Self-dual gravity and color/kinematics duality in AdS4

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Based on:

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2304.07206 with C. Armstrong, H. Goodhew, J. Mei

Motivation

- Double copy reduces complicated gravity calculations to simpler ones in gauge theory.
- Can this progress be extended to curved background like (A)dS?
- What does this teach us about flat space amplitudes?
- New connections between AdS/CFT and flat space holography?

Overview

- review of flat space double copy
- review of SDYM and SDG in flat space
- SDYM and SDG in AdS_4
- color/kinematics duality and $w_{1+\infty}$ in AdS_4
- beyond the self-dual sector
- conclusion

Double Copy in Flat Space

- 3-point gluon amplitudes square into 3-point graviton amplitudes
- Color/kinematics duality can be used to extend double copy beyond three points. Take a 4-point gluon amplitude:

$$\mathcal{A}_4 = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}, \qquad c_s + c_u + c_t = 0$$

numerators obey kinematic Jacobi: $n_s + n_t + n_u = 0$

- Squaring these numerators gives 4-point graviton amplitude:

$$\mathcal{M}_4 = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

(Bern,Carrasco,Johansson)

Self-dual Yang-Mills

- EOM: $F_{\mu\nu} = \frac{\sqrt{g}}{2} \epsilon_{\mu\nu\rho\lambda} F^{\rho\lambda}$

- Coordinates: $ds^2 = dw\, d\bar{w} - du\, dv$

$$u = it + z, \quad v = it - z,$$
$$w = x + iy, \quad \bar{w} = x - iy, \quad x^i = (u, w), \quad y^\alpha = (v, \bar{w})$$

- In lightcone gauge $A_u = 0$, self-duality constraints reduces to

$$\Box_{\mathbb{R}^4} \Phi + i \left[\partial_u \Phi, \partial_w \Phi\right] = 0$$

where $A_w = 0, \quad A_{\bar{w}} = \partial_u \Phi, \quad A_v = \partial_w \Phi$

- Hence, SDYM can be described by the following scalar theory:

$$\mathcal{L}_{\text{SDYM}} = \text{Tr}\left[\bar{\Phi}\left(\Box_{\mathbb{R}^4}\Phi + i\left[\partial_u\Phi, \partial_w\Phi\right]\right)\right]$$

where the two scalars encode opposite gluon helicities. (Bardeen,Chalmers,Siegel)

- EOM can also be written in term of Poisson bracket:

$$\Box_{\mathbb{R}^4}\Phi - \frac{i}{2}[\{\Phi, \Phi\}] = 0, \quad \{f, g\} := \partial_w f \partial_u g - \partial_u f \partial_w g = \varepsilon^{\alpha\beta}\Pi_\alpha f \Pi_\beta g$$

$$\Pi_\alpha = (\Pi_v, \Pi_{\bar{w}}) = (\partial_w, \partial_u)$$

Self-dual Gravity

- EOM: $R_{\mu\nu\rho\sigma} = \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu}{}^{\eta\lambda}R_{\eta\lambda\rho\sigma}$
- Metric: $ds^2 = dw\, d\bar{w} - du\, dv + h_{\mu\nu}\, dx^\mu dx^\nu$
- In lightcone gauge $h_{u\mu} = 0$, self-duality constraints reduce to

$$\Box_{\mathbb{R}^4}\phi - \{\{\phi,\phi\}\} = 0, \;\; \{\{f,g\}\} = \frac{1}{2}\varepsilon^{\alpha\beta}\{\Pi_\alpha f, \Pi_\beta g\},$$

where $h_{i\mu} = 0, \quad h_{\alpha\beta} = \Pi_\alpha \Pi_\beta \phi$ (Plebanski)

- Color/kinematics duality becomes manifest in self-dual sector (Montiero,O'Connell):

$$\Phi \to \phi, \qquad \frac{i}{2}[\{\ ,\ \}] \to \{\{\ ,\ \}\}$$

- Feynman rules: $V_{\text{SDYM}} = \frac{1}{2} X(k_1, k_2)\, f^{a_1 a_2 a_3}$

$$V_{\text{SDG}} = \frac{1}{2} X(k_1, k_2)^2$$

  where $X(k_1, k_2) = k_{1u} k_{2w} - k_{1w} k_{2u}$

- Kinematic Jacobi: $0 = X(k_1, k_2)\, X(k_3, k_1 + k_2) + \text{cyclic}$

- Kinematic algebra can be lifted to $w_{1+\infty}$ algebra (Monteiro), which plays an important role in flat space holography (Strominger).

AdS_4 metric

- Poincaré patch of Euclidean AdS_4 with unit radius:

$$ds^2_{\mathrm{AdS}} = \frac{dt^2 + dx^2 + dy^2 + dz^2}{z^2}$$

where $0 < z < \infty$

- Light cone coordinates:

$$u = it + z, \quad v = it - z$$
$$w = x + iy, \quad \bar{w} = x - iy,$$

$$\rightarrow \quad ds^2_{\mathrm{AdS}} = \frac{4\,(dw\,d\bar{w} - du\,dv)}{(u-v)^2}$$

- Wick rotation $z \to i\eta$ gives dS_4 metric.

SDYM in AdS_4

- AdS metric is conformally flat. Conformal factor will cancel out of self-duality equation:

$$F_{\mu\nu} = \frac{\sqrt{g}}{2} \epsilon_{\mu\nu\rho\lambda} F^{\rho\lambda}$$

- Hence, SDYM EOM same as flat space.

- Need to impose boundary conditions at z=0. Momentum along z direction will not be conserved. This will be relevant when computing boundary correlators.

SDG in AdS_4

- Modify self-duality condition:

$$T_{\mu\nu\rho\sigma} = \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu}{}^{\eta\lambda}T_{\eta\lambda\rho\sigma}$$

where $T_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{3}\Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma})$ (we set $\Lambda = -3$)

- Contracting with inverse metric gives vacuum Einstein equation:

$$R_{\mu\rho} - \Lambda g_{\mu\rho} = \frac{1}{2}\sqrt{g}\epsilon_{\mu}{}^{\sigma\eta\lambda}R_{\eta\lambda\rho\sigma} = 0$$

- Ansatz: $$ds^2 = \frac{4\left(dw\,d\bar{w} - du\,dv + h_{\alpha\beta}\,dy^\alpha dy^\beta\right)}{(u-v)^2}$$

- Scalar EOM:

$$\frac{1}{u-v}\Box_{\mathbb{R}^4}\left(\frac{\phi}{u-v}\right) - \left\{\left\{\frac{\phi}{u-v}, \frac{\phi}{u-v}\right\}\right\}_* = 0$$

where $h_{\alpha\beta} = \Pi_{(\alpha}\tilde{\Pi}_{\beta)}\phi, \quad \{\{f,g\}\}_* = \frac{1}{2}\varepsilon^{\alpha\beta}\{\Pi_\alpha f, \Pi_\beta g\}_*$

- Deformed Poisson bracket:

$$\{f,g\}_* = \frac{1}{2}\varepsilon^{\alpha\beta}(\Pi_\alpha f\tilde{\Pi}_\beta g - \Pi_\alpha g\tilde{\Pi}_\beta f) \qquad \tilde{\Pi}_\alpha = (\tilde{\Pi}_v, \tilde{\Pi}_{\bar{w}}) = \left(\partial_w, \partial_u - \tfrac{4}{u-v}\right)$$

- Jacobi relation: $\{f,\{g,h\}_*\}_* + \{g,\{h,f\}_*\}_* + \{h,\{f,g\}_*\}_* = 0$

- Leibniz rule: $\left\{\frac{fg}{(u-v)^2},h\right\}_* = \frac{1}{(u-v)^2} f\{g,h\}_* + \frac{1}{(u-v)^2} g\{f,h\}_*$

- Lagrangian: $\mathcal{L}_{\mathrm{SDG}} = \sqrt{g}\bar{\phi}\left(\Box_{\mathrm{AdS}} - m^2\right)\phi + 4\bar{\phi}\left\{\left\{\frac{\phi}{u-v},\frac{\phi}{u-v}\right\}\right\}_*$

  where $m^2 = -2$ (conformally coupled scalar)

- Deformed Poisson structure not manifest in previous formulations of SDG in AdS_4 (Przanowski,Krasnov,Neiman)

CK Duality for SDG in AdS_4

- Feynman rules:
$$V_{\mathrm{SDYM}} = \frac{1}{2} X(k_1, k_2) f^{a_1 a_2 a_3},$$
$$V_{\mathrm{SDG}} = \frac{1}{2} X(k_1, k_2) \tilde{X}(k_1, k_2)$$

  where $X(k_1, k_2) = k_{1u} k_{2w} - k_{1w} k_{2u}, \quad \tilde{X}(k_1, k_2) = X(k_1, k_2) - \frac{2i}{u - v} (k_1 - k_2)_w$

$$\left\{ e^{ik_1 \cdot x}, e^{ik_2 \cdot x} \right\} = X(k_1, k_2) e^{i(k_1 + k_2) \cdot x}$$
$$\left\{ e^{ik_1 \cdot x}, e^{ik_2 \cdot x} \right\}_* = \tilde{X}(k_1, k_2) e^{i(k_1 + k_2) \cdot x}$$

- Kinematic Jacobi:
$$\begin{aligned} 0 &= X(k_1, k_2) X(k_3, k_1 + k_2) + \mathrm{cyclic} \\ &= \tilde{X}(k_1, k_2) \tilde{X}(k_3, k_1 + k_2) + \mathrm{cyclic} \end{aligned}$$

- Double copy: $f^{a_1 a_2 a_3} \to \tilde{X}(k_1, k_2)$

w-infinity algebras in AdS_4

- Expand on-shell plane waves: $e^{ik\cdot x} = \sum_{a,b=0}^{\infty} \frac{(ik_u)^a (ik_w)^b}{a!b!} \mathfrak{e}_{ab}$, $k_u k_v - k_w k_{\bar{w}} = 0$

  $\rho = k_{\bar{w}}/k_u = k_v/k_w$

  where $\mathfrak{e}_{ab} = (u + \rho\bar{w})^a (w + \rho v)^b$ (Monteiro)
- Let $w^p_m = \frac{1}{2}\mathfrak{e}_{p-1+m,p-1-m}$. Then

$$\{w^p_m, w^q_n\} = (n(p-1) - m(q-1))\, w^{p+q-2}_{m+n},$$
$$\{w^p_m, w^q_n\}_* = \{w^p_m, w^q_n\} + \frac{(m+q-p-n)}{u-v} w^{p+q-3/2}_{m+n+1/2}$$

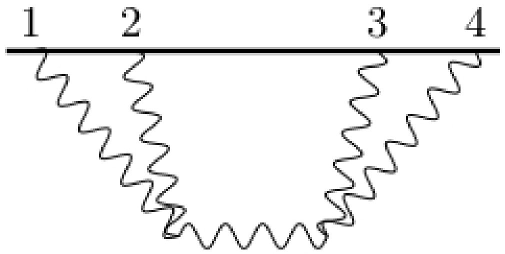
- First line is $w_{1+\infty}$ algebra. Second line contains a deformation

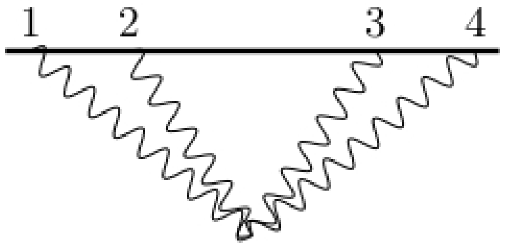
Beyond the self-dual sector

- Wick rotating AdS Witten diagrams to dS computes cosmological wavefunction (Maldacena,Pimentel,McFadden,Skenderis)
- Tree-level wavefunction for 4 gravitons first recently computed by (Bonifacio,Goodhew,Joyce,Pajer,Stefanyszyn)
- Bootstrapped using flat space limit (Maldacena,Pimentel,Raju), Cosmological Optical Theorem (Goodhew,Jazayeri,Melville,Pajer) and Manifestly Local Test (Jazayeri,Pajer,Stefanyszyn)
- dS Feynman diagrams give hundreds of thousands of terms but the bootstrap result is only about a page long.
- Combining bootstrap with double copy reduces it to only a few lines! (Armstrong,Goodhew,Lipstein,Mei)

Gluon Wavefuntion

- Diagrams:

$\psi_A =$ 

$+ \; 2 \leftrightarrow 4 \; +$ 

(axial gauge Feynman rules obtained by Liu,Tseytlin,Raju)

- s-channel wavefunction:

$$\psi_A^{(s)} = \int \frac{d\omega\,\omega}{k_s^2+\omega^2} dz\, dz'\, (KKJ)_{12}^{1/2}(z)(KKJ)_{34}^{1/2}(z')N_s$$

where $N_s = V_{12}^i H_{ij} V_{34}^j + V_c^s(\omega^2 + k_s^2)$

$$(KKJ)_{ab}^{\nu} = \frac{2}{\pi}(k_a k_b z)^{\nu} z K_{\nu}(k_a z) K_{\nu}(k_b z) J_{\nu}(\omega z)$$

Double copy ansatz

- Ansatz: $\psi^{(s)}_{\gamma,\mathrm{DC}} = \int \frac{d\omega\,\omega}{k_s^2+\omega^2} dz\, dz' \,(KKJ)^{3/2}_{12}(z)(KKJ)^{3/2}_{34}(z') \times \left(N_s^2 - \frac{1}{2}\tilde{V}^{ij}_{12} H_{ij} \tilde{V}^{kl}_{34} H_{kl} + \frac{1}{2}(\epsilon_1\cdot\epsilon_2)^2(\epsilon_3\cdot\epsilon_4)^2(\omega^2+k_s^2)^2 \right)$

  where $\tilde{V}^{ij}_{ab} = V^i_{ab} V^j_{ab}$
- Satisfies flat space limit, COT, and MLT
- Can be written in terms of deformed numerators: $N_s^\gamma = \frac{1}{2}\left(N^-_{12}N^+_{34} + N^+_{12}N^-_{34}\right)$

$$N^{\pm}_{12} = N_s + \frac{i}{\sqrt{2}}\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4\left(\omega^2+k_s^2\right) \pm \frac{1}{\sqrt{2}}\tilde{V}^{ij}_{12}H_{ij}$$
$$N^{\pm}_{34} = N_s - \frac{i}{\sqrt{2}}\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4\left(\omega^2+k_s^2\right) \pm \frac{1}{\sqrt{2}}\tilde{V}^{ij}_{34}H_{ij}$$

Corrections

- The double copy ansatz captures most terms in the 4-graviton wavefunction
- But it has spurious poles, so we add a term to cancel the poles and another to restore the MLT:

$$\psi_{\gamma}^{(s)} = \psi_{\gamma,\mathrm{DC}}^{(s)} + (\epsilon_1 \cdot \epsilon_2)^2 (\epsilon_3 \cdot \epsilon_4)^2 \left(\psi_{\mathrm{sp}}^{(s)} + \psi_{\mathrm{MLT}}^{(s)}\right)$$

$$\psi_{\mathrm{sp}}^{(s)} = -\frac{1}{2}\left(\frac{2k_1k_2k_3k_4}{(k_{12}+k_{34})^2}\left(\frac{\alpha^2}{k_{34}}+\frac{\beta^2}{k_{12}}\right)+\frac{\alpha^2k_3k_4}{k_{34}}+\frac{\beta^2k_1k_2}{k_{12}}\right)$$

$$\psi_{\mathrm{MLT}}^{(s)} = \frac{5k_1k_2k_3k_4}{E} + \frac{E}{2}(k_{12}k_{34} - 4k_1k_2 - 4k_3k_4) - \frac{1}{E}(k_1k_2 - k_3k_4)(\alpha^2 - \beta^2) - 3(\alpha^2 k_{12} + \beta^2 k_{34})$$

where $k_{ab} = k_a + k_b$, $\alpha = k_1 - k_2, \quad \beta = k_3 - k_4$

Conclusions

- SDG in AdS_4 can be described by a simple scalar theory with a deformed Poisson bracket
- Can be derived from asymmetric double copy combining the flat space kinematic algebra with a new deformed kinematic algebra
- Encodes two $w_{1+\infty}$ algebras, one of which is deformed
- Suggests a new connection between AdS/CFT and flat space holography!
- Combining double copy with bootstrap gives a compact new formula for tree-level wavefunction of four gravitons in dS_4
- We do not yet have a systematic understanding of double copy in $(A)dS_4$ but it appears to be useful

Future

- Compute boundary correlators of SDYM and SDG in AdS_4
- Investigate how they encode the double copy and $w_{1+\infty}$
- Integrability of SDG in AdS_4 (Campiglia,Nagy)
- Derive the double copy in AdS_4 by expanding around self-dual sector
- Chiral higher spin theory in AdS_4 from Moyal deformation? (Monteiro,Sharapov,Skvortsov)