_{\text{YM}} \equiv \mathfrak{colour} \otimes \mathfrak{kinematic} \otimes_{\tau} \mathfrak{scalar}$$

we need to strictify YM theory

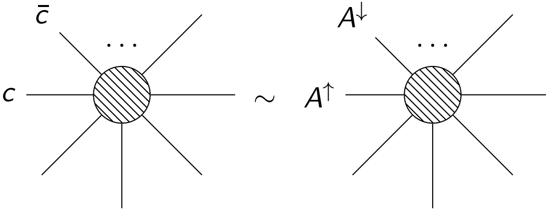
On-shell CK duality for BRST fields

- We start from a YM action manifesting CK duality for tree level, transverse gluons scattering amplitude (TW action)
- Scattering amplitudes with external arbitrary polarised gluons: to compensate CK violations, we need terms proportional to $\partial_\mu A^{\mu a}$ in the action
- We can generate these terms recursively, with an appropriate choice of gauge and redefining the NL field
- At four point, the correction to YM Lagrangian is

$$g^2 \Bigg\{ (\partial^\rho A^b_\rho) A^{c\mu} \frac{1}{\Box} \left[(\partial^\nu A^d_\mu) A^e_\nu \right] + \\ + \bar{c}^b \, Q_{\mathrm{BRST}} \left(A^{c\mu} \frac{1}{\Box} \left[(\partial^\nu A^d_\mu) A^e_\nu \right] \right) \Bigg\} f_{ed}{}^a f_{acb}$$

On-shell CK duality for BRST fields

- Next, we consider scattering amplitude with external ghost-antighost pairs
- On-shell Ward identity relates scattering amplitudes with $(k+1)$ ghost-antighost pairs to scattering amplitudes with k ghost-antighost pairs



- For every tree amplitude with n legs and k pairs ghost-antighost, recursively redefine gauge-fixing sector and ghost sector
- At four point, no further corrections to YM Lagrangian are needed

On-shell CK duality for BRST fields

- The correction to YM Lagrangian comes with a canonical strictification, given by the colour structure
- At four point, we have

$$
\begin{aligned}
\mathscr{L}_4^{\mathrm{YM,\,st}} = \; & \tfrac{1}{2}\tilde{A}_{a\mu}\Box\tilde{A}^{\mu a} - \tilde{\bar{c}}_a\Box\tilde{c}^a + \tfrac{1}{2}\tilde{b}_a\Box\tilde{b}^a + \xi\,\tilde{b}_a\sqrt{\Box}\,\partial_\mu\tilde{A}^{\mu a} - \\
& - g f_{abc}\tilde{\bar{c}}^a\partial^\mu(\tilde{A}^b_\mu\tilde{c}^c) - \\
& - \tfrac{1}{2}\tilde{G}^{\mu\nu\kappa}_a\Box\tilde{G}^a_{\mu\nu\kappa} + g f_{abc}\left(\partial_\mu\tilde{A}^a_\nu + \tfrac{1}{\sqrt{2}}\partial^\kappa\tilde{G}^a_{\kappa\mu\nu}\right)\tilde{A}^{\mu b}\tilde{A}^{\nu c} - \\
& - \tilde{K}^\mu_{1a}\Box\tilde{\bar{K}}^{1a}_\mu - \tilde{K}^\mu_{2a}\Box\tilde{\bar{K}}^{2a}_\mu - \\
& - g f_{abc}\left\{\tilde{K}^{a\mu}_1(\partial^\nu\tilde{A}^b_\mu)\tilde{A}^c_\nu + [(\partial^\kappa\tilde{A}^a_\kappa)\tilde{A}^{b\mu} + \tilde{\bar{c}}^a\partial^\mu\tilde{c}^b]\tilde{\bar{K}}^{1c}_\mu\right\} + \\
& + g f_{abc}\left\{\tilde{K}^{a\mu}_2\left[(\partial^\nu\partial_\mu\tilde{c}^b)\tilde{A}^c_\nu + (\partial^\nu\tilde{A}^b_\mu)\partial_\nu\tilde{c}^c\right] + \tilde{\bar{c}}^a\tilde{A}^{b\mu}\tilde{\bar{K}}^{2c}_\mu\right\}
\end{aligned}
$$

Lifting on-shell CK duality to off-shell

- Off-shell, we can have CK violation proportional to p_i^2
- These can be compensated order by order with terms $F_i \Box \Phi_i$, where Φ_i can be A, c or $\bar{c}$
- We can generate these terms with a non-local field redefinition

$$\Phi_i \ \mapsto \Phi_i + \sum F_i$$

- The Jacobian determinants for the required field redefinitions lead to additional counterterms that must be included in the renormalization
- The new action can be strictified up to the relevant order

Double copy of YM theory

$\mathfrak{colour} \otimes \mathfrak{colour} \otimes \mathfrak{scalar}$
Biadjoint
scalar theory
$\longleftarrow$
$\mathfrak{colour} \otimes \mathfrak{kinematic} \otimes_\tau \mathfrak{scalar}$
Yang–Mills
theory
$\longrightarrow$
$\mathfrak{kinematic} \otimes_\tau \mathfrak{kinematic} \otimes_\tau \mathfrak{scalar}$
$\mathcal{N}=0$
supergravity

Double copy of YM theory

- We start with our cubic theory

$$\mathscr{L} = \frac{1}{2}\Phi^{\alpha i} g_{\alpha\beta} G_{ij} \Box \Phi^{\beta j} + \frac{1}{3!}\Phi^{\alpha i} f_{\alpha\beta\gamma} F_{ijk} \Phi^{\beta j}\Phi^{\gamma k}$$

$$(Q\Phi)^{\alpha i} = \delta^{\alpha}_{\beta} q^{i}_{j} \Phi^{\beta j} + \frac{1}{2} f^{\alpha}_{\beta\gamma} Q^{i}_{jk} \Phi^{\beta j}\Phi^{\gamma k} + \dots$$

- We double copy both Lagrangian and BRST operator

$$\mathscr{L}_{\mathrm{DC}} = \frac{1}{2}\Phi^{i' i} G_{i'j'} G_{ij} \Box \Phi^{j' j} + \frac{1}{3!}\Phi^{i' i} F_{i'j'k'} F_{ijk} \Phi^{j' j}\Phi^{k' k}$$

$$(Q_{\mathrm{DC}}\Phi)^{i' i} = \dots$$

- Is the new theory consistent? Do we obtain a new BRST operator $Q_{\mathrm{DC}}^2 = 0$, $Q_{\mathrm{DC}} S_{\mathrm{DC}} = 0$?
- If F_{ijk} satisfies the same algebraic relations of $f_{\alpha\beta\gamma}$, i.e. if we have off-shell CK, **yes**

Double copy: field content

Antisymmetric sector

fields				anti-fields		
factorisation	$\lvert - \rvert_{\text{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim	factorisation	$\lvert - \rvert_{\mathfrak{L}}$	dim
$\tilde{\lambda} = -[\mathsf{g},\mathsf{g}]\mathsf{s}_x \frac{1}{2}\tilde{\lambda}(x)$	2	-1	$\frac{d}{2}-3$	$\tilde{\lambda}^+ = -[\mathsf{a},\mathsf{a}]\mathsf{s}_x^+ \frac{1}{2}\tilde{\lambda}^+(x)$	4	$\frac{d}{2}+3$
$\tilde{\Lambda} = [\mathsf{g},\mathsf{v}^\mu]\mathsf{s}_x \frac{1}{\sqrt{2}}\tilde{\Lambda}_\mu(x)$	1	0	$\frac{d}{2}-2$	$\tilde{\Lambda}^+ = [\mathsf{a},\mathsf{v}^\mu]\mathsf{s}_x^+ \frac{1}{\sqrt{2}}\tilde{\Lambda}^+_\mu$	3	$\frac{d}{2}+2$
$\tilde{\gamma} = [\mathsf{g},\mathsf{n}]\mathsf{s}_x \frac{1}{\sqrt{2}}\tilde{\gamma}(x)$	1	0	$\frac{d}{2}-2$	$\tilde{\gamma}^+ = [\mathsf{a},\mathsf{n}]\mathsf{s}_x^+ \frac{1}{\sqrt{2}}\tilde{\gamma}^+(x)$	3	$\frac{d}{2}+2$
$\tilde{B} = [\mathsf{v}^\mu,\mathsf{v}^\nu]\mathsf{s}_x \frac{1}{2\sqrt{2}}\tilde{B}_{\mu\nu}(x)$	0	1	$\frac{d}{2}-1$	$\tilde{B}^+ = [\mathsf{v}^\mu,\mathsf{v}^\nu]\mathsf{s}_x^+ \frac{1}{2\sqrt{2}}\tilde{B}^+_{\mu\nu}(x)$	2	$\frac{d}{2}+1$
$\tilde{\alpha} = [\mathsf{n},\mathsf{v}^\mu]\mathsf{s}_x \frac{1}{\sqrt{2}}\tilde{\alpha}_\mu(x)$	0	1	$\frac{d}{2}-1$	$\tilde{\alpha}^+ = [\mathsf{n},\mathsf{v}^\mu]\mathsf{s}_x^+ \frac{1}{\sqrt{2}}\tilde{\alpha}^+_\mu(x)$	2	$\frac{d}{2}+1$
$\tilde{\epsilon} = -[\mathsf{g},\mathsf{a}]\mathsf{s}_x \frac{1}{\sqrt{2}}\tilde{\epsilon}(x)$	0	1	$\frac{d}{2}-1$	$\tilde{\epsilon}^+ = -[\mathsf{g},\mathsf{a}]\mathsf{s}_x^+ \frac{1}{\sqrt{2}}\tilde{\epsilon}^+(x)$	2	$\frac{d}{2}+1$
$\tilde{\bar{\Lambda}} = [\mathsf{a},\mathsf{v}^\mu]\mathsf{s}_x \frac{1}{\sqrt{2}}\tilde{\bar{\Lambda}}_\mu(x)$	-1	2	$\frac{d}{2}$	$\tilde{\bar{\Lambda}}^+ = [\mathsf{g},\mathsf{v}^\mu]\mathsf{s}_x^+ \frac{1}{\sqrt{2}}\tilde{\bar{\Lambda}}^+_\mu(x)$	1	$\frac{d}{2}$
$\tilde{\bar{\gamma}} = [\mathsf{a},\mathsf{n}]\mathsf{s}_x \frac{1}{\sqrt{2}}\tilde{\bar{\gamma}}(x)$	-1	2	$\frac{d}{2}$	$\tilde{\bar{\gamma}}^+ = [\mathsf{g},\mathsf{n}]\mathsf{s}_x^+ \frac{1}{\sqrt{2}}\tilde{\bar{\gamma}}^+(x)$	1	$\frac{d}{2}$
$\tilde{\bar{\lambda}} = -[\mathsf{a},\mathsf{a}]\mathsf{s}_x \frac{1}{2}\tilde{\bar{\lambda}}(x)$	-2	3	$\frac{d}{2}+1$	$\tilde{\bar{\lambda}}^+ = -[\mathsf{g},\mathsf{g}]\mathsf{s}_x^+ \frac{1}{2}\tilde{\bar{\lambda}}^+(x)$	0	$\frac{d}{2}-1$

Double copy: field content

Symmetric sector

fields				anti-fields		
factorisation	$\lvert - \rvert_{\mathrm{gh}}$	$\lvert - \rvert_{\mathfrak{L}}$	dim	factorisation	$\lvert - \rvert_{\mathfrak{L}}$	dim
$\tilde{X} = (\mathsf{g}, \mathsf{v}^{\mu})\mathsf{s}_x \frac{1}{\sqrt{2}}\tilde{X}_{\mu}(x)$	1	0	$\frac{d}{2} - 2$	$\tilde{X}^{+} = (\mathsf{a}, \mathsf{v}^{\mu})\mathsf{s}_x^{+} \frac{1}{\sqrt{2}}\tilde{X}_{\mu}^{+}(x)$	3	$\frac{d}{2} + 2$
$\tilde{\beta} = (\mathsf{g}, \mathsf{n})\mathsf{s}_x \frac{1}{\sqrt{2}}\tilde{\beta}(x)$	1	0	$\frac{d}{2} - 2$	$\tilde{\beta}^{+} = (\mathsf{a}, \mathsf{n})\mathsf{s}_x^{+} \frac{1}{\sqrt{2}}\tilde{\beta}^{+}(x)$	3	$\frac{d}{2} + 2$
$\tilde{h} = (\mathsf{v}^{\mu}, \mathsf{v}^{\nu})\mathsf{s}_x \frac{1}{2\sqrt{2}}\tilde{h}_{\mu\nu}(x)$	0	1	$\frac{d}{2} - 1$	$\tilde{h}^{+} = (\mathsf{v}^{\mu}, \mathsf{v}^{\nu})\mathsf{s}_x^{+} \frac{1}{2\sqrt{2}}\tilde{h}_{\mu\nu}^{+}(x)$	2	$\frac{d}{2} + 1$
$\tilde{\varpi} = -(\mathsf{n}, \mathsf{v}^{\mu})\mathsf{s}_x \frac{1}{\sqrt{2}}\tilde{\varpi}_{\mu}(x)$	0	1	$\frac{d}{2} - 1$	$\tilde{\varpi}^{+} = -(\mathsf{n}, \mathsf{v}^{\mu})\mathsf{s}_x^{+} \frac{1}{\sqrt{2}}\tilde{\varpi}_{\mu}^{+}(x)$	2	$\frac{d}{2} + 1$
$\tilde{\pi} = (\mathsf{n}, \mathsf{n})\mathsf{s}_x \frac{1}{2\sqrt{2}}\tilde{\pi}(x)$	0	1	$\frac{d}{2} - 1$	$\tilde{\pi}^{+} = (\mathsf{n}, \mathsf{n})\mathsf{s}_x^{+} \frac{1}{2\sqrt{2}}\tilde{\pi}^{+}(x)$	2	$\frac{d}{2} + 1$
$\tilde{\delta} = -(\mathsf{g}, \mathsf{a})\mathsf{s}_x \frac{1}{\sqrt{2}}\tilde{\delta}(x)$	0	1	$\frac{d}{2} - 1$	$\tilde{\delta}^{+} = -(\mathsf{g}, \mathsf{a})\mathsf{s}_x^{+} \frac{1}{\sqrt{2}}\tilde{\delta}^{+}(x)$	2	$\frac{d}{2} + 1$
$\tilde{\bar{X}} = (\mathsf{a}, \mathsf{v}^{\mu})\mathsf{s}_x \frac{1}{\sqrt{2}}\tilde{\bar{X}}_{\mu}(x)$	-1	2	$\frac{d}{2}$	$\tilde{\bar{X}}^{+} = (\mathsf{g}, \mathsf{v}^{\mu})\mathsf{s}_x^{+} \frac{1}{\sqrt{2}}\tilde{\bar{X}}_{\mu}(x)$	1	$\frac{d}{2}$
$\tilde{\bar{\beta}} = (\mathsf{a}, \mathsf{n})\mathsf{s}_x \frac{1}{\sqrt{2}}\tilde{\bar{\beta}}(x)$	-1	2	$\frac{d}{2}$	$\tilde{\bar{\beta}}^{+} = (\mathsf{g}, \mathsf{n})\mathsf{s}_x^{+} \frac{1}{\sqrt{2}}\tilde{\bar{\beta}}^{+}(x)$	1	$\frac{d}{2}$

...+ auxiliary fields

Double copy: action

$$
\begin{aligned}
S_{\mathrm{DC}} \;=\; \int \mathrm{d}^d x \Big\{ & \tfrac{1}{4}\tilde{B}_{\mu\nu}\Box\tilde{B}^{\mu\nu} - \tilde{\tilde{\bar{\Lambda}}}_\mu\Box\tilde{\Lambda}^\mu + \tfrac{1}{2}\tilde{\alpha}_\mu\Box\tilde{\alpha}^\mu - \\
& - \tfrac{\tilde{\xi}^2}{2}(\partial^\mu\tilde{\alpha}_\mu)^2 + \tfrac{1}{2}\tilde{\epsilon}\Box\tilde{\epsilon} - \tilde{\tilde{\bar{\lambda}}}\Box\tilde{\lambda} - \tilde{\tilde{\bar{\gamma}}}\Box\tilde{\gamma} + \tilde{\xi}\tilde{\alpha}^\nu\sqrt{\Box}\partial^\mu\tilde{B}_{\mu\nu} + \\
& + \tilde{\xi}\tilde{\gamma}\sqrt{\Box}\partial_\mu\tilde{\tilde{\bar{\Lambda}}}^\mu - \tilde{\xi}\tilde{\tilde{\bar{\gamma}}}\sqrt{\Box}\partial_\mu\tilde{\Lambda}^\mu + \\
& + \tfrac{1}{4}\tilde{h}_{\mu\nu}\Box\tilde{h}^{\mu\nu} - \tilde{\tilde{\bar{X}}}_\mu\Box\tilde{X}^\mu + \tfrac{1}{2}\tilde{\varpi}_\mu\Box\tilde{\varpi}^\mu + \\
& + \tfrac{\tilde{\xi}^2}{2}(\partial^\mu\tilde{\varpi}_\mu)^2 - \tfrac{1}{2}\tilde{\delta}\Box\tilde{\delta} + \tfrac{1}{4}\tilde{\pi}\Box\tilde{\pi} - \tilde{\tilde{\bar{\beta}}}\Box\tilde{\beta} + \\
& + \tilde{\xi}\tilde{\varpi}^\nu\sqrt{\Box}\partial^\mu\tilde{h}_{\mu\nu} + \tilde{\xi}\tilde{\pi}\sqrt{\Box}\partial_\mu\tilde{\varpi}^\mu + \tfrac{1}{2}\tilde{\xi}^2\tilde{\pi}\partial_\mu\partial_\nu\tilde{h}^{\mu\nu} + \\
& + \tilde{\xi}\tilde{\beta}\sqrt{\Box}\partial_\mu\tilde{\tilde{\bar{X}}}^\mu - \tilde{\xi}\tilde{\tilde{\bar{\beta}}}\sqrt{\Box}\partial_\mu\tilde{X}^\mu \Big\} + \ldots
\end{aligned}
$$

Double copy: action

$$
\begin{aligned}
S_{\mathrm{DC}} \;=\; \int \mathrm{d}^d x \Big\{ & \tfrac{1}{4}\tilde{B}_{\mu\nu}\Box\tilde{B}^{\mu\nu} - \tilde{\tilde{\Lambda}}_\mu\Box\tilde{\Lambda}^\mu + \tfrac{1}{2}\tilde{\alpha}_\mu\Box\tilde{\alpha}^\mu - \\
& - \tfrac{\tilde{\xi}^2}{2}(\partial^\mu\tilde{\alpha}_\mu)^2 + \tfrac{1}{2}\tilde{\epsilon}\Box\tilde{\epsilon} - \tilde{\tilde{\lambda}}\Box\tilde{\lambda} - \tilde{\tilde{\gamma}}\Box\tilde{\gamma} + \tilde{\xi}\tilde{\alpha}^\nu\sqrt{\Box}\partial^\mu\tilde{B}_{\mu\nu} + \\
& + \tilde{\xi}\tilde{\gamma}\sqrt{\Box}\partial_\mu\tilde{\tilde{\Lambda}}^\mu - \tilde{\xi}\tilde{\tilde{\gamma}}\sqrt{\Box}\partial_\mu\tilde{\Lambda}^\mu + \\
& + \tfrac{1}{4}\tilde{h}_{\mu\nu}\Box\tilde{h}^{\mu\nu} - \tilde{\tilde{X}}_\mu\Box\tilde{X}^\mu + \tfrac{1}{2}\tilde{\varpi}_\mu\Box\tilde{\varpi}^\mu + \\
& + \tfrac{\tilde{\xi}^2}{2}(\partial^\mu\tilde{\varpi}_\mu)^2 - \tfrac{1}{2}\tilde{\delta}\Box\tilde{\delta} + \tfrac{1}{4}\tilde{\pi}\Box\tilde{\pi} - \tilde{\tilde{\beta}}\Box\tilde{\beta} + \\
& + \tilde{\xi}\tilde{\varpi}^\nu\sqrt{\Box}\partial^\mu\tilde{h}_{\mu\nu} + \tilde{\xi}\tilde{\pi}\sqrt{\Box}\partial_\mu\tilde{\varpi}^\mu + \tfrac{1}{2}\tilde{\xi}^2\tilde{\pi}\partial_\mu\partial_\nu\tilde{h}^{\mu\nu} + \\
& + \tilde{\xi}\tilde{\beta}\sqrt{\Box}\partial_\mu\tilde{\tilde{X}}^\mu - \tilde{\xi}\tilde{\tilde{\beta}}\sqrt{\Box}\partial_\mu\tilde{X}^\mu \Big\} + \ldots
\end{aligned}
$$

Antisymmetric sector

Double copy: action

$$
\begin{aligned}
S_{\mathrm{DC}} \;=\; \int \mathrm{d}^d x \Big\{ & \tfrac{1}{4}\tilde{B}_{\mu\nu}\Box\tilde{B}^{\mu\nu} - \tilde{\tilde{\bar{\Lambda}}}_\mu\Box\tilde{\Lambda}^\mu + \tfrac{1}{2}\tilde{\alpha}_\mu\Box\tilde{\alpha}^\mu - \\
& - \tfrac{\tilde{\xi}^2}{2}(\partial^\mu\tilde{\alpha}_\mu)^2 + \tfrac{1}{2}\tilde{\epsilon}\Box\tilde{\epsilon} - \tilde{\tilde{\bar{\lambda}}}\Box\tilde{\lambda} - \tilde{\tilde{\bar{\gamma}}}\Box\tilde{\gamma} + \tilde{\xi}\tilde{\alpha}^\nu\sqrt{\Box}\partial^\mu\tilde{B}_{\mu\nu} + \\
& + \tilde{\xi}\tilde{\gamma}\sqrt{\Box}\partial_\mu\tilde{\tilde{\bar{\Lambda}}}^\mu - \tilde{\xi}\tilde{\tilde{\bar{\gamma}}}\sqrt{\Box}\partial_\mu\tilde{\Lambda}^\mu + \\
& + \tfrac{1}{4}\tilde{h}_{\mu\nu}\Box\tilde{h}^{\mu\nu} - \tilde{\tilde{\bar{X}}}_\mu\Box\tilde{X}^\mu + \tfrac{1}{2}\tilde{\varpi}_\mu\Box\tilde{\varpi}^\mu + \\
& + \tfrac{\tilde{\xi}^2}{2}(\partial^\mu\tilde{\varpi}_\mu)^2 - \tfrac{1}{2}\tilde{\delta}\Box\tilde{\delta} + \tfrac{1}{4}\tilde{\pi}\Box\tilde{\pi} - \tilde{\tilde{\bar{\beta}}}\Box\tilde{\beta} + \\
& + \tilde{\xi}\tilde{\varpi}^\nu\sqrt{\Box}\partial^\mu\tilde{h}_{\mu\nu} + \tilde{\xi}\tilde{\pi}\sqrt{\Box}\partial_\mu\tilde{\varpi}^\mu + \tfrac{1}{2}\tilde{\xi}^2\tilde{\pi}\partial_\mu\partial_\nu\tilde{h}^{\mu\nu} + \\
& + \tilde{\xi}\tilde{\beta}\sqrt{\Box}\partial_\mu\tilde{\tilde{\bar{X}}}^\mu - \tilde{\xi}\tilde{\tilde{\bar{\beta}}}\sqrt{\Box}\partial_\mu\tilde{X}^\mu \Big\} + \ldots
\end{aligned}
$$

Symmetric sector

Perturbative equivalence: $\mathcal{N} = 0$ supergravity is quantum equivalent to Yang–Mills double copy

Perturbative equivalence

Antisymmetric sector is related to Kalb–Ramond theory by the following field redefinition

$$
\begin{aligned}
\tilde{\lambda} &= \lambda\,, & \tilde{\Lambda}_\mu &= \Lambda_\mu\,,\\
\tilde{\gamma} &= \sqrt{\frac{\xi}{\Box}}\left(\gamma + \frac{1-\sqrt{1-\xi}}{\xi}\partial^\mu\Lambda_\mu\right), & \tilde{B}_{\mu\nu} &= B_{\mu\nu}\,,\\
\tilde{\epsilon} &= \epsilon + \frac{1-\xi}{2\Box}\partial^\mu\alpha_\mu\,, & \tilde{\bar{\Lambda}}_\mu &= \bar{\Lambda}_\mu\,,\\
\tilde{\bar{\gamma}} &= \sqrt{\frac{\xi}{\Box}}\left(\bar{\gamma} + \frac{1-\sqrt{1-\xi}}{\xi}\partial^\mu\bar{\Lambda}_\mu\right), & \tilde{\bar{\lambda}} &= \bar{\lambda}\,,
\end{aligned}
$$

$$
\tilde{\alpha}_\mu = \sqrt{\frac{\xi}{\Box}}\left(\alpha_\mu - \partial_\mu\epsilon - \frac{1-\xi}{2\Box}\partial_\mu\partial^\nu\alpha_\nu + \frac{1-\sqrt{1-\xi}}{\xi}\partial^\nu B_{\nu\mu}\right)
$$

Analogously, symmetric sector can be related to Einstein–Hilbert gravity+dilaton

Perturbative equivalence

- Integrating out the auxiliary fields: DC and $\mathcal{N} = 0$ supergravity have the same field content and kinematic terms up to field redefinition
- The same field redefinition relates linearised BRST operators of DC and $\mathcal{N} = 0$ supergravity
- The tree level double copy holds: setting the unphysical fields to zero, DC and $\mathcal{N} = 0$ supergravity are classically equivalent
- Redefining gauge-fixing sector and using Ward identity, this can be extended to all BRST field

Perturbative equivalence

- Integrate out the auxiliary field. The discrepancies should be proportional to $\Box\Phi$ for some field Φ. Such differences can be produced with (eventually non-local) field redefinitions
- The action for Yang–Mills double copy and $\mathcal{N}=0$ supergravity now matches, and the discrepancy between $Q^{\mathcal{N}=0}_{\text{BRST}}$ and Q_{DC} is a trivial symmetry
- Conclusion: Yang–Mills double copy and $\mathcal{N}=0$ supergravity are perturbatively quantum equivalent

Future

- A better mathematical understanding of CK factorisation in terms of homotopy algebras
- Establish a web of dualities between a zoology of QFT
- Renormalisation?
- Homotopic description of open–closed string duality?

Thank you for listening!