Bootstrapping the AdS Virasoro-Shapiro amplitude

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Based on:

2204.07542, 2209.06223, 2303.08834, 2305.03593 with Luis F. Alday, João Silva

2306.12786 with Luis F. Alday

2308.03683 with Giulia Fardelli, João Silva

Why AdS/CFT?

1. String theory is a theory of quantum gravity.

   We should understand it on curved backgrounds.

2. AdS/CFT allows us to study strongly coupled gauge theories.

   Relevant for the standard model.

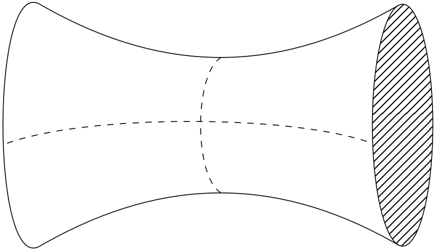
3. Having multiple descriptions for the same physical system can be extremely powerful.

   See this talk!

Anti-de Sitter space

$\mathrm{AdS}_{d+1} \in \mathbb{R}^{d,2}$

$$-X_0^2 - X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = -R_{\mathrm{AdS}}^2$$

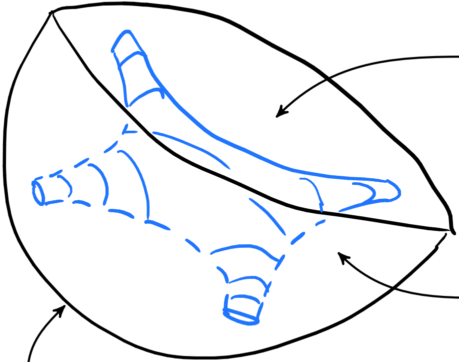


Isometry group:

$$SO(d,2) = \text{ conformal group in } \mathbb{R}^{d-1,1}$$

AdS_{d+1} has a d-dimensional conformal boundary.

1 process - 3 descriptions



5d bulk of AdS:
IIb string theory on $AdS_5 \times S^5$

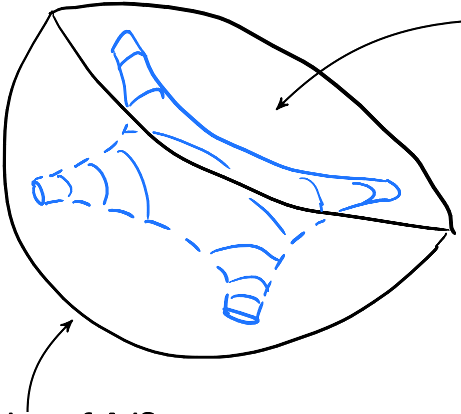
- strings on curved background

2d string world-sheet:
2d CFT???

4d boundary of AdS:
$\mathcal{N} = 4$ super Yang Mills theory

- non-abelian gauge theory
- conformal symmetry
- supersymmetry
- integrable

This talk:
Find the amplitude without quantizing the string.



5d bulk of AdS:

IIb string theory on $AdS_5 \times S^5$

- AdS radius R_{AdS}
- string length L_s
- string coupling g_s

4d boundary of AdS:

$\mathcal{N} = 4$ super Yang Mills theory

- $SU(N)$ gauge group
- coupling $\lambda = \frac{R_{\text{AdS}}}{L_s}$

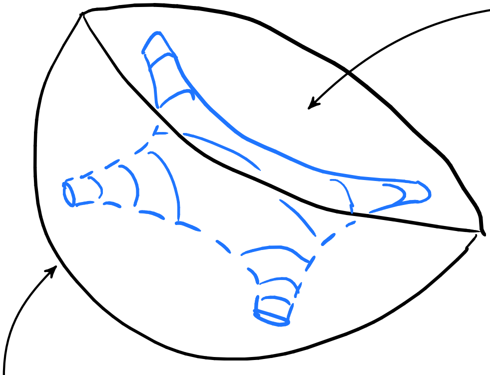
Weakly coupled strings:

$$g_s \ll 1 \quad \Leftrightarrow \quad N \gg 1$$

Expansion around flat space:

$$\frac{R_{\text{AdS}}}{L_s} \gg 1 \quad \Leftrightarrow \quad \lambda \gg 1$$

Outline



Part 1:
String scattering in flat space
($R_{\text{AdS}} \to \infty$)

Part 2:

- Boundary CFT description
- Finding the world-sheet correlator
- Checks: integrability and localization

Part 1
Flat Space Review

The Virasoro-Shapiro amplitude (flat space)

In the beginning, there was the amplitude.
[Veneziano,1968;Virasoro,1969;Shapiro,1970]

Scattering of 4 gravitons in the type IIb superstring:

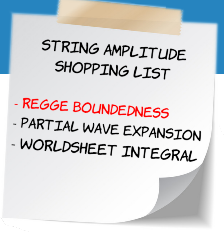
Virasoro-Shapiro amplitude

$$A^{(0)}(S,T) = -\frac{\Gamma(-S)\,\Gamma(-T)\,\Gamma(-U)}{\Gamma(S+1)\,\Gamma(T+1)\,\Gamma(U+1)}$$

$$S = -\frac{L_s^2}{4}(p_1+p_2)^2\,,\ T = -\frac{L_s^2}{4}(p_1+p_3)^2\,,\ U = -\frac{L_s^2}{4}(p_1+p_4)^2$$

$$S + T + U = 0$$

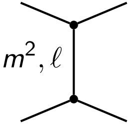
Regge boundedness (flat space)



String amplitudes have soft UV (Regge) bahaviour

$$\lim_{|S|\to\infty} A^{(0)}(S,T) \sim S^{T+\alpha_0}, \quad \mathrm{Re}(T) < 0$$

and higher spin resonances



$$= \frac{P_\ell(S)}{T - m^2} \qquad\qquad P_\ell(S) = S^\ell + O(S^{\ell-1})$$

Regge bahaviour places strong constraints on the coefficients $a_{\delta,\ell}$ in

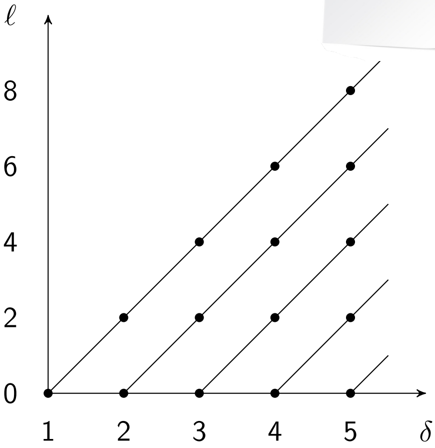
$$A^{(0)}(S,T) = \sum_{(\delta,\ell)} \frac{a_{\delta,\ell} P_\ell(S)}{T - \delta}$$

The spectrum (flat space)

The exchanged massive string spectrum is extracted via the partial wave expansion

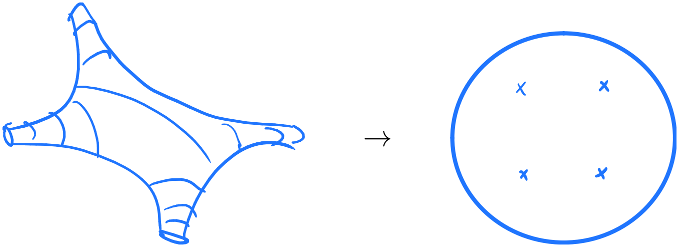
$$A^{(0)}(S,T) = \sum_{(\delta,\ell)} \frac{a_{\delta,\ell} P_\ell(S)}{T-\delta}$$

It forms linear Regge trajectories.



World-sheet integral (flat space)

The amplitude is also given by an integral over world-sheets:



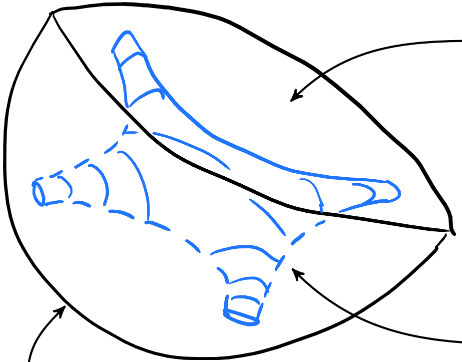
$$A^{(0)}(S,T) = \int d^2z\ |z|^{-2S-2}|1-z|^{-2T-2} G^{(0)}_{\text{tot}}(S,T,z)$$

$$G^{(0)}_{\text{tot}}(S,T,z) = \frac{1}{3}\left(\frac{1}{U^2} + \frac{|z|^2}{S^2} + \frac{|1-z|^2}{T^2}\right)$$

The integrand is a single-valued function of z!

Part 2
AdS/CFT

1 process - 3 observables



5d bulk of AdS:
scattering amplitude of 4 gravitons:

$$A(S,T) = \sum_{k=0}^{\infty} \left(\frac{L_s^2}{R_{\text{AdS}}^2}\right)^k A^{(k)}(S,T)$$

2d string world-sheet:
world-sheet integrand:

$$G_{\text{tot}}(S,T,z) = \sum_{k=0}^{\infty} \left(\frac{L_s^2}{R_{\text{AdS}}^2}\right)^k G_{\text{tot}}^{(k)}(S,T,z)$$

4d boundary of AdS:
correlator of 4 stress-tensors:

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$$

Boundary correlator to bulk amplitude

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$$

$\downarrow$ superconformal Ward identity

$$H(U,V) \qquad U=\frac{(x_1-x_2)^2(x_3-x_4)^2}{(x_1-x_3)^2(x_2-x_4)^2},\ V=\frac{(x_1-x_4)^2(x_2-x_3)^2}{(x_1-x_3)^2(x_2-x_4)^2}$$

$\downarrow$ Mellin transform

$$M(s,t)$$

$\downarrow$ Borel transform (flat space limit [Penedones;2010])

$$A(S,T) = \sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{\lambda}}\right)^k A^{(k)}(S,T)$$

$\downarrow$ world-sheet integral

$$A^{(k)}(S,T)=\int d^2z\ |z|^{-2S-2}|1-z|^{-2T-2}G^{(k)}_{\text{tot}}(S,T,z)$$

Integral transforms

Mellin transform

$$H(U,V) = \int_{-i\infty}^{i\infty} \frac{dsdt}{(4\pi i)^2} U^{\frac{s}{2}+\frac{2}{3}} V^{\frac{t}{2}-\frac{4}{3}} \Gamma\left(\frac{4}{3}-\frac{s}{2}\right)^2 \Gamma\left(\frac{4}{3}-\frac{t}{2}\right)^2 \Gamma\left(\frac{4}{3}-\frac{u}{2}\right)^2 M(s,t)$$

Borel transform

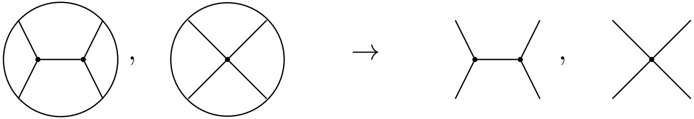
$$A(S,T) = \lambda^{\frac{3}{2}} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} e^{\alpha} \alpha^{-6} M\left(\frac{2\sqrt{\lambda}S}{\alpha}, \frac{2\sqrt{\lambda}T}{\alpha}\right)$$

The Borel transform

Borel transform

$$A(S,T) = \lambda^{\frac{3}{2}} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} e^{\alpha} \alpha^{-6} M\left(\frac{2\sqrt{\lambda}S}{\alpha}, \frac{2\sqrt{\lambda}T}{\alpha}\right)$$

1. Maps Witten diagrams to Feynman diagrams for $R_{\text{AdS}} \to \infty$ [Penedones;2010]



2. Borel summation of the low energy expansion:

$$M(s,t) = \sum_{p,q} \frac{\Gamma(6+p+q)}{\lambda^{\frac{3}{2}}} \left(\frac{s}{2\sqrt{\lambda}}\right)^p \left(\frac{t}{2\sqrt{\lambda}}\right)^q \alpha_{p,q} \quad \Rightarrow \quad A(S,T) = \sum_{p,q} S^p T^q \alpha_{p,q}$$

3. Stringy flat space limit:

$$\frac{R_{\text{AdS}}}{L_s} >> 1\,, \qquad S \sim \frac{L_s}{R_{\text{AdS}}} s \sim L_s^2 (p_1+p_2)^2 \text{ finite}$$

Bound on chaos

The bound on chaos: [Maldacena,Shenker,Stanford;2015]

- limits growth of chaos in thermal quantum systems with many degrees of freedom.
- is diagnosed with out-of-time-ordered correlators.
- applies to correlators in large N CFTs in the Regge limit.

Bound on chaos in Mellin space

$$\lim_{|s|\to\infty} |M(s,t)| \lesssim |s|^{-2}, \;\; \mathrm{Re}(t) < 2$$

Operator product expansion

We can expand $\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$ using:

Operator product expansion (OPE)

$$\mathcal{O}_2(x)\mathcal{O}_2(0) = \sum_{\mathcal{O}_{\Delta,\ell}\text{ primaries}} C_{\Delta,\ell}\; c_{\Delta,\ell}(x,\partial_y)\mathcal{O}_{\Delta,\ell}(y)\Big|_{y=0}$$

OPE data

- $\Delta =$ dimension
- $\ell =$ spin
- $C_{\Delta,\ell} =$ OPE coefficients

$M(s,t)$ has only simple poles, given by [Mack;2009], [Penedones,Silva,Zhiboedov;2019]

Poles and residues of $M(s,t)$

$$M(s,t) \sim \frac{C^2_{\Delta,\ell}\, Q_{\Delta,\ell,m}(t)}{s-(\Delta-\ell+2m)}$$

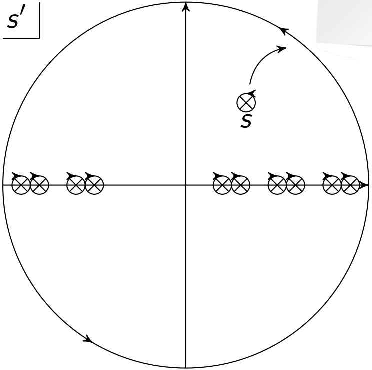
Dispersion relation

$M(s,t)$ has only OPE poles:

$$\text{poles} \sim \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s' - (\Delta - \ell + 2m)}$$

Regge bounded due to bound on chaos:

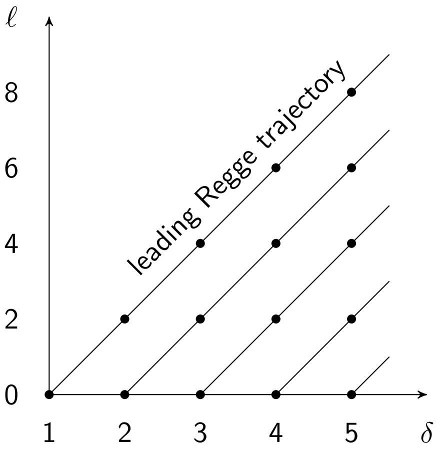
$$\lim_{|s|\to\infty} |M(s,t)| \lesssim |s|^{-2}, \ \text{Re}(t) < 2$$



$$M(s,t) = \oint_s \frac{ds'}{2\pi i} \frac{M(s',t)}{(s'-s)} = -\sum_{\text{operators}} \left(\frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s - (\Delta - \ell + 2m)} + \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{u - (\Delta - \ell + 2m)} \right)$$

Spectrum of exchanged operators

Exchanged operators: short single-trace operators of $\mathcal{N}=4$ SYM theory



known from flat space

$A^{(0)}$ data	$A^{(1)}$ data	$A^{(2)}$ data

$$\Delta_{\delta,\ell} = \lambda^{\frac{1}{4}}\Delta^{(0)}_{\delta,\ell} + \lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell} + \lambda^{-\frac{3}{4}}\Delta^{(2)}_{\delta,\ell}$$

$$C^2_{\delta,\ell} = C^{2(0)}_{\delta,\ell} + \lambda^{-\frac{1}{2}} C^{2(1)}_{\delta,\ell} + \lambda^{-1} C^{2(2)}_{\delta,\ell}$$

$\Delta^{(1)}_{\delta,\ell}$, $\Delta^{(2)}_{\delta,\ell}$ on leading trajectory known from integrability

Degeneracies in the spectrum

The amplitude encodes OPE data of multiple degenerate superprimaries.

We determined the degeneracies starting from type IIb strings in flat 10d:

$$SO(9) \to SO(4) \times SO(5) \overset{KK}{\to} SO(4) \times SO(6)$$

ℓ \ δ	1	2	3	4	5	6
						1
8					1	6
6				1	6	52
4			1	6	40	331
2		1	4	24	157	1104
0	1	2	6	22	99	547

Number of superconformal long multiplets with superprimary $\mathcal{O}_{\delta,\ell}$

- $SO(6)$ singlet
- $\Delta = 2\sqrt{\delta}\lambda^{\frac{1}{4}} + O(\lambda^0)$

Example: $\mathcal{O}_{1,0} =$ Konishi $\sim \mathrm{Tr}(\phi^I \phi_I)$

The counting was confirmed for $\delta \leq 3$ with quantum spectral curve. [Gromov,Hegedus,Julius,Sokolova;2023]

Dispersion relation $\to$ Residues

Dispersion relation for $M(s,t) \rightsquigarrow A^{(k)}(S,T)$ expanded around $S = \delta = 1, 2, \ldots$:

$$A^{(k)}(S,T) = \frac{R^{(k)}_{3k+1}(T, \delta, C^{2(0)}_{\delta,\ell})}{(S-\delta)^{3k+1}} + \ldots + \frac{R^{(k)}_{1}(T, \delta, C^{2(0)}_{\delta,\ell}, \ldots, \Delta^{(k)}_{\delta,\ell}, C^{2(k)}_{\delta,\ell})}{S-\delta} + \text{reg.}$$

Two lessons:

1. (OPE data)$^{(k-1)}$ fixes most residues of $A^{(k)}(S,T)$!
2. $G^{(k)}_{\text{tot}}(S,T,z)$ should have transcendentality $3k$:

$$\int d^2z\, |z|^{-2S-2} |1-z|^{-2T-2} \log^{3k}|z|^2 \propto \frac{1}{(S-\delta)^{3k+1}} + O\left((S-\delta)^0\right)$$

Next steps (order by order):

- Write world-sheet ansatz for $A^{(k)}(S,T)$.
- Compute its residues and match with the above to fix ansatz.

Single-valued multiple polylogarithms

MPLs:

$$L_{a_1\ldots a_{|w|}}(z) = \int_0^z \frac{dt}{t-a_1} L_{a_2\ldots a_{|w|}}(t)$$

$$L(z) = 1, \quad a_i \in \{0,1\}$$

Examples :

$$L_{0^p}(z) = \frac{1}{p!}\log^p(z)$$

$$L_{1^p}(z) = \frac{1}{p!}\log^p(1-z)$$

$$L_{0^{p-1}1}(z) = -\mathrm{Li}_p(z)$$

$$\downarrow z = 1$$

MZVs: $\zeta(n_1, n_2, \ldots)$

SVMPLs: [Brown;2004]

$$\mathcal{L}_w(z) = \sum_{\substack{w_1,w_2 \\ |w_1|+|w_2|=|w|}} c_{w,w_1,w_2} L_{w_1}(z) L_{w_2}(\bar{z})$$

$$\mathcal{L}_{0^p}(z) = \frac{1}{p!}\log^p|z|^2$$

$$\mathcal{L}_{1^p}(z) = \frac{1}{p!}\log^p|1-z|^2$$

$$\mathcal{L}_{01}(z) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log(1-\bar{z})\log|z|^2$$

$$\downarrow z = 1$$

SVMZVs: $\zeta^{\mathrm{sv}}(n_1, n_2, \ldots)$ [Brown;2013]

Toy model for strings in AdS

Polyakov action:

$$S_P = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\,g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X)$$

$$= S_{\text{flat}} + \frac{1}{R_{\text{AdS}}^2}\underbrace{\lim_{q\to 0}\frac{\partial^2}{\partial q^\mu\partial q^\nu}V^{\mu\nu}_{\text{graviton}}(q)}_{\equiv\tilde{V}} + \dots$$

AdS metric expanded around flat space:

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{R_{\text{AdS}}^2} + \cdots$$

$$h_{\mu\nu} \sim X_\mu X_\nu \sim \lim_{q\to 0}\frac{\partial^2}{\partial q^\mu\partial q^\nu}e^{iq\cdot X}$$

Amplitude:

$$A_4(p_i) \sim \int \mathcal{D}X\mathcal{D}g\, e^{-S_P}V^4_{\text{graviton}} = \int \mathcal{D}X\mathcal{D}g\, e^{-S_{\text{flat}}}\left(1 - \frac{\tilde{V}}{R_{\text{AdS}}^2} + \frac{1}{2}\frac{\tilde{V}^2}{R_{\text{AdS}}^4} + \dots\right)V^4_{\text{graviton}}$$

$$\Rightarrow \quad A_4^{(k)}(p_i) \sim \lim_{q_i\to 0}\left(\frac{\partial}{\partial q_i}\right)^{2k} A^{(0)}_{4+k}(p_i, q_i) + \dots$$

Soft gravitons in flat space

$$A_4^{(k)}(p_i) \sim \lim_{\epsilon\to 0}\left(\frac{\partial}{\epsilon\,\partial q_i}\right)^{2k} A_{4+k}^{(0)}(p_i,\epsilon q_i) + \ldots$$

Soft graviton theorem:

$$A_{n+1}(p_1,\ldots,p_n,\epsilon q) = \sum_{i=1}^{n}\left(\frac{1}{\epsilon}\frac{\varepsilon_{\mu\nu}p_i^\mu p_i^\nu}{p_i\cdot q} + \frac{\varepsilon\cdot p_i \varepsilon_\mu q_\nu J_i^{\mu\nu}}{p_i\cdot q} + O(\epsilon)\right) A_n(p_1,\ldots,p_n)$$

Flat space amplitude with k soft gravitons:

$$A_{4+k}^{(0)}(p_i,\epsilon q_i) \sim \frac{1}{\epsilon^k}A_4^{(0)}(p_i) + \frac{1}{\epsilon^{k-1}}\text{"}\partial_{p_i}\text{"}\,A_4^{(0)}(p_i) + \ldots$$

$$\sim \int d^2z|z|^{-2S-2}|1-z|^{-2T-2}\left(\frac{1}{\epsilon^k} + \frac{1}{\epsilon^{k-1}}\left(\#\log|z|^2 + \#\log|1-z|^2\right) + \ldots + \epsilon^{2k}\mathcal{L}_{|w|=3k}(z)\right)$$

$\Rightarrow \quad G_{\text{tot}}^{(k)}(S,T,z) \;\sim\;$ single-valued multiple polylogs of weight $\leq 3k$

World-sheet correlator (ansatz)

Ansatz:

$$A^{(k)}(S,T) = B^{(k)}(S,T) + B^{(k)}(U,T) + B^{(k)}(S,U)$$

$$B^{(k)}(S,T) = \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} G^{(k)}(S,T,z)$$

Assumed properties of $G^{(k)}(S,T,z)$:

- uniform transcendentality $3k$ (SVMPLs(z), SVMZVs)
- rational function in S, T with homogeneity $2k-2$
- denominator $= U^n$, $n \leq 2$
- crossing symmetry: $G^{(k)}(S,T,z) = G^{(k)}(T,S,1-z)$

Recall (flat space):

$$G^{(0)}(S,T,z) = \frac{1}{3U^2}$$

World-sheet correlator (first correction)

Symmetrised single-valued multiple polylogs:

$$\mathcal{L}^{\pm}_{w}(z) = \mathcal{L}_{w}(z) \pm \mathcal{L}_{w}(1-z) + \mathcal{L}_{w}(\bar{z}) \pm \mathcal{L}_{w}(1-\bar{z})$$

$$k = 1: \quad \text{weight 3 basis} = \text{4 symmetric} + \text{3 antisymmetric functions}$$

Solution:

$$\begin{aligned} G^{(1)}(S,T,z) = & -\frac{1}{6}\mathcal{L}^{+}_{000}(z) + 0\mathcal{L}^{+}_{001}(z) - \frac{1}{4}\mathcal{L}^{+}_{010}(z) + 2\zeta(3) \\ & + \frac{S-T}{S+T}\left(-\frac{1}{6}\mathcal{L}^{-}_{000}(z) + \frac{1}{3}\mathcal{L}^{-}_{001}(z) + \frac{1}{6}\mathcal{L}^{-}_{010}(z)\right) \end{aligned}$$

World-sheet correlator (second correction)

$k = 2:$ weight 6 basis = 25 symmetric + 20 antisymmetric functions:

$$\begin{aligned}
\vec{L}^{+} = \Big(&\mathcal{L}^{+}_{000000}(z), \mathcal{L}^{+}_{000001}(z), \mathcal{L}^{+}_{000010}(z), \mathcal{L}^{+}_{000011}(z), \mathcal{L}^{+}_{000100}(z), \mathcal{L}^{+}_{000101}(z), \mathcal{L}^{+}_{000110}(z),\\
&\mathcal{L}^{+}_{000111}(z), \mathcal{L}^{+}_{001001}(z), \mathcal{L}^{+}_{001010}(z), \mathcal{L}^{+}_{001011}(z), \mathcal{L}^{+}_{001100}(z), \mathcal{L}^{+}_{001101}(z), \mathcal{L}^{+}_{001110}(z),\\
&\mathcal{L}^{+}_{010001}(z), \mathcal{L}^{+}_{010010}(z), \mathcal{L}^{+}_{010101}(z), \mathcal{L}^{+}_{010110}(z), \mathcal{L}^{+}_{011001}(z), \mathcal{L}^{+}_{011110}(z),\\
&\zeta(3)\mathcal{L}^{+}_{000}(z), \zeta(3)\mathcal{L}^{+}_{001}(z), \zeta(3)\mathcal{L}^{+}_{010}(z), \zeta(5)\mathcal{L}^{+}_{0}(z), \zeta(3)^2\Big)\\
\vec{L}^{-} = \Big(&\mathcal{L}^{-}_{000000}(z), \mathcal{L}^{-}_{000001}(z), \mathcal{L}^{-}_{000010}(z), \mathcal{L}^{-}_{000011}(z), \mathcal{L}^{-}_{000100}(z), \mathcal{L}^{-}_{000101}(z), \mathcal{L}^{-}_{000110}(z),\\
&\mathcal{L}^{-}_{001001}(z), \mathcal{L}^{-}_{001010}(z), \mathcal{L}^{-}_{001100}(z), \mathcal{L}^{-}_{001101}(z), \mathcal{L}^{-}_{001110}(z), \mathcal{L}^{-}_{010001}(z), \mathcal{L}^{-}_{010010}(z),\\
&\mathcal{L}^{-}_{010110}(z), \mathcal{L}^{-}_{011110}(z), \zeta(3)\mathcal{L}^{-}_{000}(z), \zeta(3)\mathcal{L}^{-}_{001}(z), \zeta(3)\mathcal{L}^{-}_{010}(z), \zeta(5)\mathcal{L}^{-}_{0}(z)\Big)
\end{aligned}$$

Result:

$$G^{(2)}(S,T,z) = (S^2+T^2)\; \vec{r}_1 \cdot \vec{L}^{+} + ST\; \vec{r}_2 \cdot \vec{L}^{+} + \frac{(S^2+T^2)(S-T)}{S+T}\; \vec{r}_3 \cdot \vec{L}^{-} + \frac{ST(S-T)}{S+T}\; \vec{r}_4 \cdot \vec{L}^{-}$$

$$\vec{r}_1 = \left(-\tfrac{1}{18}, \tfrac{2971}{432}, \tfrac{13111}{3888}, -\tfrac{7271}{3888}, \ldots\right), \quad \vec{r}_2 = \ldots$$

We need to input the dimension of 1 operator ($\Delta^{(2)}_{1,0}$ = Konishi) to fix $A^{(2)}(S,T)$ completely.

OPE data

We compute $\forall \delta, \ell \qquad \# \in \mathbb{Q}$

$$k = 0: \qquad \langle C^{2(0)} \rangle_{\delta,\ell} = \#$$

$$k = 1: \qquad \sqrt{\delta}\langle C^{2(0)} \Delta^{(1)} \rangle_{\delta,\ell} = \#\,, \quad \langle C^{2(1)} \rangle_{\delta,\ell} = \#\zeta(3) + \#$$

$$k = 2: \qquad \langle C^{2(0)} (\Delta^{(1)})^2 \rangle_{\delta,\ell} = \#$$

$$\sqrt{\delta}\langle C^{2(0)} \Delta^{(2)} + C^{2(1)} \Delta^{(1)} \rangle_{\delta,\ell} = \#\zeta(3) + \#$$

$$\langle C^{2(2)} \rangle_{\delta,\ell} = \#\zeta(3)^2 + \#\zeta(5) + \#\zeta(3) + \#$$

Leading Regge trajectory:

$$\Delta\left(\tfrac{\ell}{2} + 1, \ell\right) = 2\sqrt{\tfrac{\ell}{2} + 1}\lambda^{\frac{1}{4}} - 2 + \frac{3\ell^2 + 10\ell + 16}{4\sqrt{2(\ell+2)}}\lambda^{-\frac{1}{4}}$$
$$- \frac{21\ell^4 + 144\ell^3 + 292\ell^2 + 80\ell - 128 + 96(\ell+2)^3\zeta(3)}{32(2(\ell+2))^{\frac{3}{2}}}\lambda^{-\frac{3}{4}} + O(\lambda^{-\frac{5}{4}})\,,$$

Agrees with integrability result!

[Gromov,Serban,Shenderovich,Volin;2011],[Basso;2011],[Gromov,Valatka;2011]

The low energy expansion ($S \sim T \sim 0$)
can be computed following [Vanhove,Zerbini;2018]

$$\int d^2z |z|^{-2S-2}|1-z|^{-2T-2}\mathcal{L}_w(z)$$

$$= \text{poles} + \sum_{p,q=0}^{\infty} (-S)^p(-T)^q \int \frac{d^2z}{|z|^2|1-z|^2} \underbrace{\mathcal{L}_{0^p}(z)\mathcal{L}_{1^q}(z)\mathcal{L}_w(z)}_{=\sum_{W\in 0^p ⧢ 1^q ⧢ w}\mathcal{L}_W(z)}$$

$$= \text{poles} + \sum_{p,q=0}^{\infty} (-S)^p(-T)^q \sum_{W\in 0^p ⧢ 1^q ⧢ w} \underbrace{\mathcal{L}_{0W}(1) - \mathcal{L}_{1W}(1)}$$

Single-valued multiple zeta values of weight $1+p+q+|w|$

As in flat space! [Stieberger;2013],[Brown,Dupont;Schlotterer,Schnetz;Vanhove,Zerbini;2018]

Wilson coefficients

$$A^{(k)}(S,T) = \mathrm{SUGRA}^{(k)} + 2\sum_{a,b=0}^{\infty}\left(\tfrac{1}{2}(S^2+T^2+U^2)\right)^a (STU)^b \alpha_{a,b}^{(k)}$$

We compute $\forall a, b$ $\qquad \# \in \mathbb{Q}$

$$\alpha_{a,b}^{(0)} = \sum_{k_i \text{ odd}} \#\zeta(k_1)\ldots\zeta(k_n)$$

$$\alpha_{a,b}^{(1)} = \sum_{k_i \text{ odd}} \#\zeta^{\mathrm{sv}}(k_1,k_2,k_3)\zeta(k_4)\ldots\zeta(k_n) + \ldots$$

$$\alpha_{a,b}^{(2)} = \sum_{k_i \text{ odd}} \#\zeta^{\mathrm{sv}}(k_1,k_2,k_3,k_4,k_5)\zeta(k_6)\ldots\zeta(k_n) + \ldots$$

In particular:

$$\alpha_{0,0}^{(1)} = 0\,, \quad \alpha_{1,0}^{(1)} = -\frac{22}{3}\zeta(3)^2\,, \quad \alpha_{0,0}^{(2)} = \frac{49}{4}\zeta(5)\,, \quad \alpha_{1,0}^{(2)} = \frac{4091}{16}\zeta(7)$$

Agrees with localisation result!
[Binder,Chester,Pufu,Wang;2019],[Chester,Pufu;2020],[Alday,TH,Silva;2022]

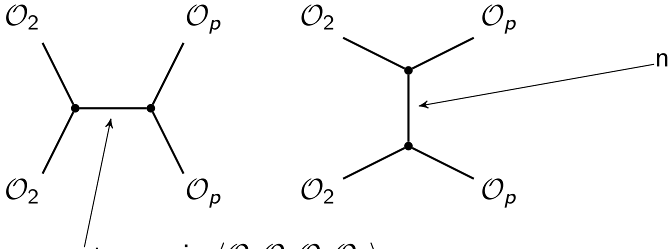
Correlators with Kaluza-Klein modes

We also computed the $O(1/\sqrt{\lambda})$ string amplitude for

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_p(x_3)\mathcal{O}_p(x_4)\rangle$$

$\mathcal{O}_p$ = KK mode
$\Delta = p = 3, 4, \ldots$
$[p, 0, 0]$ of $SO(6)$

- less crossing symmetry: $A(S, T) = A(S, U)$
- new operators:



new operators:

- odd spin
- non-zero R charge

same operators as in $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle$

World-sheet correlator for $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p\rangle$

Ansatz:

$$A^{(1)}(S,T) = B_1^{(1)}(S,T) + B_1^{(1)}(S,U) + B_1^{(1)}(U,T) + B_2^{(1)}(S,T) + B_2^{(1)}(S,U)$$

$$B_i^{(1)}(S,T) = \int d^2z|z|^{-2S-2}|1-z|^{-2T-2}G_i^{(1)}(S,T,z)\,, \quad i=1,2$$

Result:

$$\begin{aligned}
G_1^{(1)}(S,T,z) &= \frac{1}{24}\left(-p^2\mathcal{L}^+_{000}(z) + 2(p-2)p\mathcal{L}^+_{001}(z) + (p^2-2p-6)\mathcal{L}^+_{010}(z) + 48\zeta(3)\right)\\
&+ \frac{p^2(S-T)}{24(S+T)}\left(-\mathcal{L}^-_{000}(z) + 2\mathcal{L}^-_{001}(z) + \mathcal{L}^-_{010}(z)\right)\\
G_2^{(1)}(S,T,z) &= \frac{p(p-2)}{24(S+T)}\left(3S\mathcal{L}^+_{000}(z) - 2(2S+T)\mathcal{L}^+_{001}(z) - (2S+T)\mathcal{L}^+_{010}(z)\right)\\
&+ \frac{p(p-2)}{24(S+T)}\left(3S\mathcal{L}^-_{000}(z) - 2(2S-T)\mathcal{L}^-_{001}(z) - (2S-T)\mathcal{L}^-_{010}(z)\right)
\end{aligned}$$

Degeneracies of odd-spin operators

Even spin, $[0,0,0]$ of $SO(6)$:

ℓ \ δ	1	2	3	4	5	6
						1
8					1	6
6				1	6	52
4			1	6	40	331
2		1	4	24	157	1104
0	1	2	6	22	99	547

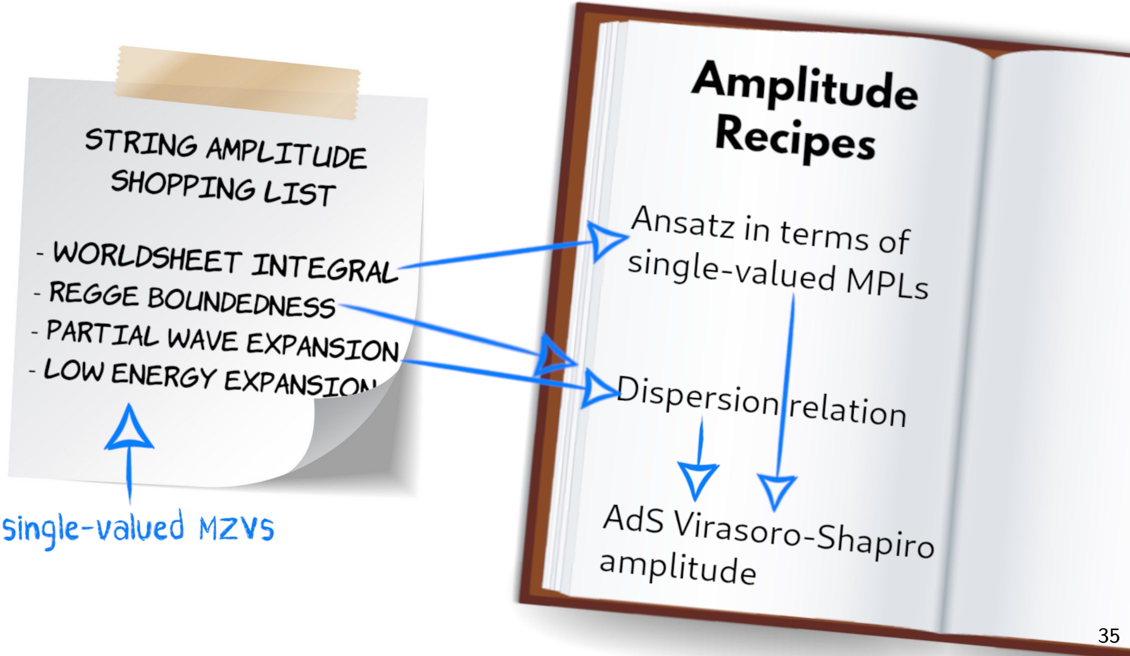
Odd spin, $[1,0,0]$ of $SO(6)$:

ℓ \ δ	1	2	3	4	5	6
9						2
7					2	32
5				2	28	316
3			2	22	206	1836
1		2	10	70	502	3536

The leading odd spin trajectory has very low degeneracies!

Good target for further study (our method, quantum spectral curve, ...).

Summary



STRING AMPLITUDE SHOPPING LIST
- WORLDSHEET INTEGRAL
- REGGE BOUNDEDNESS
- PARTIAL WAVE EXPANSION
- LOW ENERGY EXPANSION
single-valued MZVs
Amplitude Recipes
Ansatz in terms of single-valued MPLs
Dispersion relation
AdS Virasoro-Shapiro amplitude

Future directions

- Open strings / AdS Veneziano amplitude
  - Generalizations of KLT relations / single-valued map?
  - Gluon scattering on $AdS_5 \times S^5/\mathbb{Z}_2$ with $D7$ branes
  - $4d$ $\mathcal{N} = 2$ $USp(2N)$ gauge theory: localization results available [Beccaria,Korchemsky,Tseytlin;2022],[Behan,Chester,Ferrero;2023]
  - Problem: no strong coupling OPE data known for consistency checks. Integrability?
- Compute $A^{(k)}(S, T)$ directly from string theory?
  - Ramond-Ramond background flux...
  - Pure spinors?

Thank you!